Examples D in Circles

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Constructing a curve in math, we can put it in a graph. Putting it in a graph though, we need the equation of it. So constructing it, we want to find the equation of it if we are not given the equation, of course. And finding the equation, it is said that we get it, too.

Suppose $\boldsymbol{k}$ is constant, $\boldsymbol{A}$ is $\boldsymbol{k(x + 1) - 2y + 4 = 0}$, $\boldsymbol{B}$ is $\boldsymbol{2x + k(y - 3) - 10 = 0}$, and $\boldsymbol{G}$ is the point where the curves of $\boldsymbol{A}$ and $\boldsymbol{B}$ meet each other.

Then, find the trace $\boldsymbol{G}$ makes as $\boldsymbol{k}$ varies in each of two cases as follows.

0. $\boldsymbol{k}$ is real.

1. $\boldsymbol{k \leq 1}$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming k is constant, A is $k(x + 1) - 2y + 4 = 0$, B is $2x + k(y - 3) - 10 = 0$, and G is the point where the curve of A meets that of B, find the trace G makes as k varies.

From ***A***, we can get $k(x + 1) - 2y + 4 = 0 \Rightarrow k(x + 1) + (4 - 2y) = 0$, which can be an equation of a line for each value of k.

Suppose a line $x + 1 = 0$ meets another line $4 - 2y = 0$ at (s, t).

Then, we get $s + 1 = 0$, and $4 - 2t = 0$, so we get $k(s + 1) + 4 - 2t = 0$.

So if the two lines above meet at a particular point, for any value of k, the equation given indicates a line that includes the particular point.

Finding the points, we get $x + 1 = 0 \Rightarrow x = -1$, and $4 - 2y = 0 \Rightarrow y = 2$.
Thus, the two lines meet at $(-1, 2)$.
And the two lines can be put this way too: $x = -1$ and $y = 2$.
So all the lines indicated by the equation ***A*** meet each other altogether at $(-1, 2)$.

Next, from ***B***, we can get $2x + k(y - 3) - 10 = 0 \Rightarrow k(y - 3) + 2x - 10 = 0$, which can be an equation of a line for each value of k.

Suppose a line $y - 3 = 0$ meets another line $2x - 10 = 0$ at (s, t).

Then, we get $t - 3 = 0$, and $2s - 10 = 0$, so we get $k(t - 3) + 2s - 10 = 0$.

So if the two lines above meet at a particular point, for any value of k, the equation given indicates a line that includes the particular point.

Finding the points, we get $y - 3 = 0 \Rightarrow y = 3$, and $2x - 10 = 0 \Rightarrow x = 5$.
Thus, the two lines meet at $(5, 3)$.
And the two lines can be put this way too: $x = 5$ and $y = 3$.
So all the lines indicated by the equation ***B*** meet each other altogether at $(5, 3)$.

For each value of $\boldsymbol{k}$, a couple of lines get determined, one of the two is from the equation $\boldsymbol{A}$, and the other is from the equation $\boldsymbol{B}$. We have

$\boldsymbol{A}$: $\mathbf{k(x + 1) - 2y + 4 = 0 \Rightarrow kx + k - 2y + 4 = 0 \Rightarrow y = \frac{k}{2}x + \frac{k+4}{2}}$.
$\boldsymbol{B}$: $\mathbf{2x + k(y - 3) - 10 = 0 \Rightarrow 2x + ky - 3k - 10 = 0 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k}}$.

So the product of the slopes in each pair of the lines is -1, and therefore, the two lines in each pair are perpendicular to each other.

Thus, all the points at each of which the lines meet each other in a pair, form a circle.

The line segment between the two points **(-1, 2)** and **(5, 3)** is a diameter of the circle.

The center of the circle is the midpoint between the two endpoints **(-1, 2)** and **(5, 3)**.

So the center is $\mathbf{(\frac{-1+5}{2}, \frac{2+3}{2}) = (2, \frac{5}{2})}$. Suppose now, $\boldsymbol{r}$ is the radius, and $\boldsymbol{d}$ is the diameter.

Then, we get $\mathbf{d^2 = (\Delta x)^2 + (\Delta y)^2 = (-1 - 5)^2 + (2 - 3)^2 = 36 + 1 = 37}$.

We have $\boldsymbol{r = \frac{d}{2}}$, too, so we get $\boldsymbol{r^2 = \frac{d^2}{4} = \frac{37}{4}}$, and the circle is $\mathbf{(x - 2)^2 + (y - \frac{5}{2})^2 = \frac{37}{4}}$.

However, because the equations $\boldsymbol{A}$ and $\boldsymbol{B}$ cannot indicate the two lines $\mathbf{x = -1}$ and $\mathbf{y = 3}$ for any value of $\boldsymbol{k}$, the point **(-1, 3)** has to be excluded.

Therefore, the trace is $\mathbf{(x - 2)^2 + (y - \frac{5}{2})^2 = \frac{37}{4}}$ where $\mathbf{x \neq -1}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, each of $\boldsymbol{A}$ and $\boldsymbol{B}$ indicates a group of lines, so the curves are lines. Why?

