Examples C in Circles

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Constructing a curve in math, we can put it in a graph. Putting it in a graph though, we need the equation of it. So constructing it, we want to find the equation of it if we are not given the equation, of course. And finding the equation, it is said that we get it, too.

0. Find all the circles that have the same radii, and are tangent to a line $y = ax + b$ where a and b are constant.

1. Assuming a, b, c, and d are constant,

1.0. Construct a line tangent to a circle $(x - a)^2 + (y - b)^2 = c^2$.

1.1. Construct a line of slope d tangent to the circle above.

Suggestions or Solutions
To the Problem in the Example 0

Find all the circles that have the same radii, and are tangent to a line *y = ax + b* where *a* and *b* are constant.

Suppose ***A*** is the line ***y = ax + b***, ***B*** is a line ***y = ax + c***, where ***c*** is constant and ***c ≠ b***, ***r*** is the distance between the two lines ***A*** and ***B***, and ***C*** is the circle to be constructed.

Suppose also, that ***s*** and ***t*** are constant, and that a point **(*s, t*)** is in the line ***B***, and the center of a circle tangent to the line ***A***.

Then, we get **(*s, t*) = (*s, as + c*)**, which is the center of ***C***, and ***r*** is the radius of ***C***.

The distance from a point **(*u, v*)** to a line ***px + qy + r = 0*** is $\frac{|pu+qv+r|}{\sqrt{p^2+q^2}}$.

So the distance from **(*s, as + c*)** to the line ***A*** is $\frac{|as+(-1)(as+c)+b|}{\sqrt{a^2+(-1)^2}}=\frac{|b-c|}{\sqrt{a^2+1}}$.

Thus, we get

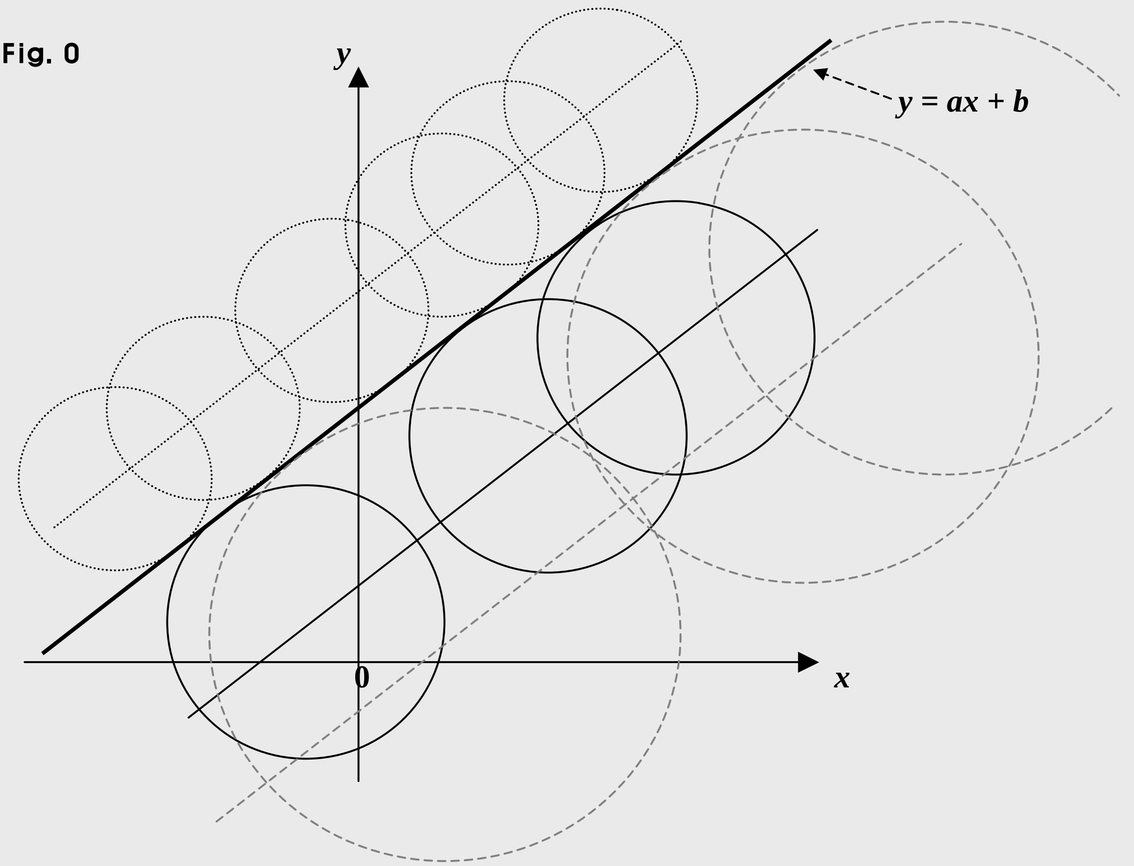
$$\frac{|b-c|}{\sqrt{a^2+1}}=r \Rightarrow \frac{(b-c)^2}{a^2+1}=r^2 \Rightarrow (b-c)^2=r^2(a^2+1) \Rightarrow b-c=\pm r\sqrt{a^2+1} \Rightarrow c=b\pm r\sqrt{a^2+1}.$$

So the circle ***C*** is $(x-s)^2+(y-as-b\pm r\sqrt{a^2+1})^2=r^2$, where ***a***, ***b***, ***r***, and ***s*** are constant.

If not quite sure of the idea behind the processes above, follow the steps below.

Finding circles, we need the centers and radii.
This problem, too, is saying that no matter what the radii may be, they all have to be the same.
The center in this problem has much to do with the radius and the line tangent to the circle.
So putting in a graph some circles of the same radii, along with a line tangent, we can see better the solution's whereabouts.

Fig. 0



Suppose now, that $\boldsymbol{A}$ is the line $\boldsymbol{y = ax + b}$.
Then, we can see that all the circles that have the same radii and are tangent to the line $\boldsymbol{A}$ have their centers in a line parallel to the line $\boldsymbol{A}$. That is, the line parallel to the line $\boldsymbol{A}$ is a particular distance away from the line $\boldsymbol{A}$, and the particular distance is the radius.
What then, is the line parallel to the line $\boldsymbol{A}$?

