Examples 7 in Parabolas

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Examples 7 in Circles

0. Suppose that

$\boldsymbol{A}$ is a circle $(\boldsymbol{x} - \boldsymbol{a}_1)^2 + (\boldsymbol{y} - \boldsymbol{b}_1)^2 = \boldsymbol{c}_1^2$, where $\boldsymbol{a}_1$, $\boldsymbol{b}_1$, and $\boldsymbol{c}_1$ are constant, and $\boldsymbol{c}_1 > \mathbf{0}$.

$\boldsymbol{B}$ is a circle $(\boldsymbol{x} - \boldsymbol{a}_2)^2 + (\boldsymbol{y} - \boldsymbol{b}_2)^2 = \boldsymbol{c}_2^2$, where $\boldsymbol{a}_2$, $\boldsymbol{b}_2$, and $\boldsymbol{c}_2$ are constant, and $\boldsymbol{c}_2 > \mathbf{0}$.

$\boldsymbol{d}$ is the distance between the centers of the two circles $\boldsymbol{A}$ and $\boldsymbol{B}$.

Then, find the relationship among $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{d}$ in each of the cases below.

0.0. The two circles meet at two points.

0.1. The two circle meet at one point (i.e., they are tangent to each other).

0.2. The two circles do not meet each other.

1. Suppose that

$\boldsymbol{A}$ is a circle $\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{c} = \mathbf{0}$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant.

$\boldsymbol{B}$ is a circle $\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{dx} + \boldsymbol{ey} + \boldsymbol{f} = \mathbf{0}$, where $\boldsymbol{d}$, $\boldsymbol{e}$, and $\boldsymbol{f}$ are constant.

Suppose also, that the two circles meet each other at two particular points.

Then, find another circle passing through the two particular points.

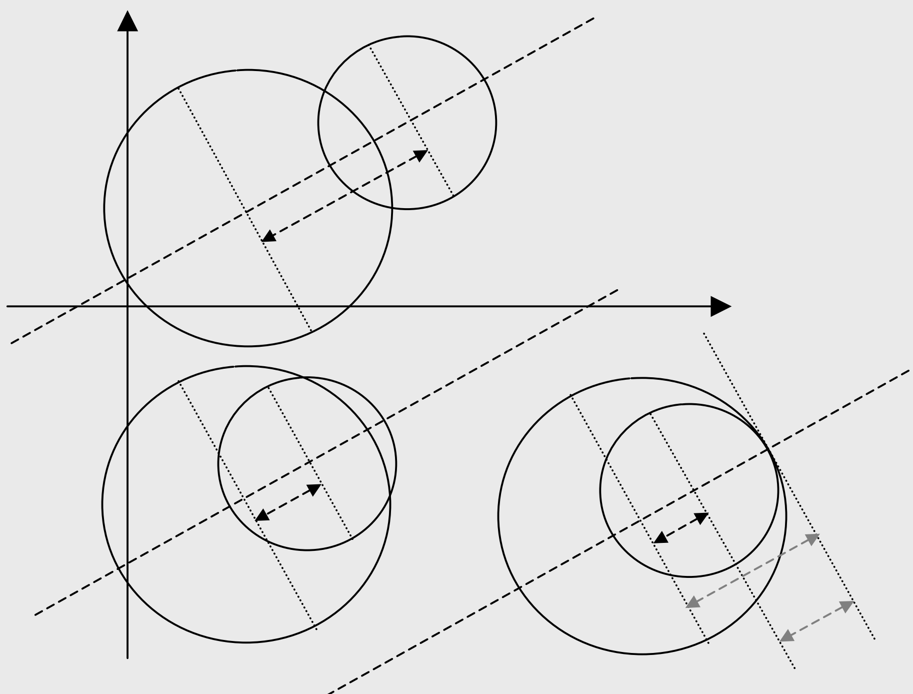
Suggestions or Solutions
To the Problem 0 in the Example 0

Suppose that
A is a circle $(x - a_1)^2 + (y - b_1)^2 = c_1^2$, where a_1, b_1, and c_1 are constant, and $c_1 > 0$.
B is a circle $(x - a_2)^2 + (y - b_2)^2 = c_2^2$, where a_2, b_2, and c_2 are constant, and $c_2 > 0$.
d is the distance between the centers of the two circles A and B.

Then, find the relationship among c_1, c_2, and d if the two circles meet at two points.

Let's first, put some circles in a graph so that two circles can meet at two points. Then, we can see better what we can do about the problem. Now, what matters in a circle?

