Examples E in Circles

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Constructing a curve in math, we can put it in a graph. Putting it in a graph though, we need the equation of it. So constructing it, we want to find the equation of it if we are not given the equation, of course. And finding the equation, it is said that we get it, too.

Suppose ***A*** and ***B*** are two points in the ***x-y*** plane, and ***G*** is an arbitrary point in a curve ***C*** in the plane. Suppose also, that the distance between ***G*** and ***A*** is twice the distance between ***G*** and ***B***. Then,

0. Find the equation of the curve ***C***.

1. Find the maximum magnitude of the slope of a line passing through a point in the curve ***C*** and the point ***A***.

Suggestions or Solutions
To the Problem in the Example 0

Suppose *A* and *B* are two points in the *x*-*y* plane, and *G* is an arbitrary point in a curve *C* in the plane. Suppose also, that the distance between *G* and *A* is twice the distance between *G* and *B*. Then, find the equation of the curve *C*.

Suppose $\boldsymbol{a}$ is a nonzero constant, $\boldsymbol{A}$ is at $\mathbf{(-2}\boldsymbol{a}\mathbf{, 0)}$, $\boldsymbol{B}$ is at $(\boldsymbol{a}, \mathbf{0})$, and $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y})$ is the arbitrary point in the curve $\boldsymbol{C}$.
Then, since we have $\frac{\overline{AG}}{\overline{BG}} = 2$, by means of the distance formula between points, we get

$$\sqrt{\{x-(-2a)\}^2+(y-0)^2} = 2\sqrt{(x-a)^2+(y-0)^2}.$$

So we get $\sqrt{(x+2a)^2+y^2} = 2\sqrt{(x-a)^2+y^2} \Rightarrow (x+2a)^2+y^2 = 4\{(x-a)^2+y^2\}$

$$\Rightarrow 4\{(x-a)^2+y^2\}-(x+2a)^2-y^2 = 4(x^2-2ax+a^2+y^2)-x^2-4ax-4a^2-y^2$$

$$= 3x^2-12ax+3y^2 = 0 \Rightarrow x^2-4ax+y^2 = 0$$

$$\Rightarrow x^2-4ax+4a^2-4a^2+y^2 = (x-2a)^2+y^2-4a^2 = 0 \Rightarrow (x-2a)^2+y^2 = 4a^2.$$

Therefore, the equation of $\boldsymbol{C}$ is $(x-2a)^2+y^2 = 4a^2$, where $\boldsymbol{a}$ is a nonzero constant.

If not quite sure of the idea behind the processes above, follow the steps below.

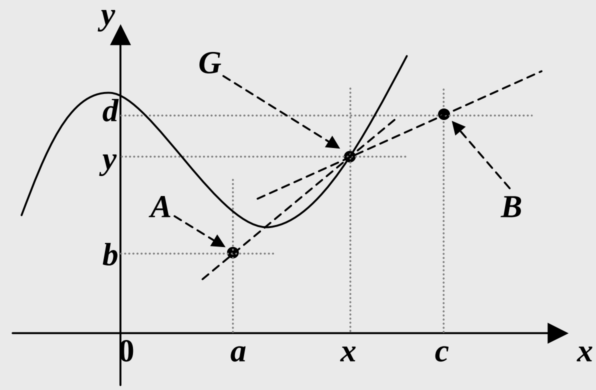
To begin with, we know $\boldsymbol{A}$ and $\boldsymbol{B}$ are just two points in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, and $\boldsymbol{G}$ is an arbitrary point in a curve $\boldsymbol{C}$ in the plane.

So assuming $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are constant, we can say $\boldsymbol{A}$ is $(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{B}$ is $(\boldsymbol{c}, \boldsymbol{d})$, and $\boldsymbol{G}$ is $(\boldsymbol{x}, \boldsymbol{y})$.