It represents all lines parallel to $\boldsymbol{A}$, and thus, is an arbitrary line parallel to the line $\boldsymbol{A}$.

So suppose that $\boldsymbol{B}$ is a line parallel to the line $\boldsymbol{A}$, $\boldsymbol{y = ax + b}$.

Then, the line $\boldsymbol{B}$ can be put in $\boldsymbol{y = ax + c}$, where $\boldsymbol{c}$ is a constant and $\boldsymbol{c \neq b}$.
And those circles that have their centers in the line $\boldsymbol{B}$ and are tangent to the line $\boldsymbol{A}$ are the solution to this problem.

So we want to find an equation that can represent all those circles, and we get to use the fact as follows.

• If we choose a point in the line ***B***, and the point is the center of a circle tangent to the line ***A***, the distance between the two lines ***A*** and ***B*** is the radius of the circle.

In fact, we can have two lines, each of which can be the line ***B***, because two lines can be the same distance away from the line ***A*** as shown in the figure below. So we can approach the solution the way as follows.

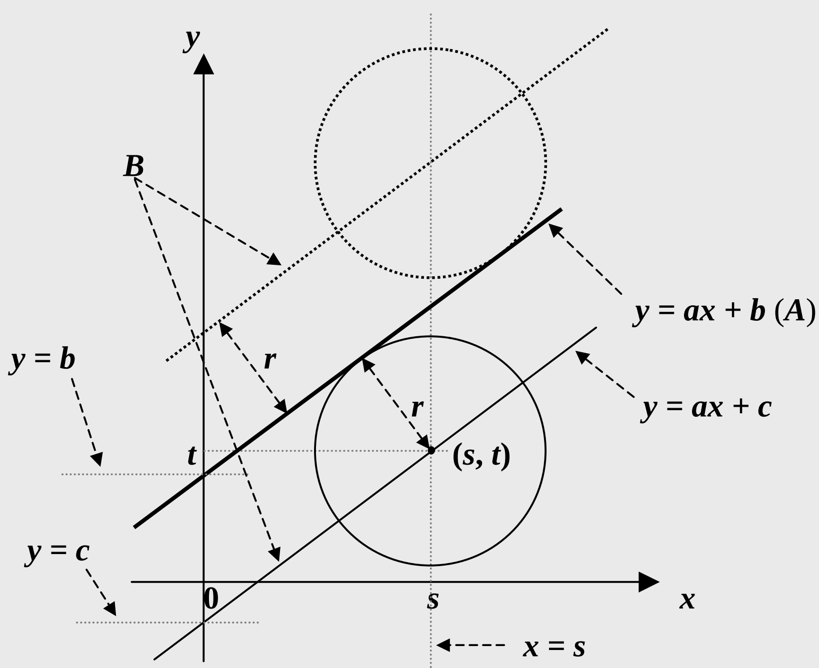
We set the value of the radius first. We can set it to ***r***, for instance, and give a value to ***r***.

Next, we specify the line ***A***. That is, we give values to ***a*** and ***b***.

Then, we will get two values of ***c***, for we can have two lines, each of which can be the line ***B*** since two lines can be the same distance away from the line ***A***. One is above the line ***A***, and the other is below the line ***A***.

Next, specifying only the ***x***-coordinate of a point in either line, we can get two centers, that is, two circles tangent to the line ***A***. One is above the line ***A***, and below is the other.

Fig. 1



Suppose now, r, s, and t are constant, and C is a circle of radius r, which is centered at a point (s, t) in the line B, and tangent to the line A. That is, C is the circle we want.

Then first, since (s, t) is in the line B, which is $y = ax + c$, we get $t = as + c$.

Thus, the center can be put this way: $(s, as + c)$.

Next, the distance between the lines A and B is the distance from the center $(s, as + c)$ to the line A. And we know the distance is the very radius r. So let's now, find the radius r.

Then, we want to get first, the distance above. We have a formula for such a distance.

• The distance from a point (u, v) to a line $px + qy + r = 0$ is $\frac{|pu+qv+r|}{\sqrt{p^2+q^2}}$.

The line is A, which is $y = ax + b$, which can be indicate by $ax - y + b = 0$, too.
The point is the center of the circle C, which is at $(s, as + c)$.

So the distance, that is, the radius is as follows. $\frac{|as+(-1)(as+c)+b|}{\sqrt{a^2+(-1)^2}} = \frac{|as-as-c+b|}{\sqrt{a^2+1}} = \frac{|b-c|}{\sqrt{a^2+1}}$.

Therefore, $(x - s)^2 + \{y - (as + c)\}^2 = \frac{(b-c)^2}{a^2+1}$ indicates the circle where the center is at $(s, t) = (s, as + c)$, and the radius is $\frac{|b-c|}{\sqrt{a^2+1}}$. And the circle is the circle C, of course.

Next, we know r is the radius of C, so we get $\frac{|b-c|}{\sqrt{a^2+1}} = r$, which is of course, the distance between the two lines A and B, too.

Let's now, get the line B, which is parallel to and r away from the line A.

Finding the line B, we want to find c since B is $y = ax + c$, and a is known since a is the slope of the line A, which is to be specified, and B is parallel to A.

Now, we have $\frac{|b-c|}{\sqrt{a^2+1}} = r$, where a, b, and r are known, since those constants are given.

So we get

$$\frac{|b-c|}{\sqrt{a^2+1}} = r \Rightarrow \frac{(b-c)^2}{a^2+1} = r^2 \Rightarrow (b-c)^2 = r^2(a^2+1) \Rightarrow b - c = \pm r\sqrt{a^2+1} \Rightarrow c = b \pm r\sqrt{a^2+1}.$$

Thus, two lines can be the line ***B***, and each of the two is a particular distance away from the line ***A***, and the particular distance is ***r***, of course.
One line is $\boldsymbol{y = ax + b + r\sqrt{a^2+1}}$, and the other is $\boldsymbol{y = ax + b - r\sqrt{a^2+1}}$.

Now, putting threads together, we have
$\boldsymbol{(x-s)^2 + \{y-(as+c)\}^2 = \frac{(b-c)^2}{a^2+1} = r^2}$, and $\boldsymbol{c = b \pm r\sqrt{a^2+1}}$.

So the circle ***C*** is $\boldsymbol{(x-s)^2 + (y - as - b \pm r\sqrt{a^2+1})^2 = r^2}$, where ***a***, ***b***, ***s***, and ***r*** are constant.