Fig. 0



The center as well as the radius matters. So let's take a closer look at those two.
Examining the radii and the distance between the centers, we can see how the radii and the distance should be related if two circles have to meet each other at two points.
First, the distance between the centers has to be less than the sum of the two radii.
However, the distance has to be greater than the difference between the two radii.
So if the two circle meet each other at two points, we need to have

$$|c_1 - c_2| < d < c_1 + c_2.$$

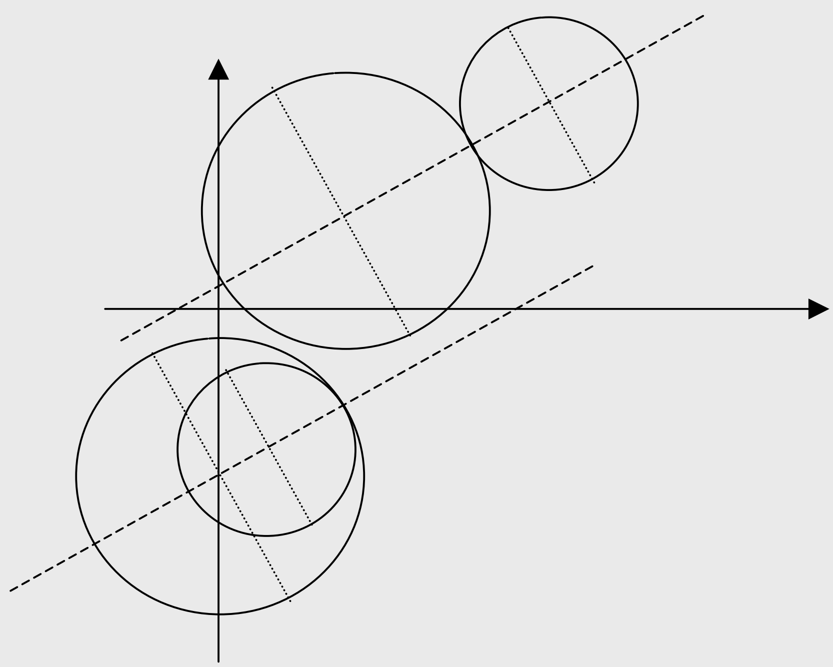
Suggestions or Solutions
To the Problem 1 in the Example 0

Suppose that
A is a circle $(x - a_1)^2 + (y - b_1)^2 = c_1^2$, where a_1, b_1, and c_1 are constant, and $c_1 > 0$.
B is a circle $(x - a_2)^2 + (y - b_2)^2 = c_2^2$, where a_2, b_2, and c_2 are constant, and $c_2 > 0$.
d is the distance between the centers of the two circles A and B.

Then, find the relationship among c_1, c_2, and d if the two circles meet at one point.

To begin with, let's put some circles in a graph so that two circles meet each other at one point. Then, we can see better where the solution can be around.

Fig. 0



Now, two circles meet each other at one point in either of two cases below.
• The distance between the centers is the same as the sum of the two radii
• The distance between the centers is the same as the difference between the two radii.

So if the two circles meet at one point, we need to have $d = |c_1 - c_2|$, or $d = c_1 + c_2$.
More specifically,

• If $d = |c_1 - c_2|$, then the two circles are tangent to each other internally.
That is, the smaller circle is tangent to the larger circle inside the larger.

• If $d = c_1 + c_2$, then the two circles are tangent to each other externally.

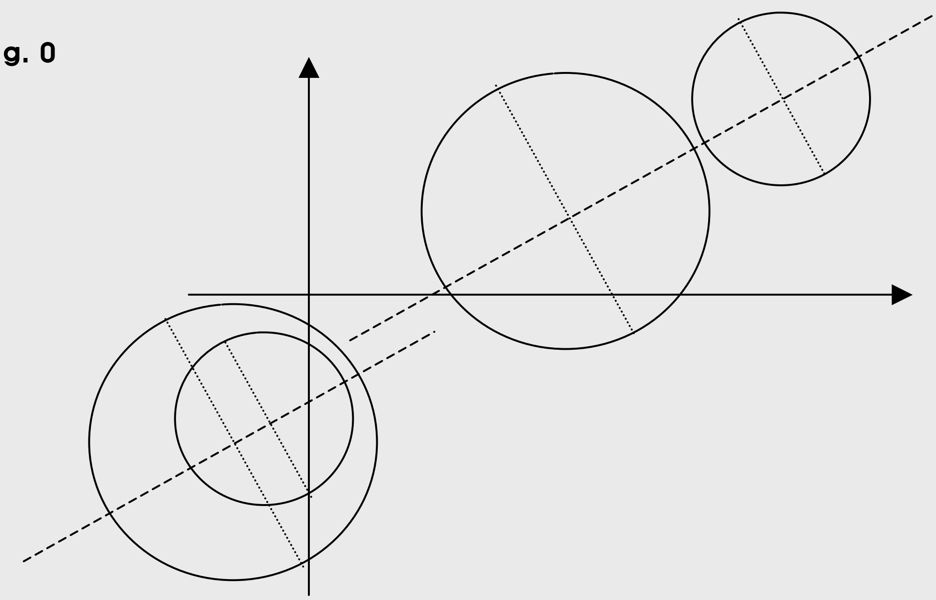
Suggestions or Solutions
To the Problem 2 in the Example 0

