Examples 8 in Circles

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See if there is any particular point the curve of each equation below passes through for any value of $\boldsymbol{k}$, which is constant, and specify the particular point if any.

0. $(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2 = 0$.

1. $9(k + 1)(x^2 + y^2) - 36(k + 2)x - 18(k + 3)y + 37k + 193 = 0$.

2. $(k + 1)(x^2 + y^2) - 2(3k + 4)x - 2(k + 2)y + 2k + 18 = 0$.

3. $(k + 1)(x^2 + y^2) - 2(k + 4)x - 2(k + 3)y + k + 21 = 0$.

4. $(k + 1)(x^2 + y^2) - 8(k + 1)x - 6(k + 1)y + 24k + 21 = 0$.

5. $(k + 1)(x^2 + y^2) - 2(2k + 3)x - 2(2k + 3)y - k + 17 = 0$.

Suggestions or Solutions
To the Problem in the Example 0

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$\mathbf{(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2 = 0.}$

$(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2$
$= k(x^2 + y^2) + 4kx + 2ky + 4k + x^2 + y^2 + 2x - 2y - 2$
$= k(x^2 + y^2 + 4x + 2y + 4) + (x^2 + y^2 + 2x - 2y - 2) = 0$, which can be an equation of a circle for each value of k.

Suppose a circle $x^2 + y^2 + 4x + 2y + 4 = 0$ meets another circle $x^2 + y^2 + 2x - 2y - 2 = 0$ at (s, t).

Then, we get $s^2 + t^2 + 4s + 2t + 4 = 0$, and $s^2 + t^2 + 2s - 2t - 2 = 0$.

So we get $k(s^2 + t^2 + 4s + 2t + 4) + s^2 + t^2 + 2s - 2t - 2 = 0$.

So if the two circles above meet at a particular point, for any value of k, the equation given indicates a circle that includes the particular point. And the same is true for another particular point, too, if the two circles share the other particular point. So if the two circles meet at particular points, for any value of k, the equation given indicates a circle that includes the particular points.   So now, finding the points, we get

first, $x^2 + y^2 + 4x + 2y + 4 = x^2 + y^2 + 2x - 2y - 2 \Rightarrow 2x + 4y + 6 = 0 \Rightarrow x + 2y + 3 = 0$
$\Rightarrow x = -2y - 3 \Rightarrow x^2 + y^2 + 2x - 2y - 2 = (-2y - 3)^2 + y^2 + 2(-2y - 3) - 2y - 2 = 0$
$\Rightarrow 4y^2 + 12y + 9 + y^2 - 4y - 6 - 2y - 2 = 5y^2 + 6y + 1 = (5y + 1)(y + 1) = 0$
$\Rightarrow y = -\frac{1}{5}$ or $y = -1$.

Next, $y = -\frac{1}{5} \Rightarrow x = -2y - 3 = \frac{2}{5} - 3 = -\frac{13}{5} \Rightarrow (-\frac{13}{5}, -\frac{1}{5})$, and
$y = -1 \Rightarrow x = -2y - 3 = -1 \Rightarrow (-1, -1)$.

Thus, the two circles meet at two points $(-1, -1)$ and $(-\frac{13}{5}, -\frac{1}{5})$, which are therefore, the particular points a circle indicated by the equation given includes for any value of k.

If not quite sure of the idea behind the processes above, follow the steps below.

The problems in this example 2 are no other than the one in the example 1 in the **Examples 7**. Doing those problems, we will see how a group of circles can share particular points, and see better how equations for circles work.

The number of circles in such a group can be infinite or finite depending on the way we set the constant k in the equation.

For instance, we can set $3 \le k < 7$ where k is integer, or can set $1 < k < 2$ where k is real.

Then in the first case, k can be 3, 4, 5, and 6 only, so the number of circles in the group can be up to 4, and in the second, since k can be any real number between 1 and 2, we can make infinite choices for k, so the group can have infinite circles.

Let's now, begin with a fact as follows.

• Suppose two circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$ meet at particular points, m and n are constants, and $m + n \neq 0$. Then, for each pair of values of m and n, a new equation $m(x^2 + y^2 + ax + by + c) + n(x^2 + y^2 + dx + ey + f) = 0$ indicates a circle passing through the particular points.

So every circle indicated by the new equation above passes through the particular points. The same is true for a circle and a line, too.

• Suppose a circle $x^2 + y^2 + ax + by + c = 0$ meets a line $ux + vy + w = 0$ at particular points. Then, for each pair of values of m and n, a new equation $m(x^2 + y^2 + ax + by + c) + n(ux + vy + w) = 0$ where $m \neq 0$ indicates a circle passes through the particular points.

So every circle indicated by the new equation above passes through the particular points.

• In particular, such a new equation can be called a linear combination of two equations.

Such two equations indicate respectively either 'two circles' or 'a circle and a line'.

So such a circle indicated by such a new equation can be said to be 'a linear combination of two circles' or 'a linear combination of a circle and a line'.

Besides, every center of every circle mentioned above is in the same line.

So for instance, if the center of every circle that is a linear combination of two circles ***A*** and ***B*** is in a line ***L***, the line ***L*** passes through the centers of the two circles ***A*** and ***B***, too.

The same is true for a linear combination of a circle and a line, also.

So for instance, if the center of every circle that is a linear combination of a circle ***C*** and a line ***N*** is in a line ***D***, the line ***D*** passes through the center of the circle ***C***, too.

In addition, the line ***D*** is perpendicular to the line ***N***.

And the similar idea applies to the case of the linear combination of two circles.

Suppose for instance, two circles ***A*** and ***B*** meet each other at two points ***P*** and ***Q***, a line ***R*** passes through ***P*** and ***Q***, and a line ***L*** passes through the centers of ***A*** and ***B***.

Then, the line ***L*** is perpendicular to the line ***R***, and has all the centers of all the circles that are linear combinations of ***A*** and ***B***.

