Examples 9 in Circles

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0. Find a line passing through two points where the following two circles meet.

$\boldsymbol{x^2 + y^2 + 4x + 2y - 4 = 0}$ and $\boldsymbol{x^2 + y^2 - 2x - 2y + 1 = 0}$.

1. See if there is any particular point the curve of each equation below passes through for any value of $\boldsymbol{k}$, which is constant, and specify the particular point if any.

1.0. $\boldsymbol{k(x^2 + y^2) - (6k - 1)x - 2(2k + 1)y + 9k + 3 = 0}$.

1.1. $\boldsymbol{k(x^2 + y^2) - (2k - 1)x - 2(3k - 1)y + 5k - 12 = 0}$.

1.2. $\boldsymbol{k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = 0}$.

Suggestions or Solutions
To the Problem in the Example 0

Find the line passing through the two points where the following two circles meet.
$\mathbf{x^2 + y^2 + 4x + 2y - 4 = 0}$ **and** $\mathbf{x^2 + y^2 - 2x - 2y + 1 = 0}$.

$$-(x^2 + y^2 + 4x + 2y - 4) + x^2 + y^2 - 2x - 2y + 1 = 0$$

$$\Rightarrow -4x - 2y + 4 - 2x - 2y + 1 = -6x - 4y + 5 = 0 \Rightarrow y = \tfrac{5}{4} - \tfrac{3}{2}x.$$

Therefore, the line is $y = \frac{5}{4} - \frac{3}{2}x$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting $\mathbf{k = -1}$ in the equation $\mathbf{k(x^2 + y^2 + 4x + 2y - 4) + (x^2 + y^2 - 2x - 2y + 1) = 0}$, we can immediately get the equation of the line this problem is asking.   Why?

We have covered the idea in **Examples 8**.

Let's now, check to see if the two circles really meet at two points.

We can see it comparing the distance between the centers and the sum of or the difference between the radii.

And we can get the centers and radii putting the equations in the standard form.

$$x^2 + y^2 + 4x + 2y - 4 = (x + 2)^2 - 4 + (y + 1)^2 - 1 - 4 = (x + 2)^2 + (y + 1)^2 - 9 = 0$$

So the center is **(-2, -1)**, and the radius is 3.

$$x^2 + y^2 - 2x - 2y + 1 = (x - 1)^2 + (y - 1)^2 - 2 + 1 = (x - 1)^2 + (y - 1)^2 - 1 = 0$$

So the center is **(1, 1)**, and the radius is 1.

Suppose now, ***D*** is the distance between the centers, ***S*** is the sum of the radii, and ***T*** is the difference between the radii. Then, we get

$$\boldsymbol{D^2 = (\Delta x)^2 + (\Delta y)^2 = \{1 - (-2)\}^2 + \{1 - (-1)\}^2 = 9 + 4 = 13},\ \boldsymbol{S = 3 + 1 = 4},\ \&\ \boldsymbol{T = 3 - 1 = 2}.$$

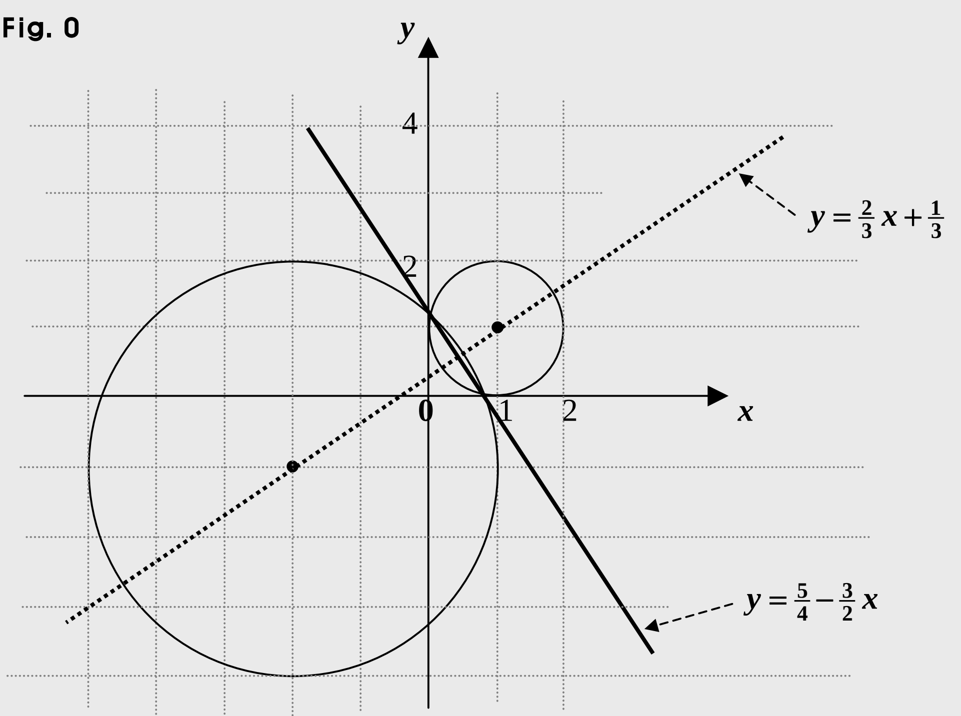
Thus, we get $\boldsymbol{S > D}$, because ***S*** and ***D*** are nonnegative and $\boldsymbol{S^2 = 16 > D^2 = 13}$.
Also, we get $\boldsymbol{T < D}$, because ***T*** and ***D*** are nonnegative and $\boldsymbol{T^2 = 4 < D^2 = 13}$.
So we get $\boldsymbol{T < D < S}$.
Therefore, the two circles do meet each other at two points, and share the two points.
If not sure, refer to the Problem 0 in the Example 0 in **Examples 8**.

Now, let's set $\boldsymbol{k = -1}$ in the equation $\boldsymbol{k(x^2 + y^2 + 4x + 2y - 4) + (x^2 + y^2 - 2x - 2y + 1) = 0}$, and actually, find the equation of the line passing through the two points where the two given circles meet each other.

$$\boldsymbol{k = -1 \Rightarrow k(x^2 + y^2 + 4x + 2y - 4) + (x^2 + y^2 - 2x - 2y + 1) = -4x - 2y + 4 - 2x - 2y + 1}$$
$$\boldsymbol{= -6x - 4y + 5 = 0 \Rightarrow y = \tfrac{5}{4} - \tfrac{3}{2}x}.$$

Let's now, put in a graph the two circles and the line passing through the two points.

Fig. 0



Of course, we can find first, the two points where the two circles meet, and then, get the equation of the line passing through the two points.