Since $\boldsymbol{k}$ is constant, we can choose one value for $\boldsymbol{k}$ at a time, so for each value of $\boldsymbol{k}$, $\boldsymbol{A}$ indicates a line, and so does $\boldsymbol{B}$. Thus, for each of $\boldsymbol{A}$ and $\boldsymbol{B}$, we can get as many lines as the number of choices we make for $\boldsymbol{k}$. So $\boldsymbol{A}$ and $\boldsymbol{B}$ each can be said to represent a group of lines.

Now, every time we choose a value for $\boldsymbol{k}$, we can determine a couple of lines using the two equations $\boldsymbol{A}$ and $\boldsymbol{B}$.
The two lines in each couple meet at a point if both are not parallel, of course.
Thus, we can determine as many points as the number of the choices for $\boldsymbol{k}$.   So what?

Those many points are in a curve, which is the trace we are after in this problem.
So every one of such many points is $\boldsymbol{G}$, which is therefore, an arbitrary point in the curve.

Thus, $\boldsymbol{G}$ can be put in $\boldsymbol{(s, t)}$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ are constant. Then, as $\boldsymbol{k}$ varies, the values of $\boldsymbol{s}$ and $\boldsymbol{t}$ change, so points keep getting made. Thus, a curve gets made of the points, and the curve can be called the trace of $\boldsymbol{G}$.   So let's now, find the trace that $\boldsymbol{G}$ makes.

To begin with, we have a fact as follows.

Suppose that $\boldsymbol{m}$ and $\boldsymbol{n}$ are constants, and that the curves of two equations $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$ meet each other at particular points.

Then, the curve of $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ meets the curves of $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$ at the particular points, too. That is, the three curves share the very particular points. In particular, the equation $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ is called a linear combination of the two equations $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$.   Note that in math, a curve can be a line, too.

Now, in the fact above, the two equations $\boldsymbol{f}$ and $\boldsymbol{g}$ can indicate a line each.
So for this problem, we can use the fact as follows.

Suppose a line $\boldsymbol{ax + by + c = 0}$ meets another line $\boldsymbol{dx + ey + f = 0}$ at a particular point, and $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant.

Then, another line $\boldsymbol{m(ax + by + c) + n(dx + ey + f) = 0}$ passes through the particular point for each pair of values of $\boldsymbol{m}$ and $\boldsymbol{n}$.

Let's now, start getting the trace.
We may want to first, check to see if each equation given in this problem can be such a linear combination as explained above.

So let's now, extract from each equation such curves as those of $\boldsymbol{f}$ and $\boldsymbol{g}$ explained above.

To begin with, for $\boldsymbol{A}$, we get $\mathbf{k(x + 1) - 2y + 4 = 0 \Rightarrow k(x + 1) + (4 - 2y) = 0}$.

Thus, the curves are two lines $\mathbf{x + 1 = 0}$ and $\mathbf{4 - 2y = 0}$; in other words, $\mathbf{x = -1}$ and $\mathbf{y = 2}$.

Next, for $\boldsymbol{B}$, we get $\mathbf{2x + k(y - 3) - 10 = 0 \Rightarrow k(y - 3) + 2x - 10 = 0}$.

So the curves are two lines $\mathbf{y - 3 = 0}$ and $\mathbf{2x - 10 = 0}$; that is, $\mathbf{y = 3}$ and $\mathbf{x = 5}$.

Therefore, we can see that

• All the lines the equation $\boldsymbol{A}$ represents meet each other altogether at a point **(-1, 2)**, because the two lines $\mathbf{x = -1}$ and $\mathbf{y = 2}$ meet each other at the point **(-1, 2)**.

• All the lines the equation $\boldsymbol{B}$ represents meet each other altogether at a point **(5, 3)**, because the two lines $\mathbf{x = 5}$ and $\mathbf{y = 3}$ meet each other at the point **(5, 3)**.

Next, let's consider every point where a pair of lines for each value of $\boldsymbol{k}$ meet each other.

For each value of $\boldsymbol{k}$, two lines get determined, one of them is from the equation $\boldsymbol{A}$, and the other is from the equation $\boldsymbol{B}$.
And we can notice that the two lines in each pair are perpendicular to each other.
Why?

For $\boldsymbol{A}$, we get $\mathbf{k(x + 1) - 2y + 4 = 0 \Rightarrow kx + k - 2y + 4 = 0 \Rightarrow y = \frac{k}{2}x + \frac{k+4}{2}}$.

For $\boldsymbol{B}$, we get $\mathbf{2x + k(y - 3) - 10 = 0 \Rightarrow 2x + ky - 3k - 10 = 0 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k}}$.

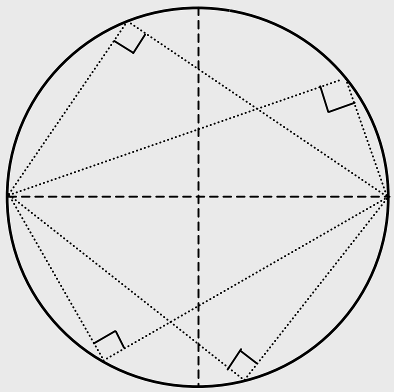
So the product of the slopes of the two lines $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\mathbf{-\frac{2}{k} \cdot \frac{k}{2} = -1}$, and therefore, the two lines in each pair for each value of $\boldsymbol{k}$ are perpendicular to each other.

Now, we can take advantage of the fact as follows.

• The arc angle for a half circle is 90^{o}.   What is the arc angle?