Fig. 0

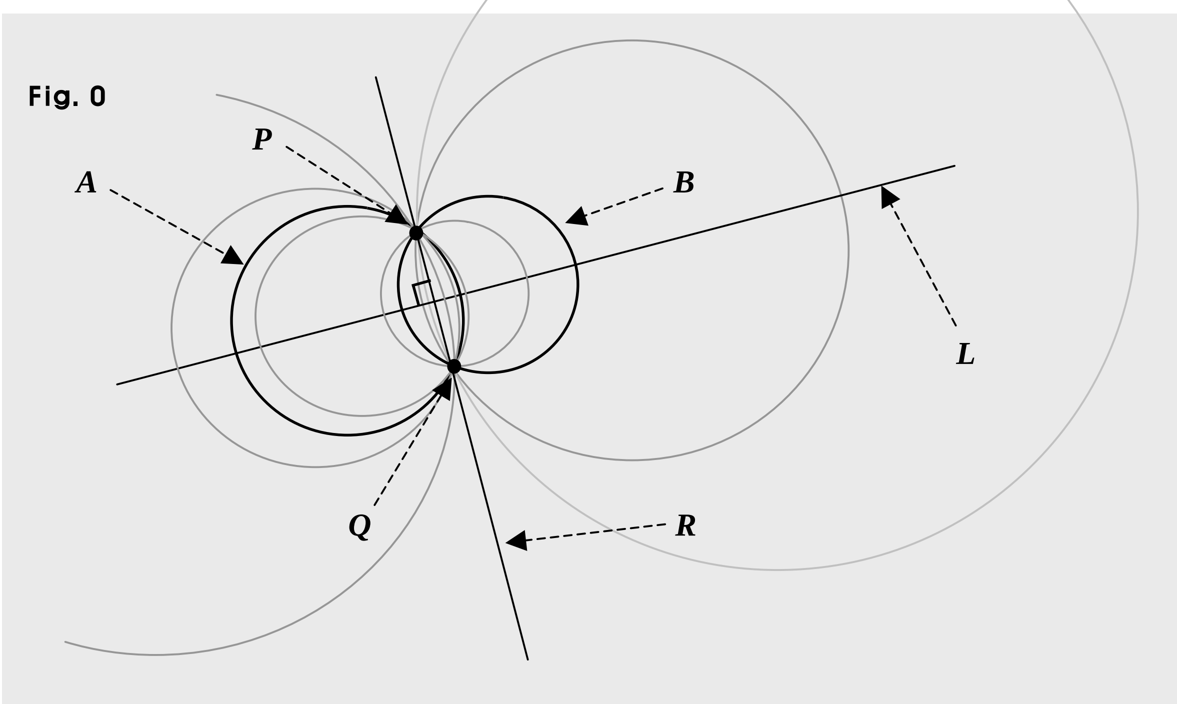
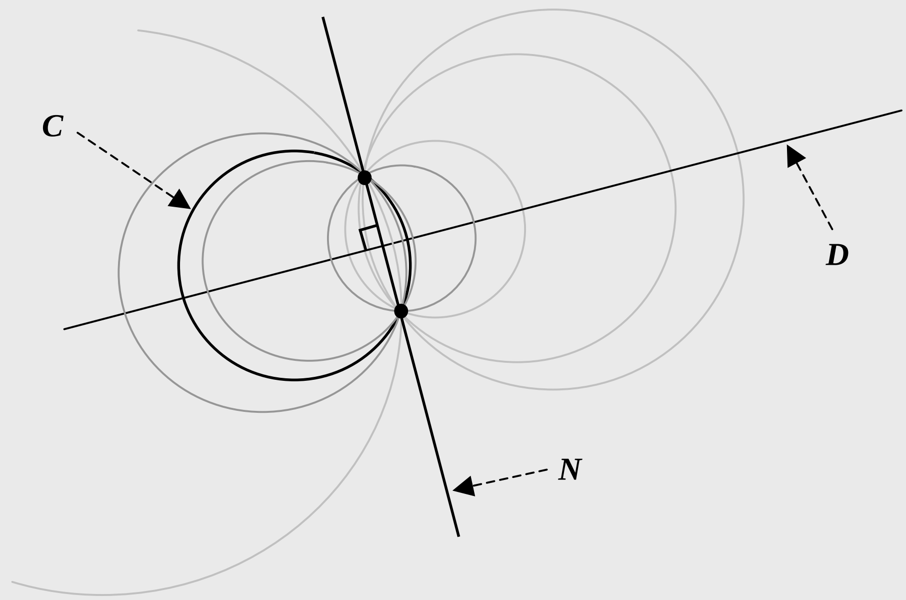


Fig. 1



Now, in this problem, we have $(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2 = 0$ for k constant. For each value of k, the equation above can indicate a circle, which can be a linear combination of two curves, which can be two circles, or can be a circle and a line.

So let's extract the two curves from the equation given, and see what the curves are.

$(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2 = 0$

$\Rightarrow k(x^2 + y^2) + x^2 + y^2 + 4kx + 2x + 2ky - 2y + 4k - 2$

$= k(x^2 + y^2) + 4kx + 2ky + 4k + x^2 + y^2 + 2x - 2y - 2$

$= k(x^2 + y^2 + 4x + 2y + 4) + (x^2 + y^2 + 2x - 2y - 2) = 0$.

Therefore, we can see that $m = k$, and $n = 1$ in the form of a linear combination of two equations, and that the curves are two circles as follows.

$x^2 + y^2 + 4x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 2y - 2 = 0$.

What if however, we just put a couple of values into k in turn, get two circles, and then, find the two points?

We can get the solution that way, too, if the problem is multiple-choice.
Doing a problem in math though, we may want to justify the solution, too.
That is, doing it mathematically, we not only provide but also explain the solution.

Assuming now, ***U*** is the first of the two circles above, and ***V*** is the second, let's find the centers and radii of ***U*** and ***V*** and then, see if the two circles ***U*** and ***V*** meet each other.

$$x^2 + y^2 + 4x + 2y + 4 = x^2 + 4x + 4 - 4 + y^2 + 2y + 1 - 1 + 4 = (x + 2)^2 + (y + 1)^2 - 1 = 0$$

So the center of ***U*** is **(-2, -1)**, and the radius is 1.

$$x^2 + 2x + y^2 - 2y - 2 = x^2 + 2x + 1 - 1 + y^2 - 2y + 1 - 1 - 2 = (x + 1)^2 + (y - 1)^2 - 4 = 0$$

So the center of ***V*** is **(-1, 1)**, and the radius is 2.

Suppose now, ***D*** is the distance between the two centers **(-2, -1)** and **(-1, 1)**, ***S*** is the sum of the two radii, ***T*** is the difference between the radii, and Δ is the difference between two coordinates. We can find ***D*** by the distance formula, of course. Then, we get

$$D^2 = (\Delta x)^2 + (\Delta y)^2 = \{-1 - (-2)\}^2 + \{1 - (-1)\}^2 = 1 + 4 = 5,\ S = 1 + 2 = 3,\ \&\ T = 2 - 1 = 1.$$

So we get $S > D$ for ***S*** and ***D*** are nonnegative and $S^2 = 9 > D^2 = 5$. So we get $T < D < S$.

Thus, we can see that the circle ***U*** meets the circle ***V*** at two particular points.

(If not sure, refer to the example 0 in **Examples 7**.)

Therefore, the equation $(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2 = 0$ can be said to represent a group of circles sharing altogether the two particular points since for each value of ***k***, the equation indicates a circle passing through the two particular points.

Let's now, get the two particular points. Then, we want to solve the system where

$(x + 2)^2 + (y + 1)^2 - 1 = 0$ and $(x + 1)^2 + (y - 1)^2 - 4 = 0$. Then, we can get first,

$(x+1)^2 + (y-1)^2 - 4 = (x+2)^2 + (y+1)^2 - 1$

$\Rightarrow (x+1)^2 - (x+2)^2 + (y-1)^2 - (y+1)^2 - 4 - (-1) = 0$

$\Rightarrow \{(x+1) - (x+2)\}\{(x+1) + (x+2)\} + \{(y-1) - (y+1)\}\{(y-1) + (y+1)\} - 3$

$= (-1)(2x+3) - 2(2y) - 3 = -2x - 4y - 6 = 0 \Rightarrow x + 2y + 3 = 0 \Rightarrow x = -2y - 3$.