Suppose that
A is a circle $(x - a_1)^2 + (y - b_1)^2 = c_1^2$, where a_1, b_1, and c_1 are constant, and $c_1 > 0$.
B is a circle $(x - a_2)^2 + (y - b_2)^2 = c_2^2$, where a_2, b_2, and c_2 are constant, and $c_2 > 0$.
d is the distance between the centers of the two circles A and B.

Then, find the relationship among c_1, c_2, and d if the two circles do not meet each other.

Let's this time, put some circles in a graph so that two circles do not meet each other.

Fig. 0



Now, we can see that two circles do not meet each other in either of two cases below.
• The distance between the centers is greater than the sum of the two radii.
• The distance between the centers is less than the difference between the two radii.

So if the two circles do not meet each other, we need to have $d < |c_1 - c_2|$, or $d > c_1 + c_2$. More specifically,

• If $d < |c_1 - c_2|$, then the two circles are away from each other internally.
That is, the smaller circle is inside the larger circle, but is not tangent to the larger.

• If $d > c_1 + c_2$, then the two circles are away from each other externally. That is, the two circles are completely outside each other.

Suggestions or Solutions
To the Problem in the Example 1

Suppose that
A is a circle $x^2 + y^2 + ax + by + c = 0$, where a, b, and c are constant.
B is a circle $x^2 + y^2 + dx + ey + f = 0$, where d, e, and f are constant.

Suppose also, the two circles meet each other at two particular points.

Then, find another circle passing through the two particular points.

Suppose first, m and n are constant, and $m + n \neq 0$.
Then, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$, too, indicates a circle.

That's because

first, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f)$
$= (m + n)x^2 + (m + n)y^2 + (am + dn)x + (bm + en)y + cm + fn$
$= (m + n)\{x^2 + y^2 + \frac{(am+dn)}{(m+n)}x + \frac{(bm+en)}{(m+n)}y + \frac{(cm+fn)}{(m+n)}\} = 0$. And we have $m + n \neq 0$.

So next, we get $x^2 + y^2 + \frac{(am+dn)}{(m+n)}x + \frac{(bm+en)}{(m+n)}y + \frac{(cm+fn)}{(m+n)} = 0$, which is a circle since a, b, c, d, e, f, m, and n are constants.

Suppose next, C is the circle $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ where $m + n \neq 0$. Suppose also, (s, t) and (u, v) are the two points where two circles A and B meet each other. Then,

first, we get $s^2 + t^2 + as + bt + c = 0$ and $s^2 + t^2 + ds + et + f = 0$ since each of the two circles A and B passes through the point (s, t).

So we get $m(s^2 + t^2 + as + bt + c) + n(s^2 + t^2 + ds + et + f) = 0$, which means C passes through the point (s, t), too.

Next, we get $u^2 + v^2 + au + bv + c = 0$ and $u^2 + v^2 + du + ev + f = 0$ since each of the two circles A and B passes through the other point (u, v), too.

So we get $\boldsymbol{m(u^2 + v^2 + au + bv + c) + n(u^2 + v^2 + du + ev + f) = 0}$, which means $\boldsymbol{C}$ passes through the point $\boldsymbol{(u, v)}$, too.

Thus, the circle $\boldsymbol{C}$ passes through the two points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$, too.

So the three circles $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ share the two points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$.

Therefore, the solution to this problem is as follows.

$\boldsymbol{m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0}$ where $\boldsymbol{m}$ & $\boldsymbol{n}$ are constants, and $\boldsymbol{m + n \neq 0}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Knowing three points not in a line, we can find a particular circle, which passes through the three points, of course.