So we can get the line this problem is asking in such a way as above, too, but it will take some time to find such two points.

However, knowing how circles (equations for circles) work, we can get such a line in no time.

In short:

$$-(x^2 + y^2 + 4x + 2y - 4) + x^2 + y^2 - 2x - 2y + 1 = 0$$

$$\Rightarrow -4x - 2y + 4 - 2x - 2y + 1 = -6x - 4y + 5 = 0 \Rightarrow y = \frac{5}{4} - \frac{3}{2}x.$$

Therefore, the line is $y = \frac{5}{4} - \frac{3}{2}x$.

Suggestions or Solutions
To the Problem 0 in the Example 1

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$k(x^2 + y^2) - (6k - 1)x - 2(2k + 1)y + 9k + 3 = 0$.

First, we can put the equation the way below.

$k(x^2 + y^2) - (6k - 1)x - 2(2k + 1)y + 9k + 3$
$= k(x^2 + y^2) - 6kx + x - 4ky - 2y + 9k + 3$
$= k(x^2 + y^2) - 6kx - 4ky + 9k + x - 2y + 3$
$= k(x^2 + y^2 - 6x - 4y + 9) + (x - 2y + 3) = 0$, which can indicate a circle for a value of k if $k \neq 0$.

Next, suppose a circle $x^2 + y^2 - 6x - 4y + 9 = 0$ meets a line $x - 2y + 3 = 0$ at (s, t).

Then, we get $s^2 + t^2 - 6s - 4t + 9 = 0$, and $s - 2t + 3 = 0$.

So we get $k(s^2 + t^2 - 6s - 4t + 9) + s - 2t + 3 = 0$.

So if the circle and the line share any particular point, for any value of k, other than 0, the equation given indicates a circle that includes the particular point.

Finding the points, we can get first, $x - 2y + 3 = 0 \Rightarrow x = 2y - 3$.

And next, we get $x^2 + y^2 - 6x - 4y + 9 = (2y - 3)^2 + y^2 - 6(2y - 3) - 4y + 9$

$= 4y^2 - 12y + 9 + y^2 - 12y + 18 - 4y + 9 = 5y^2 - 28y + 36 = (5y - 18)(y - 2) = 0$

$\Rightarrow y = 2$ or $y = \frac{18}{5}$. Thus, $y = 2 \Rightarrow x = 2y - 3 = 1$, and $y = \frac{18}{5} \Rightarrow x = 2y - 3 = \frac{36}{5} - 3 = \frac{21}{5}$.

So for any value of k, other than 0, the equation given indicates a circle that includes the two points $(1, 2)$ and $(\frac{21}{5}, \frac{18}{5})$, which are therefore, the particular points.

If not quite sure of the idea behind the processes above, follow the steps below.

This problem is just about the same as the ones in **Examples 8**.
Let's see though, what's new here.

First, note that in math, a curve can be a line, too. Next, we have a fact as follows.

Suppose $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, and the curve of $\boldsymbol{f(x, y) = 0}$ meets that of $\boldsymbol{g(x, y) = 0}$ at particular points.

Then, at the particular points, too, the curve of $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ meets the two curves of $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$.

In other words, the three curves share the particular points.

And in particular, the equation $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ is called a linear combination of the two equations $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$.

So now, getting back to the problem in this example, we want to see first, if the equation given can be such a linear combination. That is, we want to begin with extracting equations, of which the equation given is a linear combination.

$$\boldsymbol{k(x^2 + y^2) - (6k - 1)x - 2(2k + 1)y + 9k + 3 = k(x^2 + y^2) - 6kx + x - 4ky - 2y + 9k + 3}$$
$$\boldsymbol{= k(x^2 + y^2) - 6kx - 4ky + 9k + x - 2y + 3 = k(x^2 + y^2 - 6x - 4y + 9) + (x - 2y + 3) = 0}.$$

Thus, we can see two equations, one indicates a circle $\boldsymbol{x^2 + y^2 - 6x - 4y + 9 = 0}$, and the other indicates a line $\boldsymbol{x - 2y + 3 = 0}$.

So if $\boldsymbol{C}$ is the circle above, and $\boldsymbol{L}$ is the line above, the curve of the equation given is a linear combination of $\boldsymbol{C}$ and $\boldsymbol{L}$, so we can use the fact below.

Suppose a circle $\boldsymbol{x^2 + y^2 + ax + by + c = 0}$ and a line $\boldsymbol{dx + ey + f = 0}$ meet each other at particular points, $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, and $\boldsymbol{m \neq 0}$. Then, for each pair of the values of $\boldsymbol{m}$ and $\boldsymbol{n}$, the equation $\boldsymbol{m(x^2 + y^2 + ax + by + c) + n(dx + ey + f) = 0}$ indicates a circle that includes the particular points, and has its center in a line passing through the center of a circle $\boldsymbol{x^2 + y^2 + ax + by + c = 0}$ and perpendicular to a line $\boldsymbol{dx + ey + f = 0}$.

So let's now, see if the circle $\boldsymbol{C}$ and the line $\boldsymbol{L}$ share any point.

To begin with, we can get $\mathbf{x - 2y + 3 = 0 \Rightarrow x = 2y - 3}$.

Then, we get $\mathbf{x^2 + y^2 - 6x - 4y + 9 = (2y - 3)^2 + y^2 - 6(2y - 3) - 4y + 9}$

$\mathbf{= 4y^2 - 12y + 9 + y^2 - 12y + 18 - 4y + 9 = 5y^2 - 28y + 36 = (5y - 18)(y - 2) = 0}$

$\Rightarrow \mathbf{y = 2}$ or $\mathbf{y = \frac{18}{5}}$.

So we get $\mathbf{y = 2 \Rightarrow x = 2y - 3 = 1}$, and $\mathbf{y = \frac{18}{5} \Rightarrow x = 2y - 3 = \frac{36}{5} - 3 = \frac{21}{5}}$.

Therefore, the circle $\boldsymbol{C}$ meets the line $\boldsymbol{L}$ at two points $\mathbf{(1, 2)}$ and $\mathbf{(\frac{21}{5}, \frac{18}{5})}$.

It is not the case however, the equation $\mathbf{k(x^2 + y^2) - (6k - 1)x - 2(2k + 1)y + 9k + 3 = 0}$ indicates a circle for each and every value of $\boldsymbol{k}$. Why?

$\mathbf{k(x^2 + y^2) - (6k - 1)x - 2(2k + 1)y + 9k + 3 = k(x^2 + y^2 - 6x - 4y + 9) + (x - 2y + 3) = 0}$.