Thus, we get $x = -2y - 3 \Rightarrow (x+2)^2 + (y+1)^2 - 1 = (-2y - 3 + 2)^2 + (y+1)^2 - 1 = 0$

$\Rightarrow (-2y-1)^2 + (y+1)^2 - 1 = (2y+1)^2 + (y+1)^2 - 1 = 4y^2 + 4y + 1 + y^2 + 2y + 1 - 1$

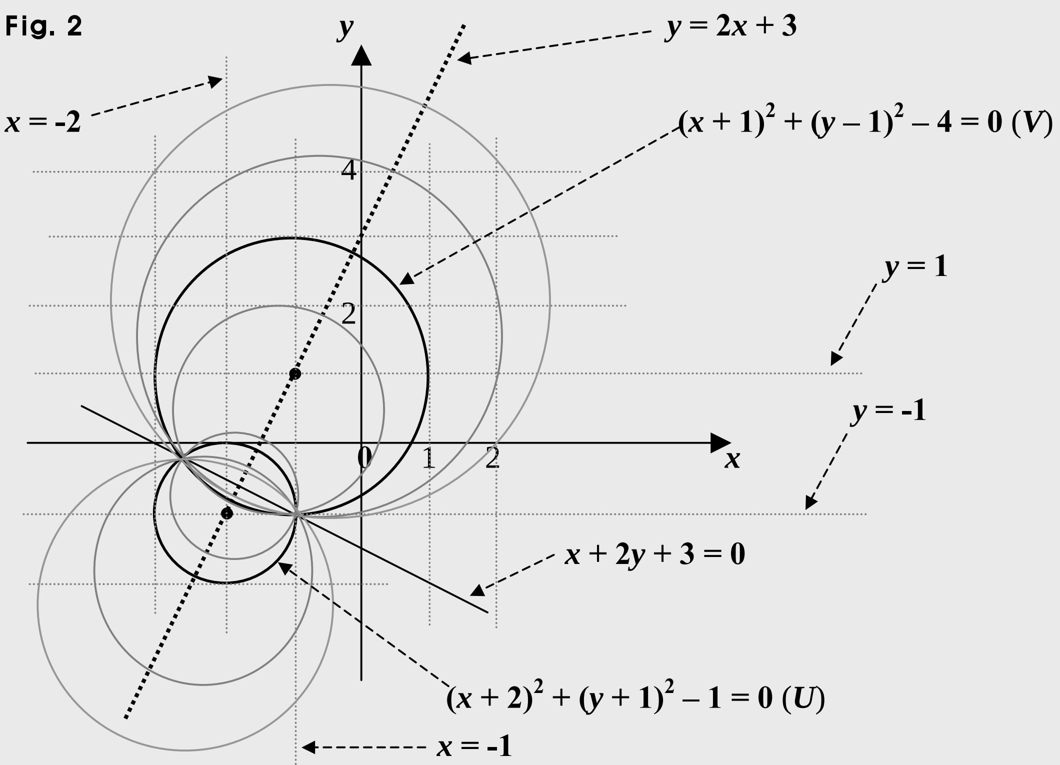
$= 5y^2 + 6y + 1 = (5y+1)(y+1) = 0 \Rightarrow y = -\frac{1}{5}$ or $y = -1$.

Therefore, $y = -\frac{1}{5} \Rightarrow x = -2y - 3 = \frac{2}{5} - 3 = -\frac{13}{5}$, and $y = -1 \Rightarrow x = -2y - 3 = -1$.

Now, all the circles in the group share altogether two points, $(-1, -1)$ and $(-\frac{13}{5}, -\frac{1}{5})$.

Let's now, put in a graph some of the circles in the group, together with U and V.

Fig. 2



In the graph above, the two circles $\boldsymbol{U}$ and $\boldsymbol{V}$ are in black. And the line $\boldsymbol{x + 2y + 3 = 0}$ is passing through the two particular points where all the circles meet altogether. Besides, the centers of all the circles are in a line perpendicular to the line $\boldsymbol{x + 2y + 3 = 0}$ and passing through the midpoint between the two particular points.

The perpendicular line is $\boldsymbol{y = 2x + 3}$, which passes through, of course, the centers of two circles $\boldsymbol{U}$ and $\boldsymbol{V}$, of which each circle in the graph is a linear combination.

Note however, of the two circles $\boldsymbol{U}$ and $\boldsymbol{V}$, the smaller one $\boldsymbol{U}$, $\boldsymbol{x^2 + y^2 + 4x + 2y + 4 = 0}$, which is the general equivalent to $\boldsymbol{(x + 2)^2 + (y + 1)^2 - 1 = 0}$ cannot be made from the given equation $\boldsymbol{(k + 1)(x^2 + y^2) + 2(2k + 1)x + 2(k - 1)y + 4k - 2 = 0}$ for any value of $\boldsymbol{k}$.

Why not?

The equation given is the same as $\boldsymbol{k(x^2 + y^2 + 4x + 2y + 4) + (x^2 + y^2 + 2x - 2y - 2) = 0}$, and for no value of $\boldsymbol{k}$, we can get $\boldsymbol{x^2 + y^2 + 4x + 2y + 4 = 0}$, which is the smaller circle $\boldsymbol{U}$.

The circle $\boldsymbol{U}$ however, can be obtained if we use such an equation as follows.

$\boldsymbol{m(x^2 + y^2 + 4x + 2y + 4) + n(x^2 + y^2 + 2x - 2y - 2) = 0}$ where $\boldsymbol{m}$ and $\boldsymbol{n}$ are constants, and $\boldsymbol{m + n \neq 0}$.

The equation above is not equivalent to the equation given, though. That's because it can indicate more circles than the equation given. That is, it is more general.

And in it, setting $\boldsymbol{m = 1}$ and $\boldsymbol{n = 0}$, we get the smaller circle $\boldsymbol{U}$, $\boldsymbol{x^2 + y^2 + 4x + 2y + 4 = 0}$.

Suggestions or Solutions
To the Problem in the Example 1

See if there is any particular point the curve of the equation below passes through for any value of *k*, which is constant, and specify the particular point if any.
$\mathbf{9(k + 1)(x^2 + y^2) - 36(k + 2)x - 18(k + 3)y + 37k + 193 = 0.}$

To begin with, we can put the equation the way below.

$\mathbf{9(k + 1)(x^2 + y^2) - 36(k + 2)x - 18(k + 3)y + 37k + 193}$
$\mathbf{= 9k(x^2 + y^2) - 36kx - 18ky + 37k + 9x^2 + 9y^2 - 72x - 54y + 193}$
$\mathbf{= k(9x^2 + 9y^2 - 36x - 18y + 37) + (9x^2 + 9y^2 - 72x - 54y + 193) = 0}$
$\mathbf{\Rightarrow k(x^2 + y^2 - 4x - 2y + \frac{37}{9}) + (x^2 + y^2 - 8x - 6y + \frac{193}{9}) = 0}$, which can be an equation of a circle for each value of $\boldsymbol{k}$.