Three points not in a line can determine one circle only. That is, one circle only can pass through such three points, and no two different circles can share such three points.

What then, about two points?

Infinitely many circles can share the two, and of course, one point, too, can be shared by that many circles. Thus, infinitely many circles can be the solution to this problem.

It doesn't mean that any circle can be the solution, though.

So let's find out about what circles can be the solution.

Suppose now, that $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, that $\boldsymbol{m + n \neq 0}$, and that curves of two equations $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$ meet each other at particular points (that is, they share the particular points).
The two equations $\boldsymbol{f}$ and $\boldsymbol{g}$ are in terms of the two variables $\boldsymbol{x}$ and $\boldsymbol{y}$, and for instance, the two equations can be $\boldsymbol{3xy + 2 = 0}$ and $\boldsymbol{x^2 - xy + 2y^3 - 1 = 0}$.

Then, the curve of another equation $mf(x, y) + ng(x, y) = 0$ meets at the particular points the curves of $f(x, y) = 0$ and $g(x, y) = 0$.

That is, the curve of $mf(x, y) + ng(x, y) = 0$, too, includes the very particular points, so all the three curves share the particular points.

Now, the same is true for circles, too. In this example, the two circles A and B meet each other at two particular points, and their equations are as follows.

$x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$.

Thus, if $m + n \neq 0$, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ is a circle, and meets two circles A and B at the two particular points. Why?

Rearranging the equation above, we get

$$m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f)$$
$$= (m + n)x^2 + (m + n)y^2 + (am + dn)x + (bm + en)y + cm + fn$$
$$= (m + n)\{x^2 + y^2 + \frac{(am+dn)}{(m+n)}x + \frac{(bm+en)}{(m+n)}y + \frac{(cm+fn)}{(m+n)}\} = 0.$$ And we have $m + n \neq 0$.

So we get $x^2 + y^2 + \frac{(am+dn)}{(m+n)}x + \frac{(bm+en)}{(m+n)}y + \frac{(cm+fn)}{(m+n)} = 0$, which is a circle since a, b, c, d, e, f, m, and n are constant, and in turn, the fractional expressions are constant, too.

Suppose next, C is the circle $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$, where $m + n \neq 0$, and the circle A meets the circle B at two points (s, t) and (u, v).

We know A is $x^2 + y^2 + ax + by + c = 0$, and B is $x^2 + y^2 + dx + ey + f = 0$.

Then, first, we get $s^2 + t^2 + as + bt + c = 0$ and $s^2 + t^2 + ds + et + f = 0$ since each of the two circles A and B passes through the point (s, t).

Thus, we get $m(s^2 + t^2 + as + bt + c) + n(s^2 + t^2 + ds + et + f) = 0$, which means the circle C passes through the point (s, t), too.

Also, we get $u^2 + v^2 + au + bv + c = 0$ and $u^2 + v^2 + du + ev + f = 0$ since each of the two circles A and B passes through the other point (u, v), too.

So we get $m(u^2 + v^2 + au + bv + c) + n(u^2 + v^2 + du + ev + f) = 0$, which means the circle C passes through the point (u, v), too.

Therefore, the curve of $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$, that is, the circle C passes through the two points (s, t) and (u, v).

So the three circles A, B and C meet altogether at the two points (s, t) and (u, v). In other words, the three circles A, B, and C share the two points (s, t) and (u, v).

Therefore, the solution to this problem is as follows.

$m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ where m & n are constants, and $m + n \neq 0$.

What if $m + n = 0$, though?

Then, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ is an equation of a line passing through the two points (s, t) and (u, v). Why?

To begin with, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f)$

$= (m + n)x^2 + (m + n)y^2 + (am + dn)x + (bm + en)y + cm + fn$

$= (am + dn)x + (bm + en)y + cm + fn = 0$, since $m + n = 0$.

Next, we know $am + dn$, $bm + en$, and $cm + fn$ are constant since a, b, c, d, e, f, m, and n are constants. So $(am + dn)x + (bm + en)y + cm + fn = 0$ is an equation of a line.

Now, since the two circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$ include respectively the point (s, t), we get $s^2 + t^2 + as + bt + c = 0$ and $s^2 + t^2 + ds + et + f = 0$.

So we get $m(s^2 + t^2 + as + bt + c) + n(s^2 + t^2 + ds + et + f) = 0$.
Thus, since $m + n = 0$, we get $(am + dn)s + (bm + en)t + cm + fn = 0$.