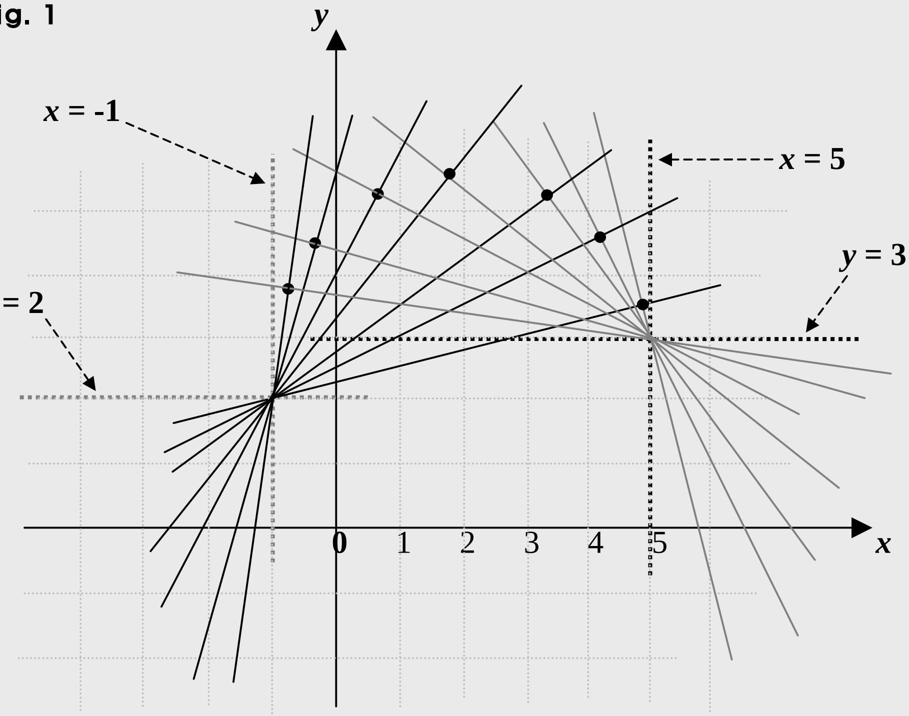
An arc angle is an angle made by two rays emitting from a point in a circle and limiting (or defining) an arc in the circle. So an arc angle is between 0^o and 180^o.
For instance, the arc angle for a half circle is 90^o.

Fig. 0



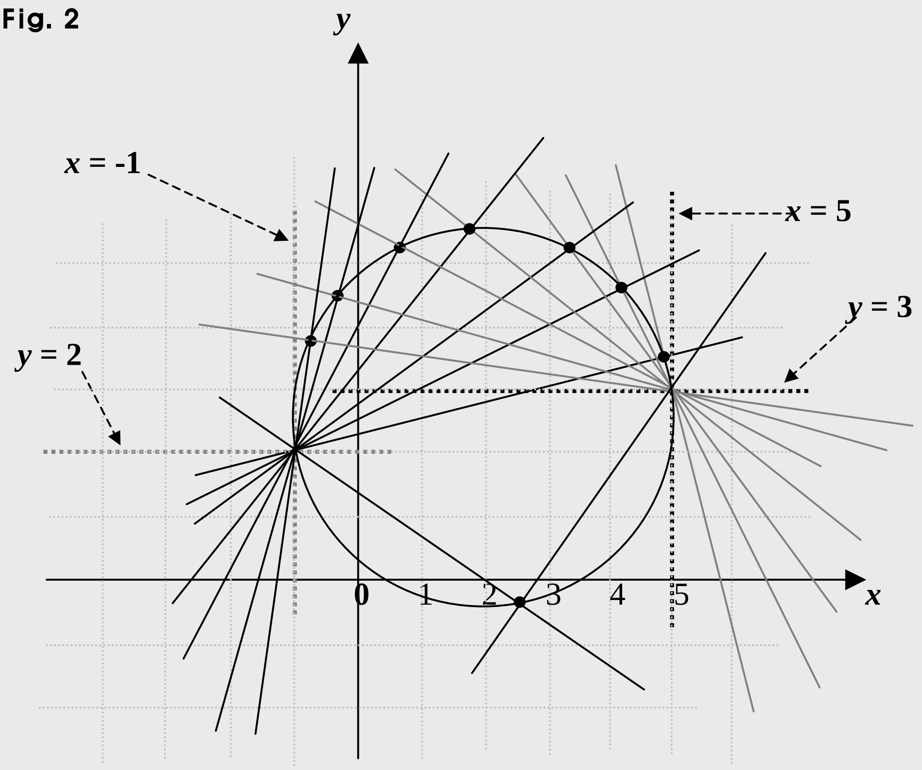
Let's now, put in a graph some pairs of the lines from the two equations.
The lines the equation ***A*** represents pass through the point **(-1, 2)**, and the lines the equation ***B*** represents pass through the point **(5, 3)**.

Fig. 1



Notice that all the points at each of which two lines in each pair meet each other, are in a circle. And the line segment between the two points **(-1, 2)** and **(5, 3)** is a diameter of the circle. That's because two lines meeting at each and every point above are perpendicular to each other. And the arc angle for a half circle is 90^o.

Fig. 2



Now, let's find the circle. Finding a circle, what do we need to have?