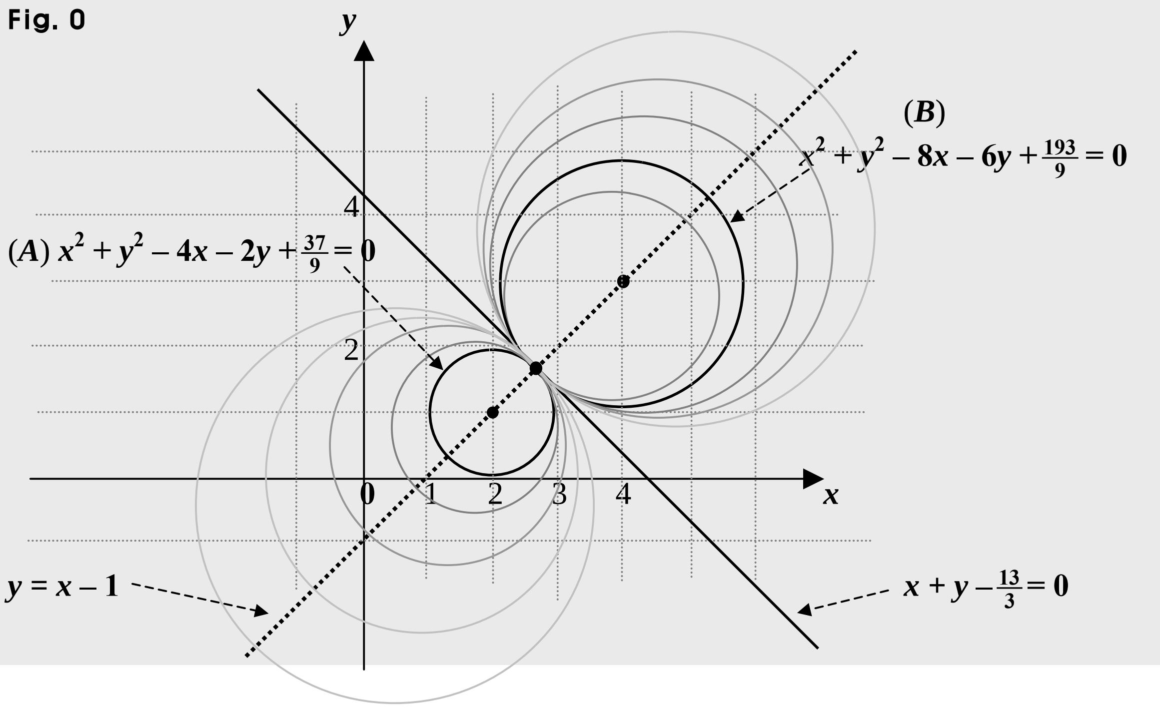
Next, suppose two circles, $\mathbf{x^2 + y^2 - 4x - 2y + \frac{37}{9} = 0}$ and $\mathbf{x^2 + y^2 - 8x - 6y + \frac{193}{9} = 0}$ meet at $\mathbf{(s, t)}$. Then, we get $\mathbf{s^2 + t^2 - 4s - 2t + \frac{37}{9} = 0}$, and $\mathbf{s^2 + t^2 - 8s - 6t + \frac{193}{9} = 0}$, so we get $\mathbf{k(s^2 + t^2 - 4s - 2t + \frac{37}{9}) + s^2 + t^2 - 8s - 6t + \frac{193}{9} = 0}$.

So if the two circles above meet at a particular point, for any value of $\boldsymbol{k}$, the equation given indicates a circle that includes the particular point. And the same is true for another particular point, too, if the two circles share the other particular point. So if the two circles meet at particular points, for any value of $\boldsymbol{k}$, the equation given indicates a circle that includes the particular points. So now, finding the points, we get

$\mathbf{x^2 + y^2 - 4x - 2y + \frac{37}{9} = x^2 + y^2 - 8x - 6y + \frac{193}{9} \Rightarrow 4x + 4y - \frac{156}{9} = 0}$
$\mathbf{\Rightarrow x + y - \frac{39}{9} = 0 \Rightarrow x = -y + \frac{13}{3}}$
$\mathbf{\Rightarrow x^2 + y^2 - 4x - 2y + \frac{37}{9} = (-y + \frac{13}{3})^2 + y^2 - 4(-y + \frac{13}{3}) - 2y + \frac{37}{9}}$
$\mathbf{= y^2 - \frac{26}{3}y + \frac{169}{9} + y^2 + 4y - \frac{52}{3} - 2y + \frac{37}{9} = 2y^2 - \frac{20}{3}y + \frac{50}{9}}$
$\mathbf{= 2(y^2 - \frac{10}{3}y + \frac{25}{9}) = 2(y - \frac{5}{3})^2 = 0 \Rightarrow y = \frac{5}{3} \Rightarrow x = -y + \frac{13}{3} = -\frac{5}{3} + \frac{13}{3} = \frac{8}{3} \Rightarrow (\frac{8}{3}, \frac{5}{3})}$.

So the two circles meet at one point $\mathbf{(\frac{8}{3}, \frac{5}{3})}$, which is therefore, the only particular point a circle indicated by the equation given includes for any value of $\boldsymbol{k}$.

Fig. 0



In the graph above, the two circles $\boldsymbol{A}$ and $\boldsymbol{B}$ are in black, and each of the other circles is a linear combination of $\boldsymbol{A}$ and $\boldsymbol{B}$.

The centers of all the circles are in a line $\boldsymbol{y = x - 1}$, which passes through the particular point, and is perpendicular to a line $\boldsymbol{x + y - \frac{13}{3} = 0}$, which passes through the particular point, too, where all the circles in the group meet each other altogether.

Note however, of the two circles $\boldsymbol{A}$ and $\boldsymbol{B}$, the smaller one $\boldsymbol{A}$, $\boldsymbol{x^2 + y^2 - 4x - 2y + \frac{37}{9} = 0}$, which is equivalent to $\boldsymbol{(x - 2)^2 + (y - 1)^2 - \frac{8}{9} = 0}$ cannot be obtained from the given equation $\boldsymbol{9(k + 1)(x^2 + y^2) - 36(k + 2)x - 18(k + 3)y + 37k + 193 = 0}$ for any value of $\boldsymbol{k}$.

The given equation can be put in $\boldsymbol{k(x^2 + y^2 - 4x - 2y + \frac{37}{9}) + (x^2 + y^2 - 8x - 6y + \frac{193}{9}) = 0}$, so for no value of $\boldsymbol{k}$, we can get $\boldsymbol{x^2 + y^2 - 4x - 2y + \frac{37}{9} = 0}$, which is the smaller circle $\boldsymbol{A}$.

The circle $\boldsymbol{A}$ however, can be obtained if we use the equation below.
$\boldsymbol{m(x^2 + y^2 - 4x - 2y + \frac{37}{9}) + n(x^2 + y^2 - 8x - 6y + \frac{193}{9}) = 0}$, where $\boldsymbol{m + n \neq 0}$.