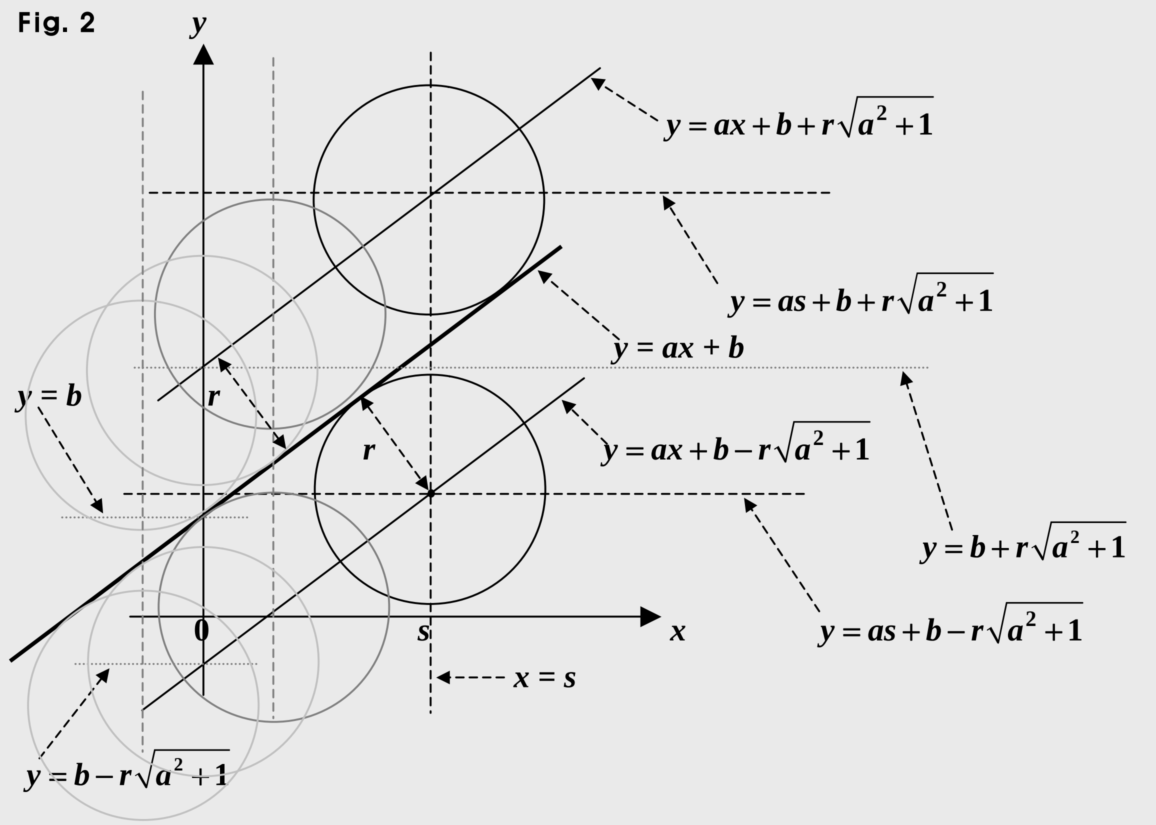


Fig. 2

Now, let's specify a specific group of circles that have the same radii, and are tangent to the line ***y* = *ax* + *b***.

The equation for those circles is $\boldsymbol{(x-s)^2 + (y - as - b \pm r\sqrt{a^2+1})^2 = r^2}$.

First, we specify the line $\boldsymbol{A}$ and the radius. That is, we give values to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{r}$.

Suppose for instance, the line $\boldsymbol{A}$ is $\boldsymbol{y = \frac{1}{2}x + 1}$, and the radius $\boldsymbol{r}$ is $\frac{\sqrt{5}}{2}$.

Then, we can get two lines parallel to the line $\boldsymbol{A}$, each of which is $\boldsymbol{r}$ away from the line $\boldsymbol{A}$.

Then, the equation for circles above can represent two circles for each value of $\boldsymbol{s}$, which is the $\boldsymbol{x}$-coordinate at the center, which is in the line $\boldsymbol{B}$.

One circle is $\boldsymbol{(x-s)^2 + (y - \frac{1}{2}s + \frac{1}{4})^2 = \frac{5}{4}}$, which is centered at $\boldsymbol{(s, \frac{1}{2}s - \frac{1}{4})}$.

The other circle is $\boldsymbol{(x-s)^2 + (y - \frac{1}{2}s - \frac{9}{4})^2 = \frac{5}{4}}$, which is centered at $\boldsymbol{(s, \frac{1}{2}s + \frac{9}{4})}$.

Thus, each of the two equations above represents a group of all circles where the centers are in one of two lines, which are parallel to and $\frac{\sqrt{5}}{2}$ away from the line $\boldsymbol{A}$, $\boldsymbol{y = \frac{1}{2}x + 1}$.

The two lines stated above can be the line $\boldsymbol{B}$.

One of the two lines is as follows. $\boldsymbol{y = ax + b + r\sqrt{a^2+1} \Rightarrow y = \frac{1}{2}x + \frac{9}{4}}$.

And the other line is as follows. $\boldsymbol{y = ax + b - r\sqrt{a^2+1} \Rightarrow y = \frac{1}{2}x - \frac{1}{4}}$.

Then, having only to specify the $\boldsymbol{x}$-coordinate at a point in either of the two lines, we get a circle tangent to the line $\boldsymbol{A}$.

That is, giving a value to $\boldsymbol{s}$ in $\boldsymbol{(x-s)^2 + (y - \frac{1}{2}s + \frac{1}{4})^2 = \frac{5}{4}}$ or in $\boldsymbol{(x-s)^2 + (y - \frac{1}{2}s - \frac{9}{4})^2 = \frac{5}{4}}$, we get such a circle, since $\boldsymbol{s}$ is the $\boldsymbol{x}$-coordinate of the point.