We can use the standard form, $\boldsymbol{(x - a)^2 + (y - b)^2 = c^2}$, where $\boldsymbol{(a, b)}$ is the center and $\boldsymbol{c}$ is the radius. So

1. Given the center and the radius, we can directly use the standard form above.

2. Given three points in the circle, we can get a system of three equations for $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. And putting each of the three points into the standard equation above, we can get an equation. So we can get three equations for the three unknowns, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Solving the system, we get the values of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

In this case, we can readily find the center and the radius.
The center is the midpoint between the two endpoints **(-1, 2)** and **(5, 3)** of the diameter.

So the center is $\left(\frac{-1+5}{2}, \frac{2+3}{2}\right) = \left(2, \frac{5}{2}\right)$. Suppose now, r is the radius, and d is the diameter.

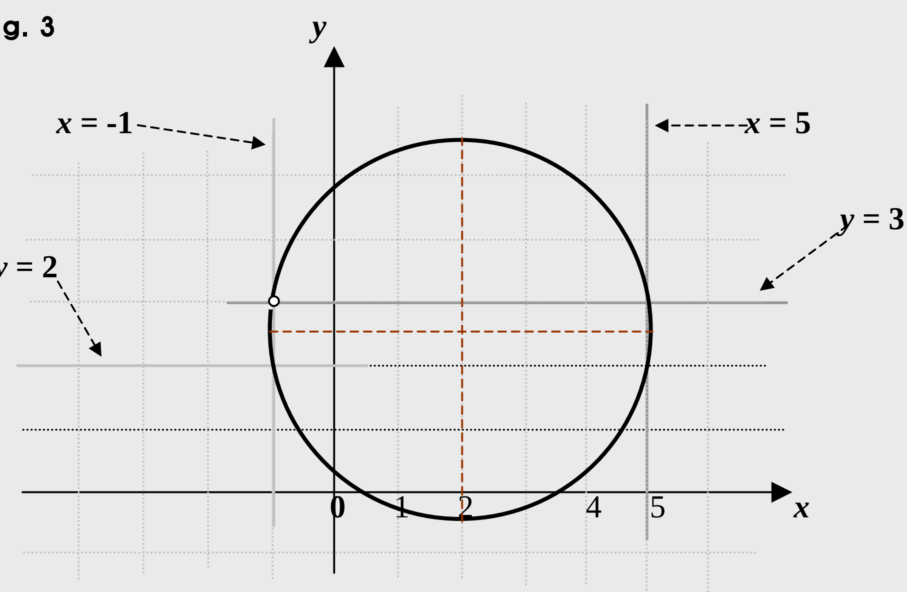
Then, we get $d^2 = (\Delta x)^2 + (\Delta y)^2 = (-1-5)^2 + (2-3)^2 = 36 + 1 = 37$.

We have $r = \frac{d}{2}$, too, so we get $r^2 = \frac{d^2}{4} = \frac{37}{4}$, and the circle is $(x-2)^2 + (y-\frac{5}{2})^2 = \frac{37}{4}$.

Is the entire circle above the trace we've been after, then?

It's not quite the case, and we may want to take a look at a graph below.

Fig. 3



We can see that a particular point in the circle has to be excluded from the graph above, and the particular point is **(-1, 3)**. That's because the two lines $x = -1$ and $y = 3$ cannot be indicated by any of the two equations A and B for any value of k. The two equations are as follows.

For A, we get $k(x+1) - 2y + 4 = 0 \Rightarrow k(x+1) + (4-2y) = 0$.

For B, we get $2x + k(y-3) - 10 = 0 \Rightarrow k(y-3) + 2x - 10 = 0$.

Therefore, the trace we want is $(x-2)^2 + (y-\frac{5}{2})^2 = \frac{37}{4}$ where $x \neq -1$.

Can the point **(-1, 3)** be in the circle, $(x-2)^2 + (y-\frac{5}{2})^2 = \frac{37}{4}$, though?

Yes, it can. Putting the point **(-1, 3)** into the equation of the circle, we get

$(-1-2)^2 + (3-\frac{5}{2})^2 = 9 + \frac{1}{4} = \frac{37}{4}$.

Suggestions or Solutions
To the Problem in the Example 1

Suppose A is $k(x + 1) - 2y + 4 = 0$, B is $2x + k(y - 3) - 10 = 0$, in each of which k is constant, $k \le 1$, and G is the point where the curve of A meets that of B.
Then, find the trace G makes as k varies.

We have $\boldsymbol{k \le 1}$, and also,

For $\boldsymbol{A}$, we get $\boldsymbol{k(x+1) - 2y + 4 = 0 \Rightarrow kx + k - 2y + 4 = 0 \Rightarrow y = \frac{k}{2}x + \frac{k+4}{2}}$.
For $\boldsymbol{B}$, we get $\boldsymbol{2x + k(y-3) - 10 = 0 \Rightarrow 2x + ky - 3k - 10 = 0 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k}}$.

So first, the slope of every line by $\boldsymbol{A}$ is $\frac{k}{2}$, and we get $\boldsymbol{k \le 1 \Rightarrow \frac{k}{2} \le \frac{1}{2}}$.

Therefore, as $\boldsymbol{k}$ gets smaller, the line gets steeper, and eventually, gets nearly vertical.