So if $\mathbf{k = 0}$, we don't get a circle but just a line.

Thus, other than 0, for each and every value of $\boldsymbol{k}$, the equation given indicates a circle that includes the two points $\mathbf{(1, 2)}$ and $\mathbf{(\frac{21}{5}, \frac{18}{5})}$.

That is, the equation given represents a group of circles sharing altogether the two points.

Let's now, put in a graph the circle $\boldsymbol{C}$ and the line $\boldsymbol{L}$, together with several circles in the group. To begin with, putting the circle $\boldsymbol{C}$ in the standard form, we get

$\mathbf{x^2 + y^2 - 6x - 4y + 9 = x^2 - 6x + 9 + y^2 - 4y + 4 - 4 = (x - 3)^2 + (y - 2)^2 - 4 = 0}$.

So the center of $\boldsymbol{C}$ is $\mathbf{(3, 2)}$, and the radius is 2.

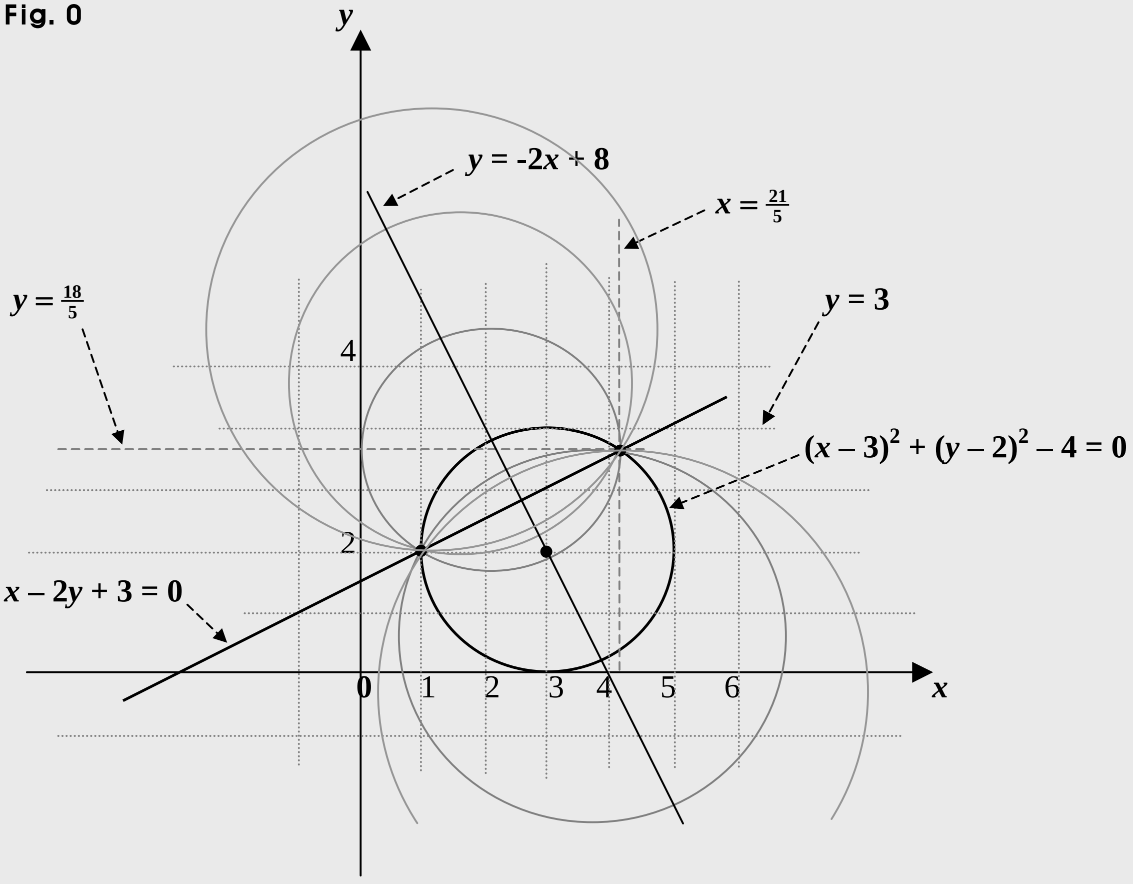
Note however. that the circle $\boldsymbol{C}$ cannot be indicated by the equation given.

We can indicate though, the circle $\boldsymbol{C}$, too, if we use an equation as follows.

$\boldsymbol{m(x^2 + y^2 - 2x - 6y + 5) + n(x + 2y - 12) = 0}$ where $\boldsymbol{m}$ and $\boldsymbol{n}$ are constants.

So in the equation above, setting $\boldsymbol{m = 1}$ and $\boldsymbol{n = 0}$, we get the circle $\boldsymbol{C}$.

Fig. 0



The centers of all the circles above are in the line $\boldsymbol{y = -2x + 8}$, which is perpendicular to the line $\boldsymbol{x - 2y + 3 = 0}$, and connects the center of the circle $\boldsymbol{(x - 3)^2 + (y - 2)^2 = 4}$ and the midpoint between the two particular points.

Suggestions or Solutions
To the Problem 1 in the Example 1

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$\boldsymbol{k(x^2 + y^2) - (2k - 1)x - 2(3k - 1)y + 5k - 12 = 0.}$

First, we can put the equation the way below.

$\boldsymbol{k(x^2 + y^2) - (2k - 1)x - 2(3k - 1)y + 5k - 12}$
$\boldsymbol{= k(x^2 + y^2) - 2kx + x - 6ky + 2y + 5k - 12}$
$\boldsymbol{= k(x^2 + y^2) - 2kx - 6ky + 5k + x + 2y - 12}$
$\boldsymbol{= k(x^2 + y^2 - 2x - 6y + 5) + (x + 2y - 12) = 0}$, which can indicate a circle for a value of $\boldsymbol{k}$ if $\boldsymbol{k \neq 0}$.

Next, suppose a circles $\boldsymbol{x^2 + y^2 - 2x - 6y + 5 = 0}$ meets a line $\boldsymbol{x + 2y - 12 = 0}$ at $\boldsymbol{(s, t)}$.

Then, we get $\boldsymbol{s^2 + t^2 - 2s - 6t + 5 = 0}$, and $\boldsymbol{s + 2t - 12 = 0}$.

So we get $\boldsymbol{k(s^2 + t^2 - 2s - 6t + 5) + s + 2t - 12 = 0}$.

So if the circle and the line share any particular point, for any value of $\boldsymbol{k}$, other than 0, the equation given indicates a circle that includes the particular point.
So let's now, find the points.