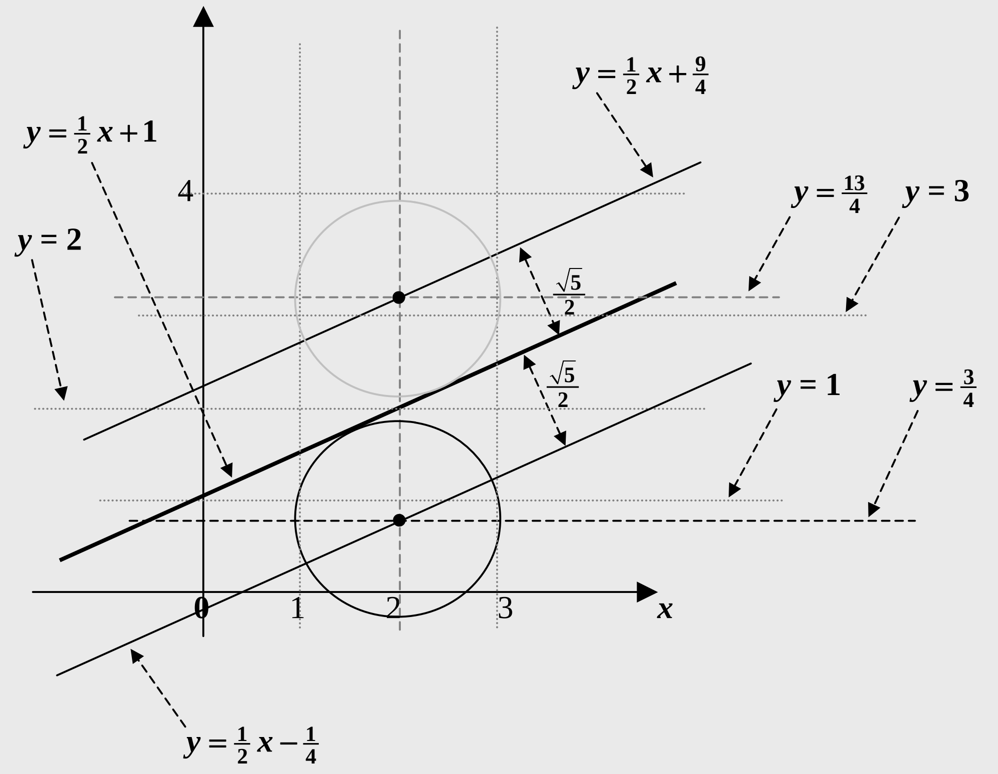
Suppose for instance, $\boldsymbol{s = 2}$. That is, the $\boldsymbol{x}$-coordinate of the point is 2.

Then, the circle we get is one of the two circles as follows.

One is $\boldsymbol{(x-2)^2 + (y - \frac{3}{4})^2 = \frac{5}{4}}$, and the other is $\boldsymbol{(x-2)^2 + (y - \frac{13}{4})^2 = \frac{5}{4}}$.

Note that for each value of $\boldsymbol{r}$, the equation above represents all the circles that are tangent to the given line $\boldsymbol{A}$, and have the same radii, which are the same as $\boldsymbol{r}$, of course.

Fig. 3



In short:

Suppose ***A*** is the line ***y = ax + b***, ***B*** is a line ***y = ax + c***, where ***c*** is constant and ***c ≠ b***, ***r*** is the distance between the two lines ***A*** and ***B***, and ***C*** is the circle to be constructed.

Suppose also, that ***s*** and ***t*** are constant, and that a point **(*s, t*)** is in the line ***B***, and the center of a circle tangent to the line ***A***.

Then, we get **(*s, t*) = (*s, as + c*)**, which is the center of ***C***, and ***r*** is the radius of ***C***.

The distance from a point **(*u, v*)** to a line ***px + qy + r* = 0** is $\frac{|pu+qv+r|}{\sqrt{p^2+q^2}}$.

So the distance from **(*s, as + c*)** to the line ***A*** is $\frac{|as+(-1)(as+c)+b|}{\sqrt{a^2+(-1)^2}}=\frac{|b-c|}{\sqrt{a^2+1}}$. Thus, we get

$$\frac{|b-c|}{\sqrt{a^2+1}}=r\Rightarrow\frac{(b-c)^2}{a^2+1}=r^2\Rightarrow(b-c)^2=r^2(a^2+1)\Rightarrow b-c=\pm r\sqrt{a^2+1}\Rightarrow c=b\pm r\sqrt{a^2+1}.$$

So the circle ***C*** is $(x-s)^2+(y-as-b\pm r\sqrt{a^2+1})^2=r^2$, where ***a***, ***b***, ***r***, and ***s*** are constant.

Suggestions or Solutions
To the Problem 0 in the Example 1

Construct a line tangent to a circle $(x - a)^2 + (y - b)^2 = c^2$, where a, b, and c are constant, and $c \neq 0$, of course.

Suppose $\boldsymbol{B}$ is a line tangent to the given circle at a point (s, t).

The center of the circle given is (a, b).

Suppose now, $\boldsymbol{A}$ is a line connecting the center (a, b) and the point (s, t).

Then, the line $\boldsymbol{B}$ includes the point (s, t), and is perpendicular to the line $\boldsymbol{A}$.

The slope the line $\boldsymbol{A}$ is $\frac{b-t}{a-s}$, and the product of the slopes of the lines $\boldsymbol{A}$ and $\boldsymbol{B}$ is -1.

So the slope of the line $\boldsymbol{B}$ is $-\frac{1}{m} = -\frac{1}{\frac{b-t}{a-s}} = -\frac{a-s}{b-t}$.

Therefore, the line $\boldsymbol{B}$ is $y - t = -\frac{a-s}{b-t}(x - s)$ since the line $\boldsymbol{B}$ includes (s, t).