So all the lines by the equation $\boldsymbol{A}$ range from a line $\boldsymbol{y = \frac{1}{2}x + \frac{5}{2}}$ to a vertical line $\boldsymbol{x = -1}$, but the vertical line $\boldsymbol{x = -1}$ cannot be made because the equation $\boldsymbol{A}$ cannot indicate it.

Next, for each value of $\boldsymbol{k}$, the slope of a line by the equation $\boldsymbol{B}$ is $-\frac{2}{k}$.

Thus, we get $\boldsymbol{k \le 1 \Rightarrow -k \ge -1 \Rightarrow (-\frac{1}{k} \le -1}$ or $\boldsymbol{-\frac{1}{k} > 0) \Rightarrow (-\frac{2}{k} \le -2}$ or $\boldsymbol{-\frac{2}{k} > 0)}$.

So all the lines indicated by $\boldsymbol{B}$ range from a line $\boldsymbol{y = -2x + 13}$ to a vertical line $\boldsymbol{x = 5}$, and then, from the vertical line $\boldsymbol{x = 5}$ to a horizontal line $\boldsymbol{y = 3}$, but the line $\boldsymbol{y = 3}$ cannot be made since the equation $\boldsymbol{B}$ cannot indicate it.

Therefore, the trace is a part of the circle $\boldsymbol{(x-2)^2 + (y - \frac{5}{2})^2 = \frac{37}{4}}$.
However, the trace can be said to be composed of three curves.
One is the lower half of the circle, and the other two are two parts of the upper half.

$$\boldsymbol{(y - \tfrac{5}{2})^2 = \tfrac{37}{4} - (x-2)^2 \Rightarrow y - \tfrac{5}{2} = \pm\sqrt{\tfrac{37}{4} - (x-2)^2}}.$$

So the lower half of the circle is $\boldsymbol{y = \frac{5}{2} - \sqrt{\frac{37}{4} - (x-2)^2}}$, where $\boldsymbol{2 - \frac{\sqrt{37}}{2} \le x \le 2 + \frac{\sqrt{37}}{2}}$.

The other two parts are above the line $y = \frac{5}{2}$, which includes a diameter of the circle. One is on the left in the upper half, and the other is on the right.

All the lines indicated by A range from a line $y = \frac{1}{2}x + \frac{5}{2}$ to a vertical line $x = -1$, but the vertical line $x = -1$ cannot be made, so the part on the left in the upper half is as follows.

$$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}, \text{ where } 2 - \frac{\sqrt{37}}{2} < x < -1.$$

For the part on the right in the upper half circle, we need to find where a line meets the circle above.

Suppose L is the line.
Then the line L is $y = -2x + 13$, and the circle is $(x-2)^2 + (y - \frac{5}{2})^2 = \frac{37}{4}$.

So first, we get $y = -2x + 13 \Rightarrow y - \frac{5}{2} = -2x + 13 - \frac{5}{2} = -2x + \frac{21}{2}$.

Thus next, we get $(x-2)^2 + (y - \frac{5}{2})^2 = (x-2)^2 + (-2x + \frac{21}{2})^2 = \frac{37}{4}$

$$\Rightarrow x^2 - 4x + 4 + 4x^2 - 42x + \frac{441}{4} = 5x^2 - 46x + \frac{457}{4} = \frac{37}{4} \Rightarrow 5x^2 - 46x + \frac{420}{4} = 0$$

$$\Rightarrow 5x^2 - 46x + 105 = 0 \Rightarrow x^2 - \frac{46}{5}x + 21 = 0.$$

Then first, L meets the circle at $(3, 5)$, so one of the two roots for the equation above is 5.

Next, suppose t is the other root. Then, we get $5 + t = \frac{46}{5} \Rightarrow t = \frac{21}{5}$.

So the part on the right in the upper half circle is as follows.

$$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}, \text{ where } \frac{21}{5} \le x < 2 + \frac{\sqrt{37}}{2}.$$

Therefore, the trace is composed of the three curves as follows.

$y = \frac{5}{2} - \sqrt{\frac{37}{4} - (x-2)^2}$, where $2 - \frac{\sqrt{37}}{2} \le x \le 2 + \frac{\sqrt{37}}{2}$, which is the lower half circle.

$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}$, where $2 - \frac{\sqrt{37}}{2} < x < -1$, which is a part of the upper half circle.

$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}$, where $\frac{21}{5} \le x < 2 + \frac{\sqrt{37}}{2}$, which is a part of the upper half circle.

If not quite sure of the idea behind the processes above, follow the steps below.

In the example 0 above, we found the facts as follows.

For A, we get $\boldsymbol{k(x+1)-2y+4=0 \Rightarrow kx+k-2y+4=0 \Rightarrow y=\frac{k}{2}x+\frac{k+4}{2}}$.

For B, we get $\boldsymbol{2x+k(y-3)-10=0 \Rightarrow 2x+ky-3k-10=0 \Rightarrow y=-\frac{2}{k}x+\frac{3k+10}{k}}$.

Every line by the equation A is a linear combination of two lines $\boldsymbol{x=-1}$ and $\boldsymbol{y=2}$.
Every line by the equation B is a linear combination of two lines $\boldsymbol{x=5}$ and $\boldsymbol{y=3}$.

So all the lines the equation A represents meet each other altogether at **(-1, 2)**, and all the lines the equation B represents meet each other altogether at **(5, 3)**.