To begin with, $\boldsymbol{x + 2y - 12 = 0 \Rightarrow x = 12 - 2y}$.

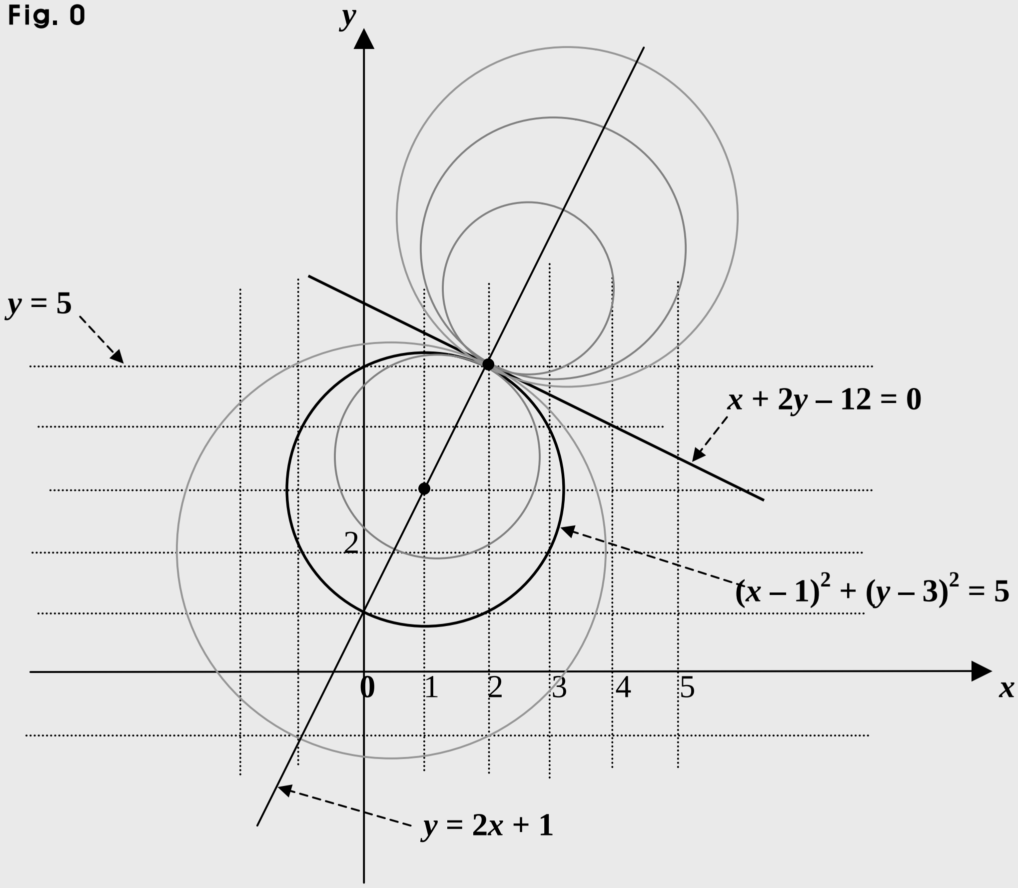
And next, we get $\boldsymbol{x^2 + y^2 - 2x - 6y + 5 = (12 - 2y)^2 + y^2 - 2(12 - 2y) - 6y + 5}$

$\boldsymbol{= 144 - 48y + 4y^2 + y^2 - 24 + 4y - 6y + 5 = 5y^2 - 50y + 125 = 0}$

$\boldsymbol{\Rightarrow y^2 - 10y + 25 = (y - 5)^2 = 0 \Rightarrow y = 5 \Rightarrow x = 12 - 2y = 2}$.

Therefore, **(2, 5)** is the particular point that is shared by each and every circle indicated by the equation given for every value of $\boldsymbol{k}$.

Fig. 0



Suggestions or Solutions
To the Problem 2 in the Example 1

See if there is any particular point the curve of the equation below passes through for any value of k, which is constant, and specify the particular point if any.
$k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = 0.$

To begin with, we can put the equation the way below.

$$k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = k(x^2 + y^2) - 6kx + x - 4ky - y + 10k + 2$$
$$= k(x^2 + y^2) - 6kx - 4ky + 10k + x - y + 2 = k(x^2 + y^2 - 6x - 4y + 10) + (x - y + 2) = 0.$$

So the given equation can be an equation of a circle for a value of k if $k \neq 0$.

Next, suppose a circles $x^2 + y^2 - 6x - 4y + 10 = 0$ meets a line $x - y + 2 = 0$ at (s, t).

Then, we get $s^2 + t^2 - 6s - 4t + 10 = 0$, and $s - t + 2 = 0$.

So we get $k(s^2 + t^2 - 6s - 4t + 10) + s - t + 2 = 0$.

So if the circle and the line share any particular point, for any value of k, other than 0, the equation given indicates a circle that includes the particular point.
Finding the particular point, we get

$$x - y + 2 = 0 \Rightarrow x = y - 2 \Rightarrow x^2 + y^2 - 6x - 4y + 10 = (y - 2)^2 + y^2 - 6(y - 2) - 4y + 10$$

$$= y^2 - 4y + 4 + y^2 - 6y + 12 - 4y + 10 = 2y^2 - 14y + 26 = 0 \Rightarrow y^2 - 7y + 13 = 0.$$

However, we have
$$y^2 - 7y + 13 = y^2 - 7y + (\tfrac{7}{2})^2 - (\tfrac{7}{2})^2 + 13 = (y - \tfrac{7}{2})^2 - \tfrac{49}{4} + 13 = (y - \tfrac{7}{2})^2 + \tfrac{3}{4} > 0.$$

Thus, the circle does not meet the line, so there is no particular point that can be shared by every circle indicated by the given equation for every nonzero k.

That is, there is no particular point that can be shared by each and every circle indicated by the given equation for every value of k.

If not quite sure of the idea behind the processes above, follow the steps below.

For this problem, too, we can use the fact below.

Suppose $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, and at particular points, the curve of $\boldsymbol{f(x, y) = 0}$ meets the curve of $\boldsymbol{g(x, y) = 0}$.

Then, at the particular points, too, the curve of $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ meets the two curves of $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$.

In other words, the three curves share the particular points.

And in particular, the equation $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ is called a linear combination of the two equations $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$.