Simplifying the equation above, we get

$$y - t = -\frac{a-s}{b-t}(x - s) \Rightarrow (b-t)(y-t) = -(a-s)(x-s) \Rightarrow (a-s)(x-s) + (b-t)(y-t) = 0.$$

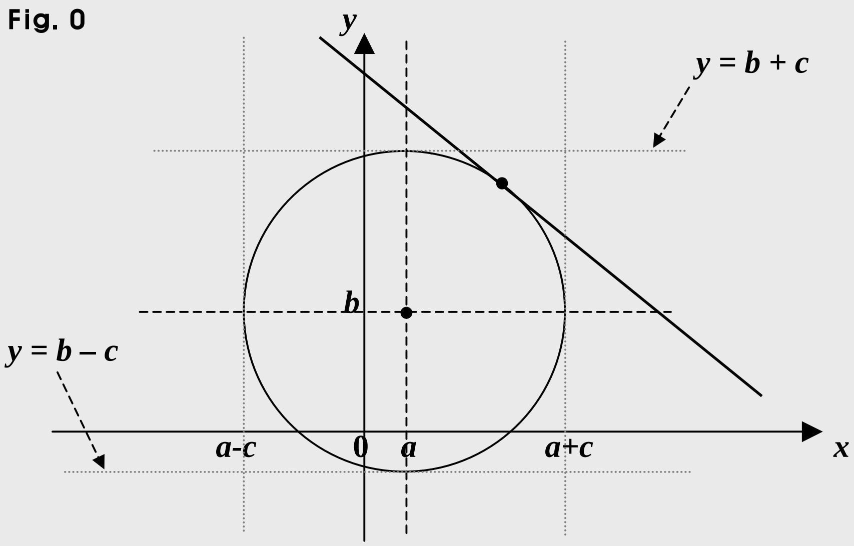
If not quite sure of the idea behind the processes above, follow the steps below.

Let's begin with putting in a graph the given circle and a line tangent to the circle.

It's a good idea to add to the graph a point indicating where the line is tangent to the circle. So we can see better what we can do about the problem.

Now, the center is at (a, b), and the radius is $|c|$. So assuming $c > 0$, putting in a graph the circle and a line, we can put them the way below.

Fig. 0



By the way, finding a line, what do we need to have?

Given a point in the line and the slope of the line, we can find the line.
Suppose for instance, a point (u, v) is in a line L in the x-y plane, and the slope is w.

Then, the line L is $y - v = w(x - u)$. (Why? Refer to **CONICS 1**.)

So we want to get a point and the slope in the line we are after in this problem.
What line are we after, though?

It is a line tangent to the circle $(x - a)^2 + (y - b)^2 = c^2$.
Where can a line be tangent to a circle, though?

A line can be tangent to a circle at a point, which is in the circle, of course.
So we may want to begin with a point where a line is tangent to a circle.

Suppose now, that

B is the line we are after,
C is the circle $(x - a)^2 + (y - b)^2 = c^2$, which is given in the problem, and
s and t are constant, and (s, t) is the point where the line B is tangent to the circle C.

Then, the tangent point **(*s, t*)** is in the tangent line ***B***, too. So next, what about the slope?

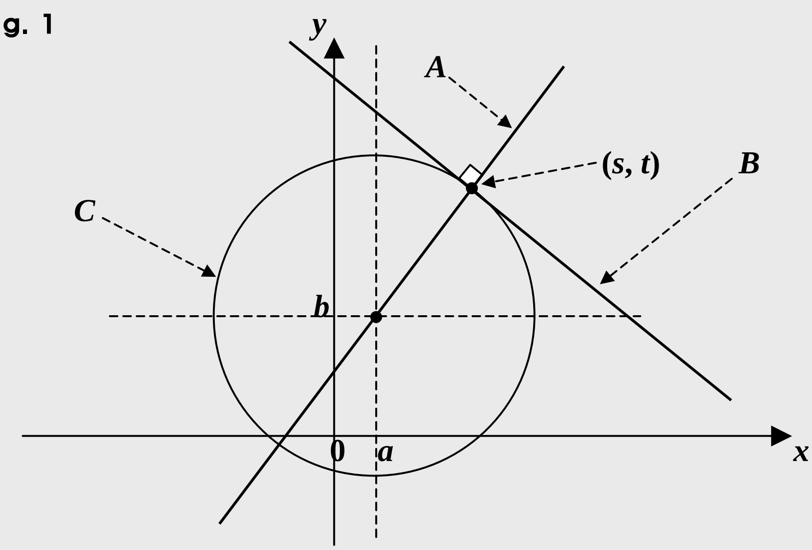
We can take advantage of a couple of facts as follows.

• A line tangent to a circle at a particular point is perpendicular to a line passing through the center of the circle and the particular point.

• Besides, the product of the slopes of two lines perpendicular to each other is -1.

Thus, finding the slope of the line connecting the center of the circle ***C*** and the tangent point **(*s, t*)**, we can immediately get the slope of the line ***B*** tangent to the circle ***C***.

Fig. 1

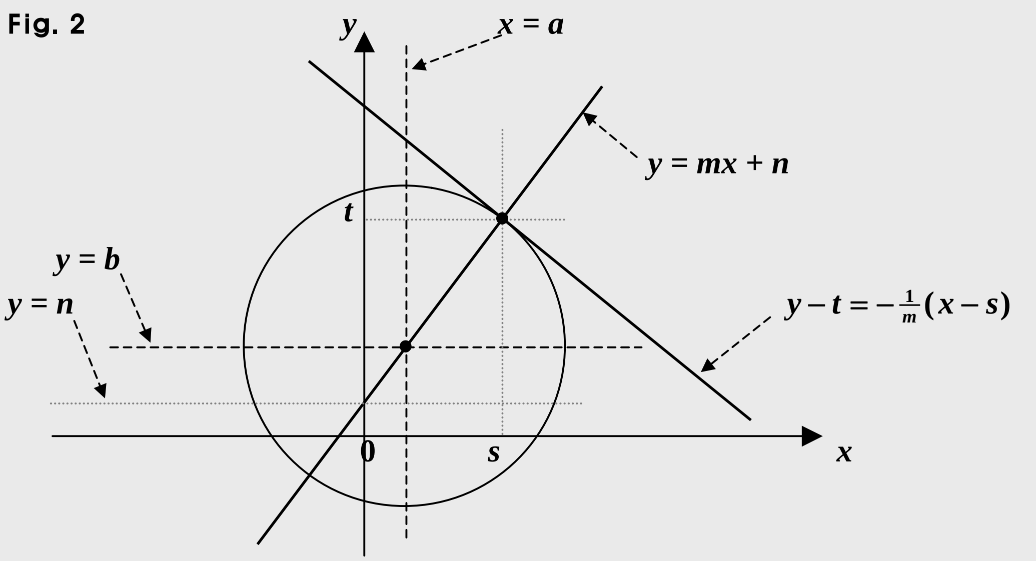


Suppose ***A*** is the line connecting the center **(*a, b*)** of the circle ***C*** and the point **(*s, t*)**.
Then, the line ***A*** is perpendicular to the line ***B*** tangent to the circle ***C*** at the point **(*s, t*)**.