For each value of k, the product of slopes of the two lines by A and B is -1, and therefore, the two lines in each pair are perpendicular to each other.

All the points at each of which two lines in each pair meet each other, are in a circle.

The line segment between the two points **(-1, 2)** and **(5, 3)** is a diameter of the circle.

Now, let's start getting the trace in the case where $\boldsymbol{k \le 1}$.

To begin with, the slope of each line by A is $\frac{k}{2}$. And we have $\boldsymbol{k \le 1}$, so we get $\frac{k}{2} \le \frac{1}{2}$.

That is, for $\boldsymbol{k \le 1}$, slopes of all the lines by the equation A are less than or equal to $\frac{1}{2}$.
For instance, as k varies, we get

$\boldsymbol{k=1 \Rightarrow y=\frac{k}{2}x+\frac{k+4}{2}=\frac{1}{2}x+\frac{5}{2}}$.

$\boldsymbol{k=0 \Rightarrow y=\frac{0}{2}x+\frac{0+4}{2}=2 \Rightarrow y=2}$.

$\boldsymbol{k=-1 \Rightarrow y=-\frac{1}{2}x+\frac{-1+4}{2}=-\frac{1}{2}x+\frac{3}{2}}$.

$\boldsymbol{k=-100 \Rightarrow y=-\frac{100}{2}x+\frac{-100+4}{2}=-50x-48}$.

So we can see that as $\boldsymbol{k}$ gets smaller, the line gets steeper, and eventually, gets nearly vertical.

Thus, all the lines by the equation $\boldsymbol{A}$ range from a line $\boldsymbol{y = \frac{1}{2}x + \frac{5}{2}}$ to a vertical line $\boldsymbol{x = -1}$, but the vertical line $\boldsymbol{x = -1}$ cannot be made because the equation $\boldsymbol{A}$ cannot indicate it.

Next, for $\boldsymbol{B}$, we get $\boldsymbol{2x + k(y - 3) - 10 = 0 \Rightarrow 2x + ky - 3k - 10 = 0 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k}}$.

So for each value of $\boldsymbol{k}$, the slope of a line by the equation $\boldsymbol{B}$ is $\boldsymbol{-\frac{2}{k}}$.

Since $\boldsymbol{k \leq 1}$, we get $\boldsymbol{-k \geq -1 \Rightarrow (-\frac{1}{k} \leq -1}$ or $\boldsymbol{-\frac{1}{k} > 0) \Rightarrow (-\frac{2}{k} \leq -2}$ or $\boldsymbol{-\frac{2}{k} > 0)}$.

Therefore, if $\boldsymbol{k \leq 1}$, we get $\boldsymbol{-\frac{2}{k} \leq -2}$ or $\boldsymbol{-\frac{2}{k} > 0}$.

That is, for $\boldsymbol{k \leq 1}$, slopes of all the lines by the equation $\boldsymbol{B}$ are less than or equal to -2, or are greater than 0.

Now, $\boldsymbol{k}$ is the denominator in $\boldsymbol{-\frac{2}{k}}$, though.   Then, $\boldsymbol{k}$ cannot be 0 even if $\boldsymbol{k \leq 1}$, can it?

Yes, it can. Actually, the equation $\boldsymbol{B}$ is $\boldsymbol{2x + k(y - 3) - 10 = 0}$, and not $\boldsymbol{y = -\frac{2}{k}x + \frac{3k+10}{k}}$. So when $\boldsymbol{k = 0}$, we get a vertical line, $\boldsymbol{x = 5}$.

Let's see now, how the lines get generated by $\boldsymbol{B}$ as $\boldsymbol{k}$ varies.

$\boldsymbol{k = 1 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k} = -2x + 13}$.

$\boldsymbol{k = 0.1 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k} = -20x + \frac{0.3+10}{0.1} = -20x + 103}$.

$\boldsymbol{k = 0.01 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k} = -200x + \frac{0.03+10}{0.01} = -200x + 1003}$.

$\boldsymbol{k = 0 \Rightarrow 2x + k(y - 3) - 10 = 0 \Rightarrow x = 5}$.

$\boldsymbol{k = -1 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k} = 2x + \frac{-3+10}{-1} = 2x - 7}$.

$\boldsymbol{k = -2 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k} = x + \frac{-6+10}{-2} = x - 2}$.

$\boldsymbol{k = -100 \Rightarrow y = -\frac{2}{k}x + \frac{3k+10}{k} = -\frac{2}{-100}x + \frac{-300+10}{-100} = \frac{1}{50}x + 2.9}$.