The equation above is not equivalent to the equation given, though. That's because it can indicate more circles than the given equation can. So it is more general. In the equation, setting $\boldsymbol{m = 1}$ and $\boldsymbol{n = 0}$, we get the smaller circle $\boldsymbol{A}$, $\boldsymbol{x^2 + y^2 - 4x - 2y + \frac{37}{9} = 0}$.

Suggestions or Solutions
To the Problem in the Example 2

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$(k + 1)(x^2 + y^2) - 2(3k + 4)x - 2(k + 2)y + 2k + 18 = 0$.

To begin with, we can put the equation the way below.

$(k + 1)(x^2 + y^2) - 2(3k + 4)x - 2(k + 2)y + 2k + 18 = 0$

$= k(x^2 + y^2) - 6kx - 2ky + 2k + x^2 + y^2 - 8x - 4y + 18$

$= k(x^2 + y^2 - 6x - 2y + 2) + (x^2 + y^2 - 8x - 4y + 18) = 0$, which can be an equation of a circle for each value of k.

Next, suppose two circles, $x^2 + y^2 - 6x - 2y + 2 = 0$ and $x^2 + y^2 - 8x - 4y + 18 = 0$ meet at (s, t).

Then, we get $s^2 + t^2 - 6s - 2t + 2 = 0$, and $s^2 + t^2 - 8s - 4t + 18 = 0$.

So we get $k(s^2 + t^2 - 6s - 2t + 2) + s^2 + t^2 - 8s - 4t + 18 = 0$.

So if the two circles above meet at a particular point, for any value of k, the equation given indicates a circle that includes the particular point. And the same is true for another particular point, too, if the two circles share the point. So if the two circles meet at particular points, for any value of k, the equation given indicates a circle that includes the particular points. Finding thus, the points now, we get

$x^2 + y^2 - 6x - 2y + 2 - (x^2 + y^2 - 8x - 4y + 18) = -6x - 2y + 2 + 8x + 4y - 18$

$= 2x + 2y - 16 = 0 \Rightarrow x + y - 8 = 0 \Rightarrow x = 8 - y$

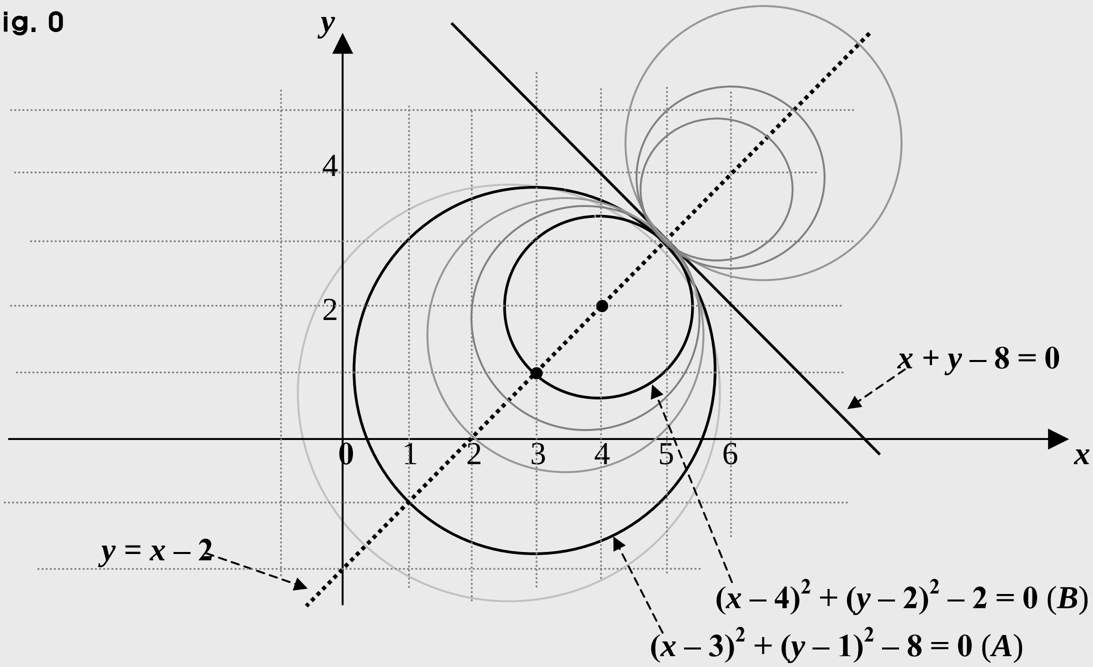
$\Rightarrow x^2 + y^2 - 6x - 2y + 2 = (8 - y)^2 + y^2 - 6(8 - y) - 2y + 2 = 0$

$\Rightarrow y^2 - 16y + 64 + y^2 - 48 + 6y - 2y + 2 = 2y^2 - 12y + 18 = 2(y^2 - 6y + 9) = 2(y - 3)^2 = 0$

$\Rightarrow y = 3 \Rightarrow x = 8 - y = 8 - 3 = 5 \Rightarrow (5, 3)$.

Thus, there is only one particular point, and it is $(5, 3)$. And let's now, put in a graph, together with A and B, some of those circles that can be indicated by the equation given.

Fig. 0



Suggestions or Solutions
To the Problem in the Example 3

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$\mathbf{(k + 1)(x^2 + y^2) - 2(k + 4)x - 2(k + 3)y + k + 21 = 0.}$

First, we can put the equation the way as follows.

$\mathbf{(k + 1)(x^2 + y^2) - 2(k + 4)x - 2(k + 3)y + k + 21}$
$\mathbf{= k(x^2 + y^2) - 2kx - 2ky + k + x^2 + y^2 - 8x - 6y + 21}$
$\mathbf{= k(x^2 + y^2 - 2x - 2y + 1) + (x^2 + y^2 - 8x - 6y + 21) = 0}$, which can be an equation of a circle for a value of k.

Next, suppose two circles $\mathbf{x^2 + y^2 - 2x - 2y + 1 = 0}$ and $\mathbf{x^2 + y^2 - 8x - 6y + 21 = 0}$ meet at $\mathbf{(s, t)}$. Then, we get $\mathbf{s^2 + t^2 - 2s - 2t + 1 = 0}$, and $\mathbf{s^2 + t^2 - 8s - 6t + 21 = 0}$,
So we get $\mathbf{k(s^2 + t^2 - 2s - 2t + 1) + s^2 + t^2 - 8s - 6t + 21 = 0}$.

So if the two circles above meet at a particular point, for any value of k, the equation given indicates a circle that includes the particular point. The same is true for another particular point, too, if the two circles share the point. So if the two circles meet at particular points, for any value of k, the equation given indicates a circle that includes the particular points.

$\mathbf{x^2 + y^2 - 2x - 2y + 1 = x^2 - 2x + 1 - 1 + y^2 - 2y + 1 - 1 + 1 = (x - 1)^2 + (y - 1)^2 - 1 = 0}$
$\Rightarrow$ The center is at $\mathbf{(1, 1)}$, and the radius is 1.

$\mathbf{x^2 + y^2 - 8x - 6y + 21 = x^2 - 8x + 16 - 16 + y^2 - 6y + 9 - 9 + 21}$
$\mathbf{= (x - 4)^2 + (y - 3)^2 - 4 = 0} \Rightarrow$ The center is at $\mathbf{(4, 3)}$, and the radius is 2.