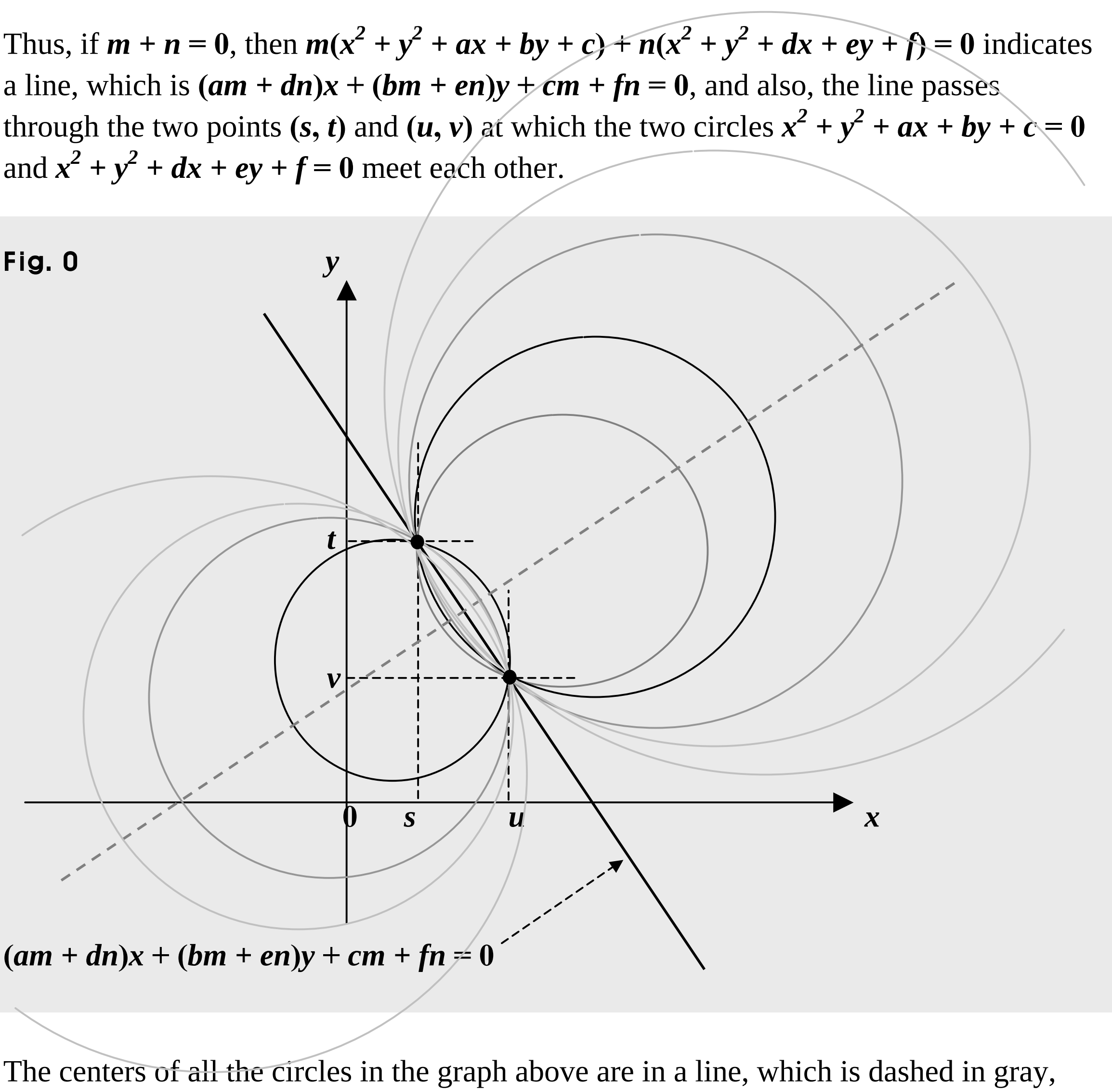
Also, since the two circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$ share the point (u, v), we get $u^2 + v^2 + au + bv + c = 0$ and $u^2 + v^2 + du + ev + f = 0$.

So we get $m(u^2 + v^2 + au + bv + c) + n(u^2 + v^2 + du + ev + f) = 0$.

Therefore, since $m + n = 0$, we get $(am + dn)u + (bm + en)v + cm + fn = 0$.

Thus, if $m + n = 0$, then $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ indicates a line, which is $(am + dn)x + (bm + en)y + cm + fn = 0$, and also, the line passes through the two points (s, t) and (u, v) at which the two circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$ meet each other.