Suppose next, ***m*** is the slope of the perpendicular line ***A***.
Then, we can put the line ***A*** in an equation $y = mx + n$, where ***m*** and ***n*** are constant.

We know that the product of slopes of two lines perpendicular to each other is –1.
So we can put for now, the line ***B*** in an equation $y - t = -\frac{1}{m}(x - s)$ since ***B*** includes **(*s, t*)**.

Fig. 2



The line $\boldsymbol{A}$ passes through $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{s}, \boldsymbol{t})$, so the slope $\boldsymbol{m} = \frac{b-t}{a-s}$.

Thus, the line $\boldsymbol{A}$ is as follows. $\boldsymbol{y - b = \frac{b-t}{a-s}(x - a)}$, where $\frac{b-t}{a-s}$ is the slope.

We know the slope of the line $\boldsymbol{B}$ is $-\frac{1}{m}$, so the slope of the line $\boldsymbol{B}$ is $-\frac{1}{\frac{b-t}{a-s}} = -\frac{a-s}{b-t}$.

Therefore, the line $\boldsymbol{B}$ is $\boldsymbol{y - t = -\frac{a-s}{b-t}(x - s)}$ since the line $\boldsymbol{B}$ passes through $(\boldsymbol{s}, \boldsymbol{t})$.

Now, we are going to simplify the equation of the line $\boldsymbol{B}$.

$$\boldsymbol{y - t = -\frac{a-s}{b-t}(x - s) \Rightarrow (b - t)(y - t) = -(a - s)(x - s) \Rightarrow (a - s)(x - s) + (b - t)(y - t) = 0}.$$

Therefore, the line $\boldsymbol{B}$ can also be indicated by $\boldsymbol{(a - s)(x - s) + (b - t)(y - t) = 0}$, which is, of course, the same as $\boldsymbol{y - t = -\frac{a-s}{b-t}(x - s)}$.

Thus, $\boldsymbol{(a - s)(x - s) + (b - t)(y - t) = 0}$ is the line tangent to a circle centered at $(\boldsymbol{a}, \boldsymbol{b})$ at a point $(\boldsymbol{s}, \boldsymbol{t})$. Such a tangent line seems to have nothing to do with the radius, though.

What matters most in a line is the slope, yet a point it passes through matters, too.
In fact, the point $(\boldsymbol{s}, \boldsymbol{t})$ the tangent line passes through pertains the radius.
What if we want to make use of $\boldsymbol{c}$, which is the radius, when we indicate the line $\boldsymbol{B}$?

Then, we can put $\boldsymbol{c}^2$ into the equation of the line $\boldsymbol{B}$.

We don't just put it into the equation of $\boldsymbol{B}$, of course.

The given circle $\boldsymbol{C}$ is $\mathbf{(}\boldsymbol{x} - \boldsymbol{a}\mathbf{)}^2 + \mathbf{(}\boldsymbol{y} - \boldsymbol{b}\mathbf{)}^2 = \boldsymbol{c}^2$, and the point $\mathbf{(}\boldsymbol{s, t}\mathbf{)}$ is in the circle $\boldsymbol{C}$.

So we get $\mathbf{(}\boldsymbol{s} - \boldsymbol{a}\mathbf{)}^2 + \mathbf{(}\boldsymbol{t} - \boldsymbol{b}\mathbf{)}^2 = \boldsymbol{c}^2$.

We can now put $(s - a)^2 + (t - b)^2$ into the equation $(s - a)(x - s) + (t - b)(y - t) = 0$.

In fact, we will come up with $(s - a)^2$ and $(t - b)^2$ out of $(s - a)(x - s) + (t - b)(y - t) = 0$. So we've got to do some algebra.

$(s - a)(x - s) + (t - b)(y - t)$
$= (s - a)(x - s + a - a) + (t - b)(y - t + b - b)$
$= (s - a)(x - a - s + a) + (t - b)(y - b - t + b)$
$= (s - a)\{(x - a) - (s - a)\} + (t - b)\{(y - b) - (t - b)\}$
$= (s - a)(x - a) - (s - a)^2 + (t - b)(y - b) - (t - b)^2 = 0$.

So we get $(s - a)(x - a) + (t - b)(y - b) = (s - a)^2 + (t - b)^2$, which is c^2.

Therefore, the equation of a line tangent at a point (s, t) to a circle of radius c centered at (a, b) is $(s - a)(x - a) + (t - b)(y - b) = c^2$.

The equation above is in fact, equivalent to the equation $(s - a)(x - s) + (t - b)(y - t) = 0$.

Now, what if the center of the circle is at the origin $(0, 0)$?

Then, $a = 0$ and $b = 0$ since the center $(a, b) = (0, 0)$.

So we need to have $a = 0$ and $b = 0$ in $(s - a)(x - a) + (t - b)(y - b) = c^2$.

Thus, we get $sx + ty = c^2$, which is the line tangent at a point (s, t) to a circle of radius c centered at the origin $(0, 0)$.
Of course, we can use the equation $(s - a)(x - s) + (t - b)(y - t) = 0$, instead.

If $a = 0$ and $b = 0$, we get $(s - a)(x - s) + (t - b)(y - t) = 0 \Rightarrow s(x - s) + t(y - t) = 0$
$\Rightarrow sx + ty = s^2 + t^2 = c^2 \Rightarrow sx + ty = c^2$. Why $s^2 + t^2 = c^2$, though?

The circle is $x^2 + y^2 = c^2$, and the circle passes through (s, t). So we get $s^2 + t^2 = c^2$.

Suggestions or Solutions
To the Problem 1 in the Example 1

Construct a line of slope of d tangent to a circle $(x - a)^2 + (y - b)^2 = c^2$, where a, b, c, and d are constant.

Suppose $\boldsymbol{C}$ is the circle given, and $\boldsymbol{A}$ is the line of slope $\boldsymbol{d}$ tangent to the circle $\boldsymbol{C}$ at a point $\boldsymbol{P(s, t)}$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ are constant. Then, the line $\boldsymbol{A}$ is $\boldsymbol{y - t = d(x - s)}$.