Suppose D is the distance between the two centers, and S is the sum of the two radii.

Then, we get $\mathbf{D^2 = (\Delta x)^2 + (\Delta y)^2 = (4 - 1)^2 + (3 - 1)^2 = 9 + 4 = 13}$, and $\mathbf{S = 1 + 2 = 3}$.

Then, we get $\mathbf{S < D}$ since S and D are nonnegative and $\mathbf{S^2 = 9 < D^2 = 13}$.

So the two circles do not meet each other, and thus, there is no particular point shared by all the circles indicated by the given equation for all values of k.

Fig. 0

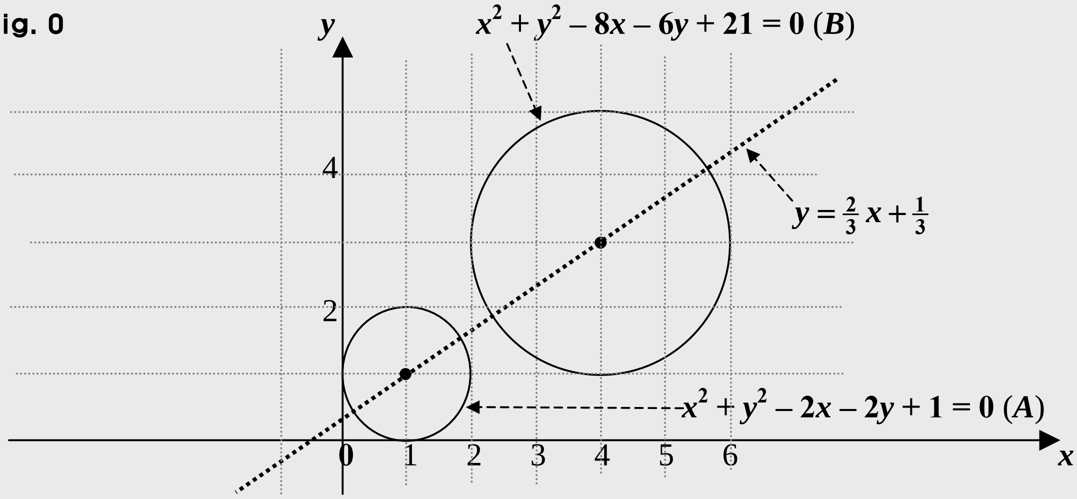
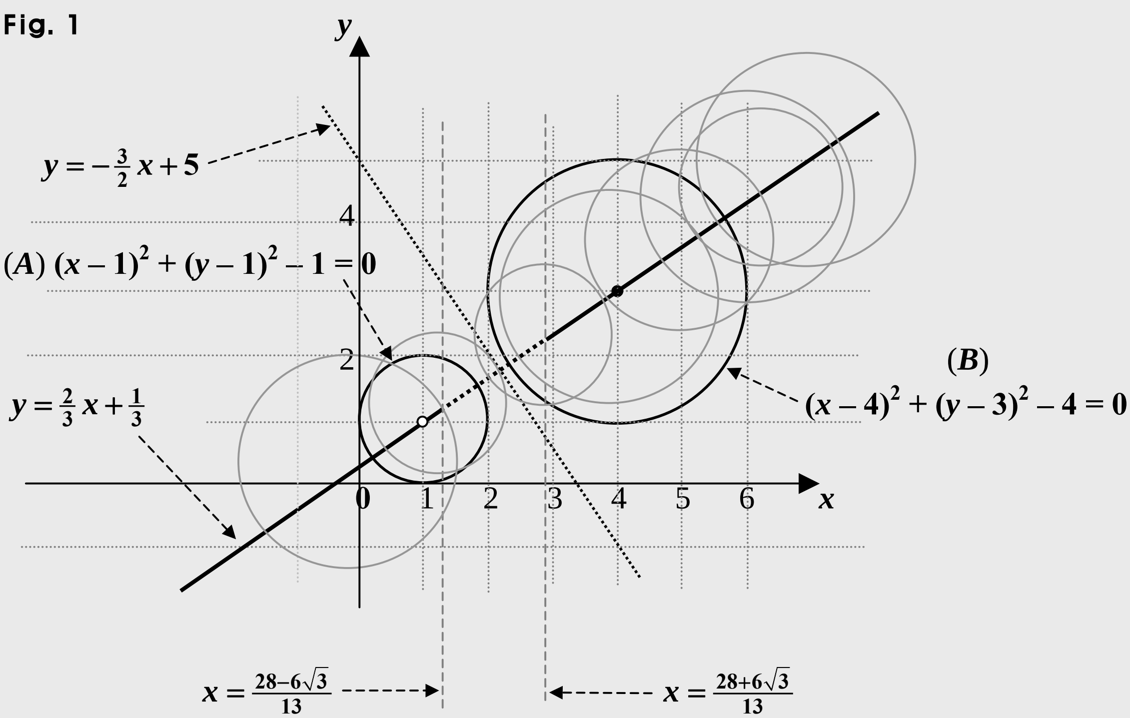


Fig. 1



Note that in the graph above, the radii of the circles in gray are not accurate at all.

That is to say that the gray circles are no more than the ones for showing how the circles in each group can be put in a graph.

The key idea is that the centers of all the circles above are in the two rays and the small line segment.

In the graph above, the two circles ***A*** and ***B*** are in black, and each of all the circles is a linear combination of the two circles ***A*** and ***B***.

Note however, for any value of ***k***, the circle ***A***, $x^2 + y^2 - 2x - 2y + 1 = 0$, which is equal to $(x - 1)^2 + (y - 1)^2 - 1 = 0$ cannot be made from the equation given in the problem.

The equation given can be put in $k(x^2 + y^2 - 2x - 2y + 1) + (x^2 + y^2 - 8x - 6y + 21) = 0$.

So for no value of ***k***, the equation above can indicate the circle ***A***.

However, the circle ***A*** can be obtained if we use such an equation as follows.

$m(x^2 + y^2 - 2x - 2y + 1) + n(x^2 + y^2 - 8x - 6y + 21) = 0$ where ***m*** and ***n*** are constants, and $m + n \neq 0$.

The equation above is not equivalent to the equation given, because it can indicate more circles that the equation given can.

In the equation, setting $m = 1$ and $n = 0$, we get the circle ***A***, $x^2 + y^2 - 2x - 2y + 1 = 0$.

Suggestions or Solutions
To the Problem in the Example 4

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$(k + 1)(x^2 + y^2) – 8(k + 1)x – 6(k + 1)y + 24k + 21 = 0.$

First, we can put the equation the way below.

$(k + 1)(x^2 + y^2) – 8(k + 1)x – 6(k + 1)y + 24k + 21$
$= k(x^2 + y^2) – 8kx – 6ky + 24k + x^2 + y^2 – 8x – 6y + 21$
$= k(x^2 + y^2 – 8x – 6y + 24) + (x^2 + y^2 – 8x – 6y + 21) = 0$, which can indicate a circle for a value of k.