Fig. 0



The centers of all the circles in the graph above are in a line, which is dashed in gray, passes through the midpoint between the two points (s, t) and (u, v), and is perpendicular to the line $(am + dn)x + (bm + en)y + cm + fn = 0$ passing through (s, t) and (u, v).

What about a circle and a line? That is, what if a circle meets a line at particular points?

The same is true for a circle and a line, too.

Suppose now, a circle $x^2 + y^2 + ax + by + c = 0$ meets a line $ux + vy + w = 0$ at particular points.

Then, an equation $m(x^2 + y^2 + ax + by + c) + n(ux + vy + w) = 0$ where $m \neq 0$ indicates a circle, and the circle passes through the particular points, too. That is to say that the two circles and the line share the particular points.

Why does the equation above indicate a circle, though?

To begin with, $m(x^2 + y^2 + ax + by + c) + n(ux + vy + w) = 0$

$\Rightarrow m(x^2 + y^2) + (ma + nu)x + (mb + nv)y + mc + nw = 0$

$\Rightarrow x^2 + y^2 + \frac{(am+nu)}{m}x + \frac{(bm+nv)}{m}y + \frac{(cm+nw)}{m} = 0.$

Next, we know $\frac{(am+nu)}{m}$, $\frac{(bm+nv)}{m}$, and $\frac{(cm+nw)}{m}$ are constant since a, b, c, u, v, w, m, and n are constant. So the equation above is an equation of circle.

In fact, the equation $m(x^2 + y^2 + ax + by + c) + n(ux + vy + w) = 0$ where $m \neq 0$ can be said to represent a group of circles share the particular points. In other words, for a pair of values for m and n, the equation indicates one circle, for another pair, the equation indicates another circle, and each of all the circles passes through the particular points.

The same is true for the equation below, too.

$m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ where m & n are constants, and $m + n \neq 0$.

Suppose C is the equation above, and a circle $x^2 + y^2 + ax + by + c = 0$ meets another circle $x^2 + y^2 + dx + ey + f = 0$ at particular points.

Then, the equation C can be said to represent a group of circles that share the particular points. That is, for one pair of values of m and n, the equation C indicates one circle, for another pair, C indicates another circle, and so forth, and each of all the circles passes through the particular points.

Now, we can simplify the whole story above into such a fact as follows.

Suppose two circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$ meet each other at particular points, and m and n are constants.

Then, if $m + n \neq 0$, for each pair of the values of m and n, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ indicates another circle passing through the particular points, and the center is in the line passing through the centers of the two circles. And the line is perpendicular to the line passing through the particular points.

In short:

Suppose first, m and n are constant, and $m + n \neq 0$.

Then, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$, too, indicates a circle.

That's because

First, $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f)$
$= (m + n)x^2 + (m + n)y^2 + (am + dn)x + (bm + en)y + cm + fn$
$= (m+n)\{x^2 + y^2 + \frac{(am+dn)}{(m+n)}x + \frac{(bm+en)}{(m+n)}y + \frac{(cm+fn)}{(m+n)}\} = 0$. And we have $m + n \neq 0$.

So next, we get $x^2 + y^2 + \frac{(am+dn)}{(m+n)}x + \frac{(bm+en)}{(m+n)}y + \frac{(cm+fn)}{(m+n)} = 0$, which is a circle since a, b, c, d, e, f, m, and n are constants.

Suppose next, C is the circle $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ where $m + n \neq 0$. Suppose also, (s, t) and (u, v) are the two points where two circles A and B meet each other. Then,

first, we get $s^2 + t^2 + as + bt + c = 0$ and $s^2 + t^2 + ds + et + f = 0$ since each of the two circles A and B passes through the point (s, t).

So we get $m(s^2 + t^2 + as + bt + c) + n(s^2 + t^2 + ds + et + f) = 0$, which means C passes through the point (s, t), too.

Next, we get $u^2 + v^2 + au + bv + c = 0$ and $u^2 + v^2 + du + ev + f = 0$ since each of the two circles A and B passes through the other point (u, v), too.

So we get $m(u^2 + v^2 + au + bv + c) + n(u^2 + v^2 + du + ev + f) = 0$, which means C passes through the point (u, v), too.

Thus, the circle C passes through the two points (s, t) and (u, v), too.

So the three circles A, B, and C share the two points (s, t) and (u, v).

Therefore, the solution to this problem is as follows.

$m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ where m & n are constants, and $m + n \neq 0$.