So we want to check to see if the equation given can be such a linear combination.
That is, we get to extract equations, which the equation given is a linear combination of.

$$\boldsymbol{k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = k(x^2 + y^2) - 6kx + x - 4ky - y + 10k + 2}$$

$$\boldsymbol{= k(x^2 + y^2) - 6kx - 4ky + 10k + x - y + 2 = k(x^2 + y^2 - 6x - 4y + 10) + (x - y + 2) = 0}.$$

So we can see two equations, one indicates a circle $\boldsymbol{x^2 + y^2 - 6x - 4y + 10 = 0}$, and the other indicates a line $\boldsymbol{x - y + 2 = 0}$.

So suppose $\boldsymbol{C}$ is the circle above, and $\boldsymbol{L}$ is the line above. Then, the curve of the equation given is a linear combination of $\boldsymbol{C}$ and $\boldsymbol{L}$, so we can use such a fact as follows.

Suppose a circle $\boldsymbol{x^2 + y^2 + ax + by + c = 0}$ meets a line $\boldsymbol{dx + ey + f = 0}$ at a particular point, $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, and $\boldsymbol{m \neq 0}$. Then, for each pair of the values of $\boldsymbol{m}$ and $\boldsymbol{n}$, the equation $\boldsymbol{m(x^2 + y^2 + ax + by + c) + n(dx + ey + f) = 0}$ indicates another circle that includes the particular point, and has its center in a line passing through the center of a circle $\boldsymbol{x^2 + y^2 + ax + by + c = 0}$ and perpendicular to a line $\boldsymbol{dx + ey + f = 0}$.

Let's see now, if the circle $\boldsymbol{C}$ meets the line $\boldsymbol{L}$.

To begin with, we can get $\mathbf{x - y + 2 = 0 \Rightarrow x = y - 2}$.

So next, we get $\mathbf{x^2 + y^2 - 6x - 4y + 10 = (y - 2)^2 + y^2 - 6(y - 2) - 4y + 10}$

$\mathbf{= y^2 - 4y + 4 + y^2 - 6y + 12 - 4y + 10 = 2y^2 - 14y + 26 = 0}$. So we get $\mathbf{y^2 - 7y + 13 = 0}$.

However, $\mathbf{y^2 - 7y + 13 = y^2 - 7y + (\tfrac{7}{2})^2 - (\tfrac{7}{2})^2 + 13 = (y - \tfrac{7}{2})^2 - \tfrac{49}{4} + 13 = (y - \tfrac{7}{2})^2 + \tfrac{3}{4} > 0}$.

In other words, we have $\mathbf{y^2 - 7y + 13 \neq 0}$. Therefore, we can see that there is no solution to the equation $\mathbf{y^2 - 7y + 13 = 0}$, so the circle does not meet the line.

We can check the same using the discriminant, too, of course.

Finding the discriminant of $\mathbf{au^2 + bu + c = 0}$, we get $\mathbf{b^2 - 4ac}$, and if the discriminant is negative, the equation has no root, so it has no solution.

However, the equation given can represent many circles. That is, other than 0, for each of some values of $\boldsymbol{k}$, the equation given still indicates a circle. That's because we have

$\mathbf{k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = k(x^2 + y^2 - 6x - 4y + 10) + (x - y + 2) = 0}$.

So if $\mathbf{k \neq 0}$, we can get a circle. And if $\mathbf{k = 0}$, of course, we just get a line $\mathbf{x - y + 2 = 0}$.

Let's now, put in a graph the circle $\boldsymbol{C}$ and the line $\boldsymbol{L}$.

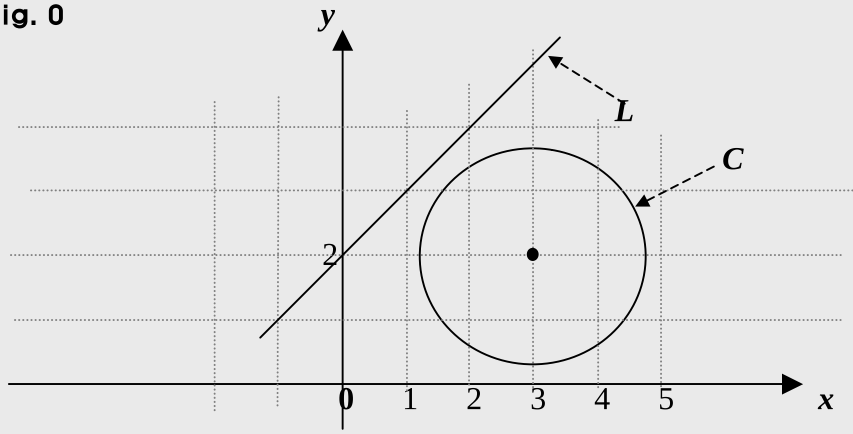
To begin with, putting the circle $\boldsymbol{C}$ in the standard form, we get

$\mathbf{x^2 + y^2 - 6x - 4y + 10 = x^2 - 6x + 9 - 9 + y^2 - 4y + 4 - 4 + 10}$

$\mathbf{= (x - 3)^2 + (y - 2)^2 - 3 = 0 \Rightarrow (x - 3)^2 + (y - 2)^2 = 3}$.

So the center of $\boldsymbol{C}$ is at $\mathbf{(3, 2)}$, and the radius is $\mathbf{\sqrt{3}}$.

Fig. 0



Now, for <u>some</u> values of $\boldsymbol{k}$, the equation $\boldsymbol{k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = 0}$ indicates a circle. So it is not the case where it indicate a circle for every value of $\boldsymbol{k}$.

For what value of $\boldsymbol{k}$ then, can the given equation indicate a circle?

Let's first, put the given equation in the standard form so that we can see the radius.

To begin with, $\boldsymbol{k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = 0}$

$$\Rightarrow \; \boldsymbol{x^2 + y^2 - \tfrac{6k-1}{k}x - \tfrac{4k+1}{k}y + \tfrac{10k+2}{k} = 0 \Rightarrow x^2 + y^2 - \tfrac{2(6k-1)}{2k}x - \tfrac{2(4k+1)}{2k}y + \tfrac{10k+2}{k} = 0}.$$

And next, setting $\boldsymbol{a = \tfrac{6k-1}{2k}}$, $\boldsymbol{b = \tfrac{4k+1}{2k}}$, and $\boldsymbol{c = \tfrac{10k+2}{k}}$, we get $\boldsymbol{x^2 + y^2 - 2ax - 2by + c = 0}$

$\Rightarrow \boldsymbol{(x - a)^2 + (y - b)^2 + c - a^2 - b^2 = 0 \Rightarrow (x - a)^2 + (y - b)^2 = a^2 + b^2 - c}$.

Thus, $\boldsymbol{a^2 + b^2 - c}$ has to be positives since a radius squared is positive.