Suppose also, $\boldsymbol{B}$ is a line perpendicular to the line $\boldsymbol{A}$ and connecting the center $\boldsymbol{(a, b)}$ and the point $\boldsymbol{P}$. Then, $\boldsymbol{B}$ is a line of slope $-\frac{1}{d}$.

Besides, the line $\boldsymbol{B}$ has $\boldsymbol{(a, b)}$ and $\boldsymbol{(s, t)}$, so the slope of the line $\boldsymbol{B}$ can be put in $\frac{t-b}{s-a}$, too. Thus, we get $-\frac{1}{d} = \frac{t-b}{s-a}$.

Meanwhile, $\boldsymbol{(s, t)}$ is in the circle $\boldsymbol{C}$, too, so we get $\boldsymbol{(s - a)^2 + (t - b)^2 = c^2}$.

Thus, we get a system for $\boldsymbol{s}$ and $\boldsymbol{t}$, where $-\frac{1}{d} = \frac{t-b}{s-a}$, and $\boldsymbol{(s - a)^2 + (t - b)^2 = c^2}$.

Then, we get $-\frac{1}{d} = \frac{t-b}{s-a} \Rightarrow s - a = -d(t - b) \Rightarrow (s - a)^2 + (t - b)^2 = d^2(t - b)^2 + (t - b)^2 = c^2$

$\Rightarrow (t - b)^2(1 + d^2) = c^2 \Rightarrow (t - b)^2 = \frac{c^2}{1+d^2} \Rightarrow t - b = \pm\frac{c}{\sqrt{1+d^2}}$.

So we now, have an equivalent system where $\boldsymbol{s - a = -d(t - b)}$ and $t - b = \pm\frac{c}{\sqrt{1+d^2}}$.

Then first, we get $t - b = \frac{c}{\sqrt{1+d^2}} \Rightarrow s - a = -d(t - b) = -\frac{cd}{\sqrt{1+d^2}} \Rightarrow s - a = -\frac{cd}{\sqrt{1+d^2}}$.

So we get $s = a - \frac{cd}{\sqrt{1+d^2}}$, and $t = b + \frac{c}{\sqrt{1+d^2}}$.

Next, we get $t - b = -\frac{c}{\sqrt{1+d^2}} \Rightarrow s - a = -d(t - b) = \frac{cd}{\sqrt{1+d^2}} \Rightarrow s - a = \frac{cd}{\sqrt{1+d^2}}$.

So we get $s = a + \frac{cd}{\sqrt{1+d^2}}$, and $t = b - \frac{c}{\sqrt{1+d^2}}$.

Therefore, we get $(s,t) = (a - \frac{cd}{\sqrt{1+d^2}},\ b + \frac{c}{\sqrt{1+d^2}})$ or $(a + \frac{cd}{\sqrt{1+d^2}},\ b - \frac{c}{\sqrt{1+d^2}})$.

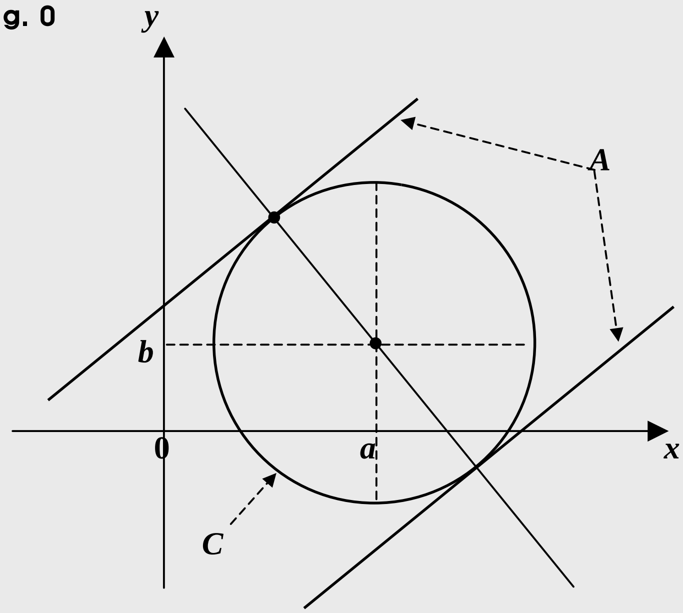
Now, the line $\boldsymbol{A}$ is initially put in $\boldsymbol{y - t = d(x - s)}$. Therefore, the line $\boldsymbol{A}$ is as follows.

$y - (b + \frac{c}{\sqrt{1+d^2}}) = d\{x - (a - \frac{cd}{\sqrt{1+d^2}})\}$ or $y - (b - \frac{c}{\sqrt{1+d^2}}) = d\{x - (a + \frac{cd}{\sqrt{1+d^2}})\}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose ***C*** is the circle given, and ***A*** is the line we want to construct.
Then, ***A*** is the line of slope ***d***, and is tangent to the circle ***C***.
We can have two lines though, that are tangent to a circle and have the same slope.
So putting in a graph the ideas above, we can see better how the idea goes.

Fig. 0



Now, we can see in the graph above, that two lines tangent to the circle ***C*** can have the same slope. So the two lines can be the line ***A***, and thus, can be the solution.

So let's find the two lines. Finding a line though, what do we need?

- We need two points in the line, or we need a point in the line, together with the slope.

We know the tangent line ***A*** has a slope of ***d***, so we need to get a point in the line ***A***.
The solution to this problem is in fact, about points where lines are tangent to a circle.
A line and a circle can be tangent to each other at a point.

So at two points respectively, the circle ***C*** is tangent to two lines, which have the same slope, which is ***d***. We may want to begin with therefore a point where a line is tangent to a circle. Getting the tangent point, since we know the slope, we can find the line tangent.

Suppose now, that the line ***A*** is tangent to the circle ***C*** at a point ***P*(*s*, *t*)**, where ***s*** and ***t*** are constant, of course.

Then, we want to find the point ***P***, which is one of the points in the circle ***C***, of course.

And finding the point ***P***, we find the coordinates of the point ***P***, which are ***s*** and ***t***.

So we can expect that we are going to get a system of two equations for ***s*** and ***t***.

To begin with, we can take advantage of a couple of facts as follows.

• A line tangent to a circle at a particular point is perpendicular to a line passing through the center of the circle and the particular point.