Thus, we can see that

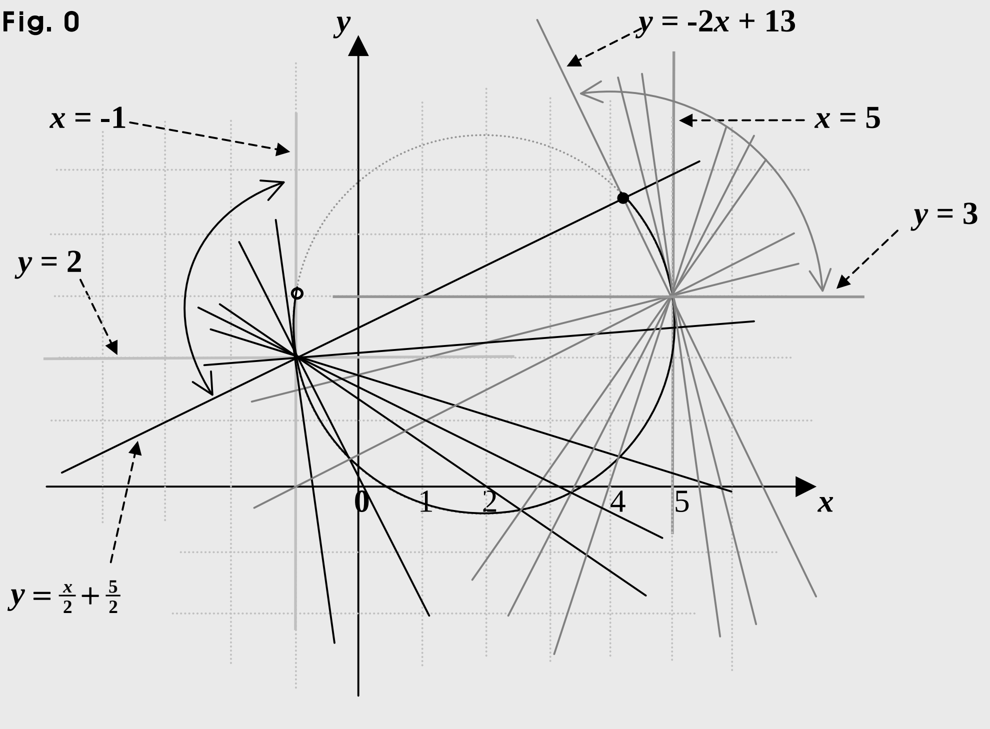
As ***k*** gets smaller staying positive, the line gets steeper (i.e., the slope is negative with bigger magnitude), and eventually, the line gets vertical.

And then, as ***k*** gets smaller being negative, the line gets less steep (i.e., the slope is positive with smaller magnitude), and eventually, the line gets almost horizontal.

Therefore, all the lines indicated by the equation ***B*** range from a line $\boldsymbol{y = -2x + 13}$ to a vertical line $\boldsymbol{x = 5}$, and then, range from the vertical line $\boldsymbol{x = 5}$ to a horizontal line $\boldsymbol{y = 3}$, but the horizontal line $\boldsymbol{y = 3}$ cannot be made since the equation ***B*** cannot indicate it.

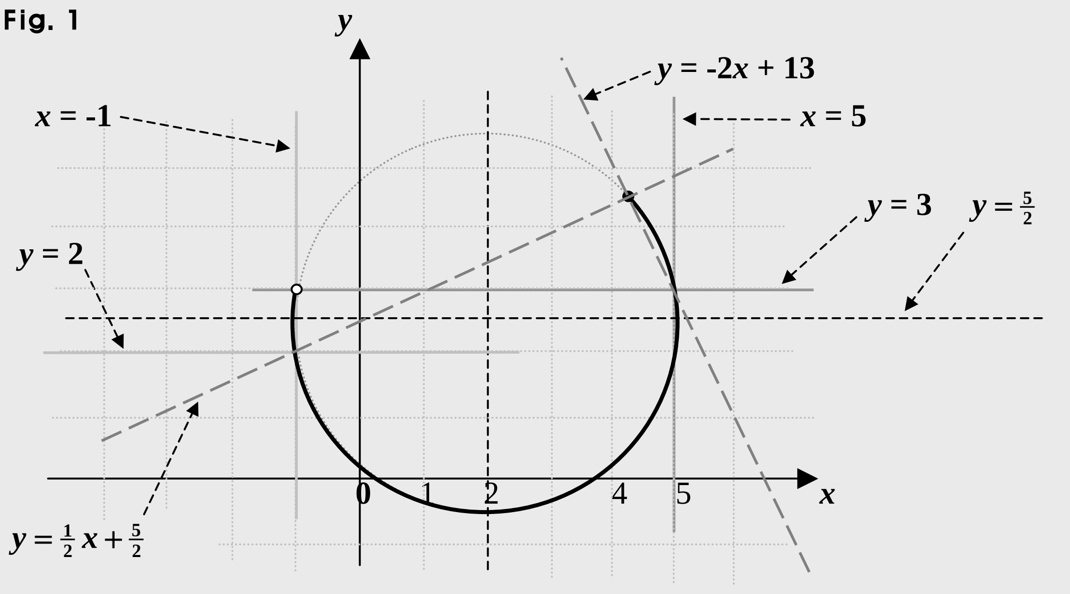
Let's now, put in a graph some of the lines by the two equations, and see what the trace looks like.

Fig. 0



Then, we can see that the trace is the part of the circle $(x-2)^2 + (y-\frac{5}{2})^2 = \frac{37}{4}$.
More specifically, the trace looks like the one in the figure below.

Fig. 1



The trace can be said to be composed of three curves.

One is the lower half of the circle, and the other two are two parts of the upper half.

The center of the circle is $\mathbf{(2, \frac{5}{2})}$, and the radius is $\frac{\sqrt{37}}{2}$.