So we get $\boldsymbol{a^2 + b^2 - c = \tfrac{(6k-1)^2}{4k^2} + \tfrac{(4k+1)^2}{4k^2} - \tfrac{10k+2}{k} = \tfrac{(6k-1)^2 + (4k+1)^2 - 4k(10k+2)}{4k^2} > 0}$.

Thus, $\boldsymbol{(6k-1)^2 + (4k+1)^2 - 4k(10k+2) = 36k^2 - 12k + 1 + 16k^2 + 8k + 1 - 40k^2 - 8k}$

$\boldsymbol{= 12k^2 - 12k + 2 = 2(6k^2 - 6k + 1) > 0 \Rightarrow 6k^2 - 6k + 1 > 0}$.

So we need to have $\boldsymbol{6k^2 - 6k + 1 = 6(k^2 - k) + 1 = 6(k^2 - k + \frac{1}{4} - \frac{1}{4}) + 1}$

$\boldsymbol{= 6(k - \frac{1}{2})^2 - \frac{3}{2} + 1 = 6(k - \frac{1}{2})^2 - \frac{1}{2} > 0 \Rightarrow (k - \frac{1}{2})^2 - \frac{1}{12} > 0}$.

Meanwhile, we have a factorization identity where $\boldsymbol{P^2 - Q^2 = (P + Q)(P - Q)}$.

So we get $\boldsymbol{\{(k - \frac{1}{2}) + \sqrt{\frac{1}{12}}\}\{(k - \frac{1}{2}) - \sqrt{\frac{1}{12}}\} > 0 \Rightarrow \{(k - \frac{1}{2}) + \frac{1}{2\sqrt{3}}\}\{(k - \frac{1}{2}) - \frac{1}{2\sqrt{3}}\} > 0}$

$\boldsymbol{\Rightarrow \{(k - \frac{1}{2}) + \frac{\sqrt{3}}{6}\}\{(k - \frac{1}{2}) - \frac{\sqrt{3}}{6}\} > 0 \Rightarrow (k - \frac{3-\sqrt{3}}{6})(k - \frac{3+\sqrt{3}}{6}) > 0}$

$\boldsymbol{\Rightarrow k > \frac{3+\sqrt{3}}{6} \approx 0.789}$ or $\boldsymbol{k < \frac{3-\sqrt{3}}{6} \approx 0.211}$.

Therefore, when $\boldsymbol{k > \frac{3+\sqrt{3}}{6}}$ or $\boldsymbol{k < \frac{3-\sqrt{3}}{6}}$, the given equation where

$\boldsymbol{k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = 0}$ can indicate a circle.

The centers of all the circles that can be indicated by the equation above are in a line, which is perpendicular to the line $\boldsymbol{L}$, which is $\boldsymbol{x - y + 2 = 0}$, and also, passes through the center of the circle $\boldsymbol{C}$, which is $\boldsymbol{x^2 + y^2 - 6x - 4y + 10}$, which is $\boldsymbol{(x - 3)^2 + (y - 2)^2 = 3}$.

Suppose $\boldsymbol{N}$ is the perpendicular line above. Then, $\boldsymbol{N}$ passes through all the centers.

So let's now, get the line $\boldsymbol{N}$.   To begin with, we have a fact below.

Given the slope and a point in a line, we can readily find the line.

So if $\boldsymbol{p}$ is the slope, and $\boldsymbol{(s, t)}$ is a point in a line, the line is $\boldsymbol{y - t = p(x - s)}$.

Now, the line $\boldsymbol{N}$ is perpendicular to the line $\boldsymbol{L}$, $\boldsymbol{x - y + 2 = 0}$, where the slope is 1.

So the slope of $\boldsymbol{N}$ is **-1** because the product of the slopes of two lines perpendicular to each other is -1, and the slope of $\boldsymbol{L}$ is 1.

Next, the line $\boldsymbol{N}$ includes the center of the circle $\boldsymbol{C}$, and the center of $\boldsymbol{C}$ is **(3, 2)**.

So the line $\boldsymbol{N}$ is $\boldsymbol{y - 2 = -(x - 3)}$, which equals $\boldsymbol{y = -x + 5}$, in which the centers of all the circles that are linear combinations of $\boldsymbol{C}$ and $\boldsymbol{L}$ are located.

And yet, not the entire line can be the location where all the centers can be put.

That's because we have the limitation on the constant $\boldsymbol{k}$ in the equation given.

So let's find in what part of the line $\boldsymbol{N}$ all the centers can be put.

To begin with, we have put the given equation in such a from as follows.

$\boldsymbol{(x - a)^2 + (y - b)^2 = a^2 + b^2 - c}$ where $\boldsymbol{a = \frac{6k-1}{2k}}$, $\boldsymbol{b = \frac{4k+1}{2k}}$, and $\boldsymbol{c = \frac{10k+2}{k}}$.

So each of the centers is at $\boldsymbol{(a, b)}$.

And finding the extent of $\boldsymbol{a}$, we can see the extent of the $\boldsymbol{x}$-coordinate of the center.

We have $\boldsymbol{a = \frac{6k-1}{2k} = 3 - \frac{1}{2k}}$ for $\boldsymbol{k > \frac{3+\sqrt{3}}{6}}$ or $\boldsymbol{k < \frac{3-\sqrt{3}}{6}}$. So to begin with, we can get

$\boldsymbol{k > \frac{3+\sqrt{3}}{6} \Rightarrow 2k > \frac{3+\sqrt{3}}{3} \Rightarrow 0 < \frac{1}{2k} < \frac{3}{3+\sqrt{3}} = \frac{3(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{3-\sqrt{3}}{2} \Rightarrow 0 < \frac{1}{2k} < \frac{3-\sqrt{3}}{2}}$.

Why do we get not only $\boldsymbol{\frac{1}{2k} < \frac{3}{3+\sqrt{3}}}$ but $\boldsymbol{0 < \frac{1}{2k}}$, too?

That's because $\boldsymbol{k}$ is positive since $\boldsymbol{k > \frac{3+\sqrt{3}}{6}}$, so we get $\boldsymbol{\frac{1}{2k} > 0}$, too.

Thus, we get $\boldsymbol{0 < \frac{1}{2k} < \frac{3-\sqrt{3}}{2} \Rightarrow 0 > -\frac{1}{2k} > -\frac{3-\sqrt{3}}{2} \Rightarrow 3 > 3 - \frac{1}{2k} > 3 - \frac{3-\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}}$.

So we get $\boldsymbol{3 > a > \frac{3+\sqrt{3}}{2} \approx 2.366}$.