Next, suppose two circles, $x^2 + y^2 – 8x – 6y + 24 = 0$ and $x^2 + y^2 – 8x – 6y + 21 = 0$ meet each other at (s, t).

Then, we get $s^2 + t^2 – 8s – 6t + 24 = 0$, and $s^2 + t^2 – 8s – 6t + 21 = 0$.

So we get $k(s^2 + t^2 – 8s – 6t + 24) + s^2 + t^2 – 8s – 6t + 21 = 0$.

So if the two circles above meet at a particular point, for any value of k, the equation given indicates a circle that includes the particular point. However, the two circles are concentric circles, so they do not meet each other. Therefore, there is no particular point shared by every circle indicated by the given equation for every value of k.

Let's see now, for what value of k, the equation can indicate a circle.

Simplifying it first, we get $(k + 1)(x^2 + y^2) – 8(k + 1)x – 6(k + 1)y + 24k + 21 = 0$

$\Rightarrow x^2 + y^2 - \frac{8(k+1)}{k+1}x - \frac{6(k+1)}{k+1}y + \frac{24k+21}{k+1} = x^2 + y^2 - 8x - 6y + \frac{24k+21}{k+1} = 0.$

Then, first of all, since a denominator cannot be 0, we need to have $k \neq -1$.

If $k = -1$ in fact, the equation doesn't even hold. Let's now set $c = \frac{24k+21}{k+1}$. Then, we get

$x^2 + y^2 – 8x – 6y + c = 0 \Rightarrow (x – 4)^2 + (y – 3)^2 – 25 + c = 0 \Rightarrow (x – 4)^2 + (y – 3)^2 = 25 – c.$

Then, if the equation above indicates a circle, we need to have $\mathbf{25 - c > 0}$.

So we get $\mathbf{25 - c = 25 - \frac{24k+21}{k+1} = \frac{25k+25-24k-21}{k+1} = \frac{k+4}{k+1} = 1 + \frac{3}{k+1} > 0 \Rightarrow \frac{3}{k+1} > -1}$.

Now, since $(\boldsymbol{k} + \mathbf{1})$ can be negative, multiply both sides by $\mathbf{(k+1)^2}$ so that the direction of the inequality sign does not change. Then, we get

$\mathbf{3(k+1) > -(k+1)^2 \Rightarrow 3k + 3 > -k^2 - 2k - 1 \Rightarrow k^2 + 5k + 4 > 0 \Rightarrow (k+4)(k+1) > 0}$.

So if $\mathbf{k > -1}$ or $\mathbf{k < -4}$, then for each value of $\boldsymbol{k}$, the equation given can indicate a circle.

Next, let's find the extent of the radius in the equation $\mathbf{(x-4)^2 + (y-3)^2 = 25 - c}$.

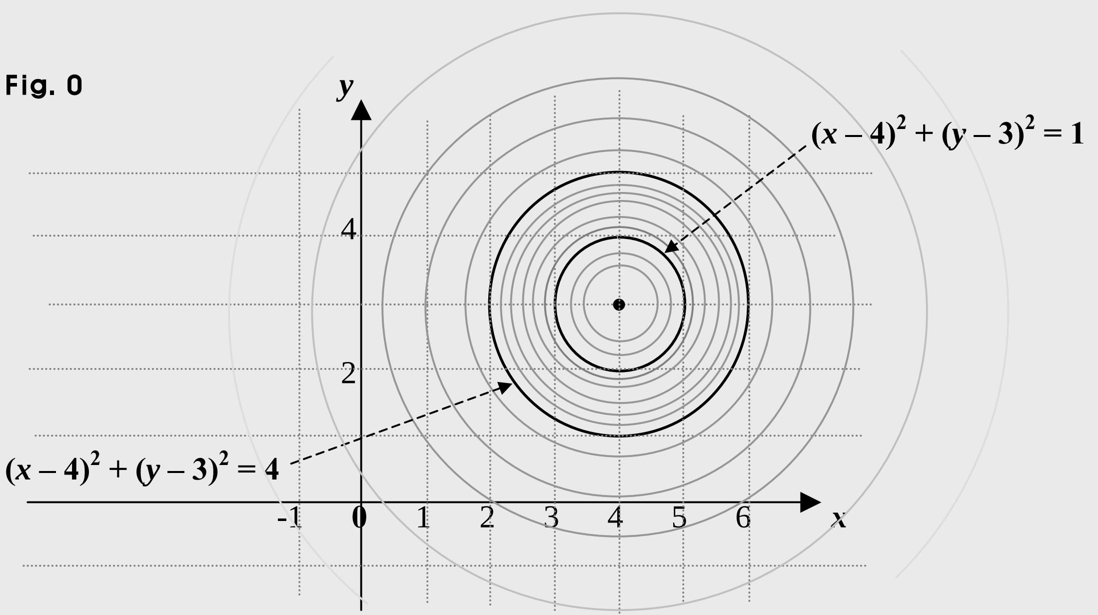
The radius squared is $\mathbf{25 - c = 1 + \frac{3}{k+1}}$ for $\mathbf{k > -1}$ or $\mathbf{k < -4}$. So we get

$\mathbf{k > -1 \Rightarrow 1 + \frac{3}{k+1} > 1 \Rightarrow 25 - c > 1}$.

$\mathbf{k < -4 \Rightarrow 0 < 1 + \frac{3}{k+1} < 1 \Rightarrow 0 < 25 - c < 1}$.

Therefore, the radius is greater than 1 for $\mathbf{k > -1}$, but is between 0 and 1 for $\mathbf{k < -4}$.
Let's now, put in a graph some of the circles concentric.

Fig. 0



In the graph, every circle in gray is a linear combination of the two circles in black.

Note that the smaller circle in black, $\boldsymbol{x^2 + y^2 - 8x - 6y + 24 = (x - 4)^2 + (y - 3)^2 - 1 = 0}$ cannot be made from $\boldsymbol{k(x^2 + y^2 - 8x - 6y + 24) + (x^2 + y^2 - 8x - 6y + 21) = 0}$ for any value of $\boldsymbol{k}$.

However, the smaller one, too, can be obtained if we use such an equation as follows.

$\boldsymbol{m(x^2 + y^2 - 8x - 6y + 24) + n(x^2 + y^2 - 8x - 6y + 21) = 0}$ where $\boldsymbol{m}$ and $\boldsymbol{n}$ are constants.

Setting $\boldsymbol{m = 1}$ and $\boldsymbol{n = 0}$, we get the smaller of the two.

Suggestions or Solutions
To the Problem in the Example 5

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$\boldsymbol{(k + 1)(x^2 + y^2) - 2(2k + 3)x - 2(2k + 3)y - k + 17 = 0.}$

First, we can put the equation the way below.

$(k + 1)(x^2 + y^2) - 2(2k + 3)x - 2(2k + 3)y - k + 17$
$= k(x^2 + y^2) - 4kx - 4ky - k + x^2 + y^2 - 6x - 6y + 17$
$= k(x^2 + y^2 - 4x - 4y - 1) + (x^2 + y^2 - 6x - 6y + 17) = 0$, which can be an equation of a circle for each value of k.