Thus, the circle is $(x-2)^2+(y-\frac{5}{2})^2=\frac{37}{4}$, and we get

$$(y-\tfrac{5}{2})^2=\tfrac{37}{4}-(x-2)^2 \Rightarrow y-\tfrac{5}{2}=\pm\sqrt{\tfrac{37}{4}-(x-2)^2}.$$

So the lower half of the circle is $y=\frac{5}{2}-\sqrt{\frac{37}{4}-(x-2)^2}$, where $2-\frac{\sqrt{37}}{2}\le x\le 2+\frac{\sqrt{37}}{2}$.

Let's now, move on to the other two parts in the upper half of the circle.

Both parts are above the horizontal line $y=\frac{5}{2}$, which includes a diameter of the circle. One of the two is on the left in the upper half, and the other is on the right.

Now, all the lines by the equation $\boldsymbol{A}$ range from a line $y=\frac{1}{2}x+\frac{5}{2}$ to a vertical line x = **-1**, but the vertical line x = **-1** cannot be made, for the equation $\boldsymbol{A}$ cannot indicate it.

The equation of the upper half circle is $\boldsymbol{y = \frac{5}{2} + \sqrt{\frac{37}{4} + (x-2)^2}}$ for $\boldsymbol{2 - \frac{\sqrt{37}}{2} \le x \le 2 + \frac{\sqrt{37}}{2}}$.

So the part on the left is as follows. $\boldsymbol{y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}}$, where $\boldsymbol{2 - \frac{\sqrt{37}}{2} < x < -1}$.

Next, for the other part, that is, the part on the right in the upper half circle, we need to know where a line and the circle meet each other.

Suppose $\boldsymbol{L}$ is the line.
Then, the line $\boldsymbol{L}$ is $\boldsymbol{y = -2x + 13}$, and of course, the circle is $\boldsymbol{(x-2)^2 + (y - \frac{5}{2})^2 = \frac{37}{4}}$.

So first, we get $\boldsymbol{y = -2x + 13 \Rightarrow y - \frac{5}{2} = -2x + 13 - \frac{5}{2} = -2x + \frac{21}{2}}$.

Thus next, we get $\boldsymbol{(x-2)^2 + (y - \frac{5}{2})^2 = (x-2)^2 + (-2x + \frac{21}{2})^2 = \frac{37}{4}}$

$$\boldsymbol{\Rightarrow x^2 - 4x + 4 + 4x^2 - 42x + \frac{441}{4} = 5x^2 - 46x + \frac{457}{4} = \frac{37}{4} \Rightarrow 5x^2 - 46x + \frac{420}{4} = 0}$$

$$\boldsymbol{\Rightarrow 5x^2 - 46x + 105 = 0 \Rightarrow x^2 - \frac{46}{5}x + 21 = 0}.$$

Now, being tangent to the circle, the line $\boldsymbol{L}$ meets the circle at one point, so the final quadratic equation above has a double root, and thus, can be put in a complete square.

On the other hand, if the line $\boldsymbol{L}$ meets the circle at two points, the final equation has two different roots.

Now, the line $\boldsymbol{L}$ is indicated by the equation $\boldsymbol{B}$ for $\boldsymbol{k = 1}$, and we know all the lines the equation $\boldsymbol{B}$ represents meet each other altogether at **(5, 3)**, which is in the circle. So the line $\boldsymbol{L}$ meets the circle at the point **(5, 3)**, too.

So if the final quadratic equation above has two different roots, we can immediately see that one of the two is 5. What about the other root, then?

We can readily find the other root using the fact as follows.

- The sum of the two roots of the equation $\boldsymbol{ax^2 + bx + c = 0}$ is $-\frac{b}{a}$.

Suppose the final equation has two different roots, and $\boldsymbol{t}$ is the other root.

Then, we get $\frac{46}{5} = 5 + t \Rightarrow t = \frac{21}{5}$, which is not 5, and therefore, is the other root.

Meanwhile, we get $x^2 - \frac{46}{5}x = x^2 - \frac{46}{5}x + (\frac{23}{5})^2 - (\frac{23}{5})^2 = (x - \frac{23}{5})^2 - (\frac{23}{5})^2$.

So we get $x^2 - \frac{46}{5}x + 21 = (x - \frac{23}{5})^2 - (\frac{23}{5})^2 + 21 = (x - \frac{23}{5})^2 - \frac{529}{25} + \frac{525}{25}$

$= (x - \frac{23}{5})^2 - \frac{4}{25} = 0 \Rightarrow (x - \frac{23}{5})^2 = \frac{4}{25} \Rightarrow x - \frac{23}{5} = \pm\frac{2}{5} \Rightarrow x = \frac{23}{5} \pm \frac{2}{5} \Rightarrow x = 5$ or $\frac{21}{5}$.

So the part on the right in the upper half circle is as follows.

$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}$, where $\frac{21}{5} \le x < 2 + \frac{\sqrt{37}}{2}$.

Therefore, the trace is composed of the three curves as follows.

$y = \frac{5}{2} - \sqrt{\frac{37}{4} - (x-2)^2}$, where $2 - \frac{\sqrt{37}}{2} \le x \le 2 + \frac{\sqrt{37}}{2}$, which is the lower half circle.

$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}$, where $2 - \frac{\sqrt{37}}{2} < x < -1$, which is a part of the upper half circle.

$y = \frac{5}{2} + \sqrt{\frac{37}{4} - (x-2)^2}$, where $\frac{21}{5} \le x < 2 + \frac{\sqrt{37}}{2}$, which is a part of the upper half circle.