Next, let's move on to $\boldsymbol{k < \frac{3-\sqrt{3}}{6}}$. Then, we get $\boldsymbol{k < \frac{3-\sqrt{3}}{6} \Rightarrow 2k < \frac{3-\sqrt{3}}{3}}$.

So we get $\boldsymbol{\frac{1}{2k} > \frac{3}{3-\sqrt{3}}}$ or $\boldsymbol{\frac{1}{2k} \leq 0}$. Why do we get $\boldsymbol{\frac{1}{2k} \leq 0}$, too, though?

That's because we have $\boldsymbol{k} < \frac{3-\sqrt{3}}{6}$, so $\boldsymbol{k}$ can be $\leq \mathbf{0}$.

Now, beginning with $\frac{1}{2k} \leq 0$, we get $\frac{1}{2k} \leq 0 \Rightarrow -\frac{1}{2k} \geq 0 \Rightarrow 3 - \frac{1}{2k} \geq 3 \Rightarrow a \geq 3$.

Next, we get $\frac{1}{2k} > \frac{3}{3-\sqrt{3}} = \frac{3(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})} = \frac{3+\sqrt{3}}{2} \Rightarrow \frac{1}{2k} > \frac{3+\sqrt{3}}{2} \Rightarrow -\frac{1}{2k} < -\frac{3+\sqrt{3}}{2}$

$\Rightarrow 3 - \frac{1}{2k} < 3 - \frac{3+\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2} \Rightarrow a < \frac{3-\sqrt{3}}{2} \approx 0.634$.

Therefore, we get $a \geq 3$ or $a < \frac{3-\sqrt{3}}{2} \approx 0.634$.

Thus, we now have $3 > a > \frac{3+\sqrt{3}}{2} \approx 2.366$, $a \geq 3$, or $a < \frac{3-\sqrt{3}}{2} \approx 0.634$.

Putting threads together, we have $a > \frac{3+\sqrt{3}}{2} \approx 2.366$ or $a < \frac{3-\sqrt{3}}{2} \approx 0.634$.

Now, since $\boldsymbol{a}$ is the $\boldsymbol{x}$-coordinate of each center, all the centers are in two rays as follows.

$y = -x + 5$ for $x > \frac{3+\sqrt{3}}{2} \approx 2.366$ or $x < \frac{3-\sqrt{3}}{2} \approx 0.634$.

Note that the circle C, $x^2 + y^2 - 6x - 4y + 10 = 0$, that is, $(x - 3)^2 + (y - 2)^2 = 3$ cannot be made by the equation given. That's because we have

$$k(x^2 + y^2) - (6k - 1)x - (4k + 1)y + 10k + 2 = k(x^2 + y^2 - 6x - 4y + 10) + (x - y + 2) = 0.$$

So if $k \neq 0$, we can get a circle. And if $k = 0$, of course, we just get a line, $x - y + 2 = 0$.

However, the circle C can be obtained if we use such an equation as follows.

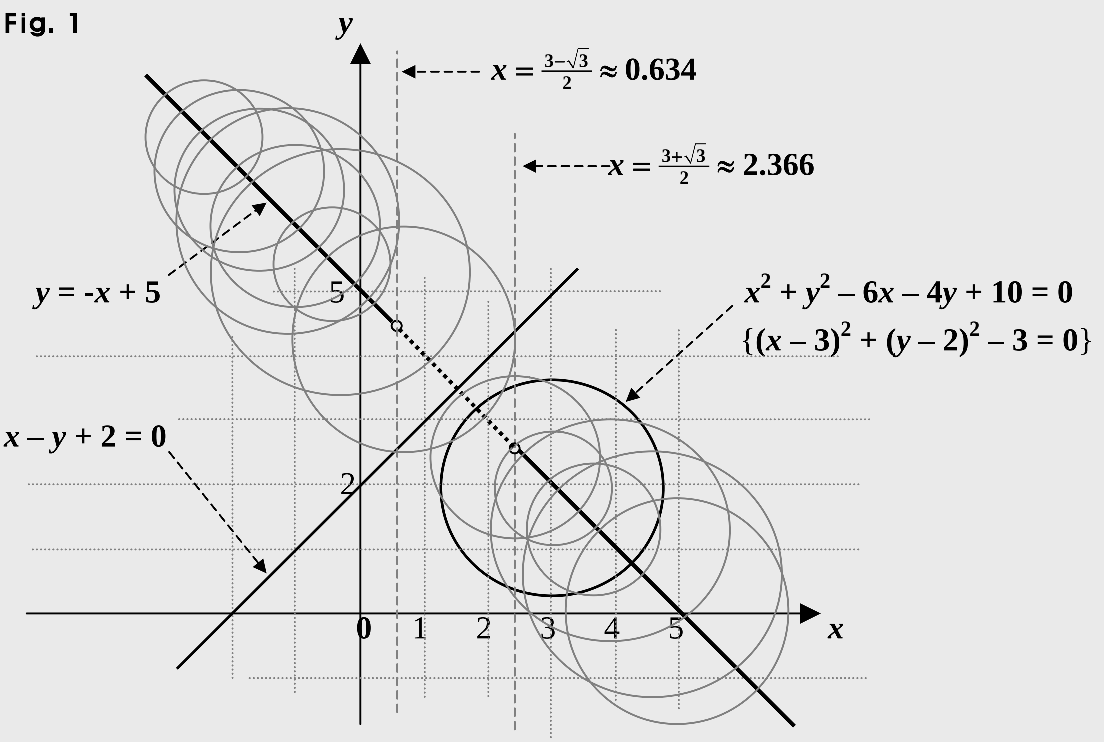
$m(x^2 + y^2 - 6x - 4y + 10) + n(x - y + 2) = 0$ where m and n are constants.

So setting $m = 1$ and $n = 0$, we get the circle C.

Let's now, put in a graph some circles each of which is a linear combination of the circle C and the line L, together with the circle C and the line L.

Note that in the graph above, the radii of the circles in gray are not accurate at all. That is, the gray circles are no more than the ones for showing how the circles can be put in a graph. The key idea is that the centers of all the circles are on the two rays.

Fig. 1



Now, what if one of the curves in such a linear combination as above is not a circle?

Suppose for instance, the equation given is as follows.

$\boldsymbol{k(x^2 + y^2) - 2(k - 1)x - 2(3k - 1)y + 12k - 1 = 0}$.