Next, suppose two circles $x^2 + y^2 - 4x - 4y - 1 = 0$ and $x^2 + y^2 - 6x - 6y + 17 = 0$ meet at (s, t).

Then, we get $s^2 + t^2 - 4s - 4t - 1 = 0$, and $s^2 + t^2 - 6s - 6t + 17 = 0$.

So we get $k(s^2 + t^2 - 4s - 4t - 1) + s^2 + t^2 - 6s - 6t + 17 = 0$.

So if the two circles above meet at a particular point, for any value of k, the equation given indicates a circle that includes the particular point.

Te same is true for another particular point, too, if the two circles share the point.

So if the two circles meet at particular points, for any value of k, the equation given indicates a circle that includes the particular points.

Getting the centers and radii of two circles, we can see if they can meet each other.

$x^2 + y^2 - 4x - 4y - 1 = x^2 - 4x + 4 - 4 + y^2 - 4y + 4 - 4 - 1 = (x - 2)^2 + (y - 2)^2 - 9 = 0$
So the center is $(2, 2)$, and the radius is 3.

$x^2 + y^2 - 6x - 6y + 17 = x^2 - 6x + 9 - 9 + y^2 - 6y + 9 - 9 + 17$
$= (x - 3)^2 + (y - 3)^2 - 1 = 0$. So the center is $(3, 3)$, and the radius is 1.

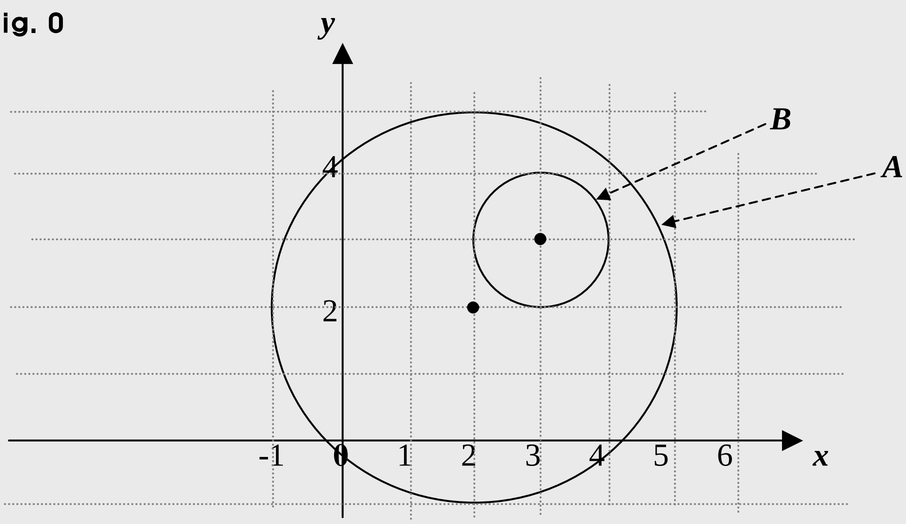
Suppose next, D is the distance between the two centers, and T is the difference between the radii. Then, we get

$$D^2 = (\Delta x)^2 + (\Delta y)^2 = (3 - 2)^2 + (3 - 2)^2 = 1 + 1 = 2$$
$$T = 3 - 1 = 2 > D = \sqrt{2} \;\; \Rightarrow T > D.$$

So one of the two circles is inside the other, and the two circles do now meet each other.

Therefore, there is no particular point each and every circle indicated by the equation given includes for every value of k.

Fig. 0



Now, let's find out for what value of k, the equation given can indicate a circle.

Then, simplifying the equation first, we get

$$(k + 1)(x^2 + y^2) - 2(2k + 3)x - 2(2k + 3)y - k + 17 = 0$$
$$\Rightarrow x^2 + y^2 - \frac{2(2k+3)}{k+1}x - \frac{2(2k+3)}{k+1}y + \frac{17-k}{k+1} = 0.$$

Then, first of all, we need to have $k \neq -1$ since a denominator cannot be 0.

Let's now, set $a = \frac{2k+3}{k+1}$, and $b = \frac{17-k}{k+1}$. Then, we get

$\boldsymbol{x^2 + y^2 - 2ax - 2ay + b = 0}$

$\boldsymbol{\Rightarrow x^2 - 2ax + a^2 - a^2 + y^2 - 2ay + a^2 - a^2 + b = (x - a)^2 + (y - a)^2 - 2a^2 + b = 0}$

$\boldsymbol{\Rightarrow (x - a)^2 + (y - a)^2 = b - 2a^2}$.

So $\boldsymbol{b - 2a^2}$ has to be positive. Thus, we need to have

$$\boldsymbol{b - 2a^2 = \tfrac{17-k}{k+1} - \tfrac{2(2k+3)^2}{(k+1)^2} = \tfrac{(17-k)(k+1)-2(2k+3)^2}{(k+1)^2} > 0 \Rightarrow (17-k)(k+1) - 2(2k+3)^2 > 0}$$

$$\boldsymbol{\Rightarrow -k^2 + 16k + 17 - 2(4k^2 + 12k + 9) = -9k^2 - 8k - 1 > 0 \Rightarrow 9k^2 + 8k + 1 < 0}.$$

Meanwhile, $\boldsymbol{9k^2 + 8k + 1 = 9(k^2 + \tfrac{8}{9}k + \tfrac{16}{81} - \tfrac{16}{81}) + 1 = 9(k + \tfrac{4}{9})^2 - \tfrac{16}{9} + 1 = 9(k + \tfrac{4}{9})^2 - \tfrac{7}{9}}$.

So we get $\boldsymbol{9(k + \tfrac{4}{9})^2 - \tfrac{7}{9} < 0 \Rightarrow (k + \tfrac{4}{9})^2 - \tfrac{7}{81} < 0}$.

Thus, we get $\boldsymbol{(k + \tfrac{4}{9})^2 - \tfrac{7}{81} = \{(k + \tfrac{4}{9}) + \tfrac{\sqrt{7}}{9}\}\{(k + \tfrac{4}{9}) - \tfrac{\sqrt{7}}{9}\} = (k + \tfrac{4+\sqrt{7}}{9})(k + \tfrac{4-\sqrt{7}}{9}) < 0}$.

So we get $\boldsymbol{-\tfrac{4+\sqrt{7}}{9} < k < -\tfrac{4-\sqrt{7}}{9}}$. We have $\boldsymbol{k \neq -1}$, too, though.

However, we have $\boldsymbol{-\tfrac{4+\sqrt{7}}{9} > -1}$, also.

So $\boldsymbol{k}$ can't be -1 in the case where $\boldsymbol{-\tfrac{4+\sqrt{7}}{9} < k < -\tfrac{4-\sqrt{7}}{9}}$.

Therefore, if $\boldsymbol{-\tfrac{4+\sqrt{7}}{9} < k < -\tfrac{4-\sqrt{7}}{9}}$,

the equation $\boldsymbol{(k + 1)(x^2 + y^2) - 2(2k + 3)x - 2(2k + 3)y - k + 17 = 0}$ where $\boldsymbol{k}$ is constant can indicate a circle.