• Besides, the product of the slopes of two lines perpendicular to each other is -1.

Thus, we may want to begin with a line passing through the center and a tangent point.

So suppose now, that ***B*** is the line that is perpendicular to the tangent line ***A***, and passes through the center **(*a*, *b*)** and the point ***P*(*s*, *t*)**.

Then, using the two lines ***A*** and ***B***, we should be able to get a system of two equations for the two constants ***s*** and ***t***.

First, the slope of the line ***A*** is ***d***.

And next, the line ***A*** is tangent to the circle ***C*** at ***P*(*s*, *t*)**, and thus, has the point ***P*(*s*, *t*)**.

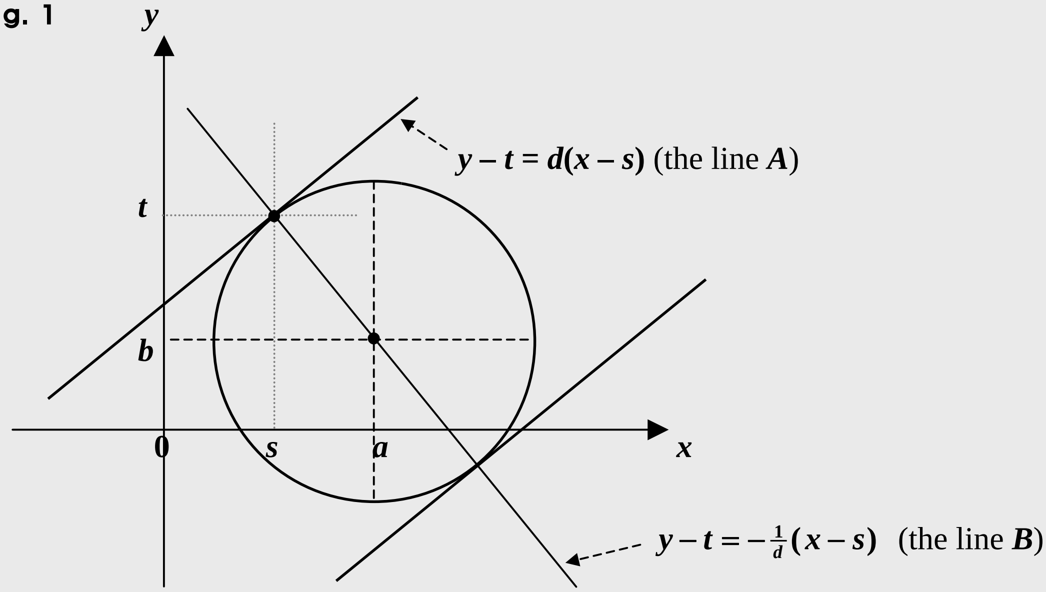
So the line ***A*** can be put in an equation as follows. $y - t = d(x - s)$.

Next, the line ***B*** is perpendicular to the line ***A***, which is a line of slope ***d***.

So ***B*** is a line of slope $-\frac{1}{d}$.

And the line ***B*** has the point (s, t). So the line ***B*** can be put, for now, in $y - t = -\frac{1}{d}(x - s)$.

Fig. 1



Now, the line ***B*** connects (s, t) and (a, b), so its slope can be put in $\frac{t-b}{s-a}$, too.

Therefore, we can see that $-\frac{1}{d} = \frac{t-b}{s-a}$.

Meanwhile, the circle ***C*** is $(x - a)^2 + (y - b)^2 = c^2$.

So since (s, t) is in the circle ***C***, we get $(s - a)^2 + (t - b)^2 = c^2$.

Thus, we get a system of two equations for ***s*** and ***t*** as follows.
$-\frac{1}{d} = \frac{t-b}{s-a}$, and $(s - a)^2 + (t - b)^2 = c^2$.

So we need to do some algebra.

To begin with, we get $-\frac{1}{d} = \frac{t-b}{s-a} \Rightarrow s - a = -d(t - b)$.

Thus, we get $(s - a)^2 + (t - b)^2 = c^2 \Rightarrow d^2(t - b)^2 + (t - b)^2 = c^2$

$\Rightarrow (t - b)^2(1 + d^2) = c^2 \Rightarrow (t - b)^2 = \frac{c^2}{1+d^2} \Rightarrow t - b = \pm\frac{c}{\sqrt{1+d^2}}$.

Therefore, we now have $s - a = -d(t - b)$ and $t - b = \pm\frac{c}{\sqrt{1+d^2}}$.

Then first, we get $t - b = \frac{c}{\sqrt{1+d^2}} \Rightarrow s - a = -d(t - b) = -\frac{cd}{\sqrt{1+d^2}} \Rightarrow s - a = -\frac{cd}{\sqrt{1+d^2}}$.

So we get $s = a - \frac{cd}{\sqrt{1+d^2}}$, and $t = b + \frac{c}{\sqrt{1+d^2}}$.

So next, we get $t - b = -\frac{c}{\sqrt{1+d^2}} \Rightarrow s - a = -d(t - b) = \frac{cd}{\sqrt{1+d^2}} \Rightarrow s - a = \frac{cd}{\sqrt{1+d^2}}$.

So we get $s = a + \frac{cd}{\sqrt{1+d^2}}$, and $t = b - \frac{c}{\sqrt{1+d^2}}$.

Therefore, we get $(s, t) = (a - \frac{cd}{\sqrt{1+d^2}},\ b + \frac{c}{\sqrt{1+d^2}})$ or $(a + \frac{cd}{\sqrt{1+d^2}},\ b - \frac{c}{\sqrt{1+d^2}})$.

We know that the line A is initially indicated by $y - t = d(x - s)$.

So the line A can be either of the two lines as follows.

$y - (b + \frac{c}{\sqrt{1+d^2}}) = d\{x - (a - \frac{cd}{\sqrt{1+d^2}})\}$ and $y - (b - \frac{c}{\sqrt{1+d^2}}) = d\{x - (a + \frac{cd}{\sqrt{1+d^2}})\}$,

each of which is a line of slope d tangent to a circle $(x - a)^2 + (y - b)^2 = c^2$.

We can put the line A this way, too: $y - (b \pm \frac{c}{\sqrt{1+d^2}}) = d\{x - (a \mp \frac{cd}{\sqrt{1+d^2}})\}$.