Then, extracting the curves, we get

$$\begin{aligned}
&k(x^2 + y^2) - 2(k - 1)x - 2(3k - 1)y + 12k - 1 \\
&= k(x^2 + y^2) - 2kx + 2x - 6ky + 2y + 12k - 1 \\
&= k(x^2 + y^2) - 2kx - 6ky + 12k + 2x + 2y - 1 \\
&= k(x^2 + y^2 - 2x - 6y + 12) + (2x + 2y - 1) = 0.
\end{aligned}$$

So one is $x^2 + y^2 - 2x - 6y + 12 = 0$, and the other is a line $2x + 2y - 1 = 0$.

And next, putting the first of the two above in the standard form, we get

$\boldsymbol{x^2 + y^2 - 2x - 6y + 12 = x^2 - 2x + 1 - 1 + y^2 - 6y + 9 - 9 + 12}$

$\boldsymbol{= (x - 1)^2 + (y - 3)^2 + 2 = 0}$. So we get $\boldsymbol{(x - 1)^2 + (y - 3)^2 = -2}$, which however, does not indicate a circle. So does the equation given not indicate a circle for any value of $\boldsymbol{k}$?

It is not necessarily the case. It can still be the case where the equation can indicate a circle for some value of $\boldsymbol{k}$.

So let's check to see if there is any value for k so that the equation indicates a circle. Then, to begin with, we can put the equation this way:

$$\boldsymbol{k(x^2 + y^2) - 2(k - 1)x - 2(3k - 1)y + 12k - 1 = 0}$$

$$\boldsymbol{\Rightarrow x^2 + y^2 - \frac{2(k-1)}{k}x - \frac{2(3k-1)}{k}y + \frac{12k-1}{k} = 0}.$$

And next, setting $\boldsymbol{a = \frac{k-1}{k}}$, $\boldsymbol{b = \frac{3k-1}{k}}$, and $\boldsymbol{c = \frac{12k-1}{k}}$, we get $\boldsymbol{x^2 + y^2 - 2ax - 2by + c = 0}$.

Then, we get $\boldsymbol{x^2 + y^2 - 2ax - 2by + c = x^2 - 2ax + a^2 - a^2 + y^2 - 2by + b^2 - b^2 + c}$

$\boldsymbol{= (x - a)^2 + (y - b)^2 + c - (a^2 + b^2) = 0 \Rightarrow (x - a)^2 + (y - b)^2 = a^2 + b^2 - c}$.

Now, if the equation $\boldsymbol{(x - a)^2 + (y - b)^2 = a^2 + b^2 - c}$ indicates a circle, $\boldsymbol{a^2 + b^2 - c}$ has to be positive. If it indicates a circle, we need to have

$$\boldsymbol{a^2 + b^2 - c = \left(\frac{k-1}{k}\right)^2 + \left(\frac{3k-1}{k}\right)^2 - \frac{12k-1}{k} = \frac{(k-1)^2 + (3k-1)^2 - (12k-1)k}{k^2} > 0}.$$

Then, first, we need to have $\boldsymbol{k \neq 0}$ since a denominator cannot be 0.

Next, we need to have $\boldsymbol{(k - 1)^2 + (3k - 1)^2 - (12k - 1)k > 0}$.

Expanding the left hand side and simplifying the result, we get
$\boldsymbol{k^2 - 2k + 1 + 9k^2 - 6k + 1 - 12k^2 + k = -2k^2 - 7k + 2 > 0 \Rightarrow 2k^2 + 7k - 2 < 0}$.

Next, setting $\boldsymbol{2k^2 + 7k - 2 = 0}$, we get $\boldsymbol{k = \frac{-7\pm\sqrt{49-4\cdot2\cdot(-2)}}{4} = \frac{-7\pm\sqrt{65}}{4}}$.

So we get $\boldsymbol{\frac{-7-\sqrt{65}}{4} < k < \frac{-7+\sqrt{65}}{4}}$, since the parabola $\boldsymbol{y = 2x^2 + 7x - 2}$ is concave-up.

Therefore, we need to have $\boldsymbol{\frac{-7-\sqrt{65}}{4} < k < \frac{-7+\sqrt{65}}{4}}$, yet $\boldsymbol{k}$ cannot be 0.

So we need to have $\boldsymbol{\frac{-7-\sqrt{65}}{4} < k < 0}$, or $\boldsymbol{0 < k < \frac{-7+\sqrt{65}}{4}}$.

Thus, the given equation $\boldsymbol{k(x^2 + y^2) - 2(k - 1)x - 2(3k - 1)y + 12k - 1 = 0}$ does indicate a circle if $\boldsymbol{\frac{-7-\sqrt{65}}{4} < k < 0}$, or $\boldsymbol{0 < k < \frac{-7+\sqrt{65}}{4}}$.

Now, we have $\boldsymbol{k(x^2 + y^2) - 2(k - 1)x - 2(3k - 1)y + 12k - 1}$
$\boldsymbol{= k(x^2 + y^2 - 2x - 6y + 12) + (2x + 2y - 1) = 0}$.

So the equation can represent a group of circles each of which is a linear combination of the two curves as follows. $\boldsymbol{x^2 + y^2 - 2x - 6y + 12 = 0}$, and $\boldsymbol{2x + 2y - 1 = 0}$.

More specifically, it can represent two groups of circles.

One is for the case where $\boldsymbol{\frac{-7-\sqrt{65}}{4} < k < 0}$, and the other is for this case: $\boldsymbol{0 < k < \frac{-7+\sqrt{65}}{4}}$.

So next, let's check to see if the two curves meet each other.

Solving the system of $\boldsymbol{x^2 + y^2 - 2x - 6y + 12 = 0}$, and $\boldsymbol{2x + 2y - 1 = 0}$, we can get first,

$\boldsymbol{2x + 2y - 1 = 0 \Rightarrow y = -x + \frac{1}{2}}$. So next, we get

$$\boldsymbol{x^2 + y^2 - 2x - 6y + 12 = x^2 + (-x + \tfrac{1}{2})^2 - 2x - 6(-x + \tfrac{1}{2}) + 12 = 2x^2 + 3x + \tfrac{37}{4} = 0}.$$

Checking to see if there is any root, we can check to see if the discriminant ≥ 0.

The discriminant is $3^2 - 4\cdot2\cdot\frac{37}{4} = 9 - 74$, which is negative.

So there is no root, and thus, the two curves don't meet.

So there is no particular point every circle indicated by the given equation includes for every value allowed for $\boldsymbol{k}$.

What if the two curves did meet at particular points, though?

Then of course, all the circles in the groups stated above would meet altogether at the particular points. That is, each and every circle indicated by the given equation for every value allowed for $\boldsymbol{k}$ would pass through the particular points.