So next, coming up with a connective equation between $\boldsymbol{x}$ and $\boldsymbol{y}$, we get the equation of the curve $\boldsymbol{C}$. How then, can we get the connective equation?

Let's first, put in a graph all the three points and some curve.

Fig. 0



We have the distance ratio, $\frac{\overline{AG}}{\overline{BG}} = 2$. In other words, $\overline{AG} : \overline{BG} = 2:1$.

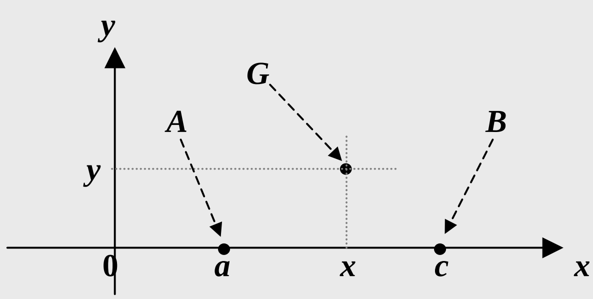
So using the distance formula, we can set up a relation among the three points in terms of the distances between them. However, putting the points in such a way as above, we get to end up with an expression $\sqrt{(x-a)^2+(y-b)^2} = 2\sqrt{(x-c)^2+(y-d)^2}$, which is not only highly unlikely to give us much help but quite complicated, too, due to too many constants involved.

It is important to note that the two points ***A*** and ***B*** are *just any two* points in a plane.

Thus, we are free to choose the positions of the two points ***A*** and ***B*** as long as ***A*** and ***B*** keep their positions (i.e., their relative positions are fixed). So let's try another graph.

Putting ***A*** and ***B*** on the ***x***-axis, we could get a graph as below.

Fig. 1



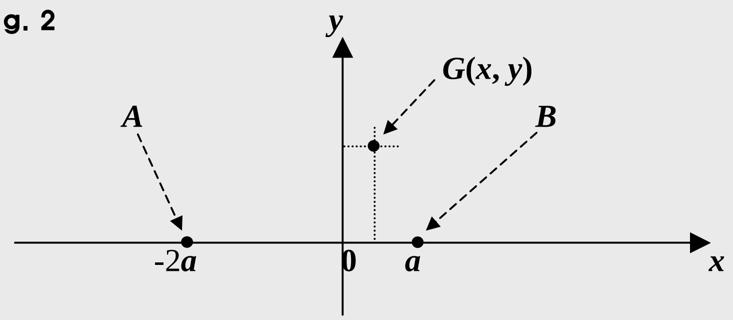
Now, the point ***A*** is $(a, 0)$, the point ***B*** is $(c, 0)$, and the arbitrary point ***G*** is (x, y).

Then, setting up the relation among the points in terms of the distances, we can get

$$\sqrt{(x-a)^2+(y-0)^2}=2\sqrt{(x-c)^2+(y-0)^2}\Rightarrow\sqrt{(x-a)^2+y^2}=2\sqrt{(x-c)^2+y^2}.$$

It looks much simpler than the previous one, which however, is still not quite likely to give us much help either. So let’s try another graph.

Fig. 2



This time, the point A is $(-2a, 0)$, the point B is $(a, 0)$, and the arbitrary point G is (x, y).

The ratio is $\frac{\overline{AG}}{\overline{BG}}=2$, which can give us $\sqrt{\{x-(-2a)\}^2+(y-0)^2}=2\sqrt{(x-a)^2+(y-0)^2}$.

Thus, we get

$$\sqrt{(x+2a)^2+y^2}=2\sqrt{(x-a)^2+y^2}\Rightarrow(x+2a)^2+y^2=4\{(x-a)^2+y^2\}$$

$$\Rightarrow 4\{(x-a)^2+y^2\}-(x+2a)^2-y^2=4(x^2-2ax+a^2+y^2)-x^2-4ax-4a^2-y^2$$

$$=3x^2-12ax+3y^2=0\Rightarrow x^2-4ax+y^2=0$$

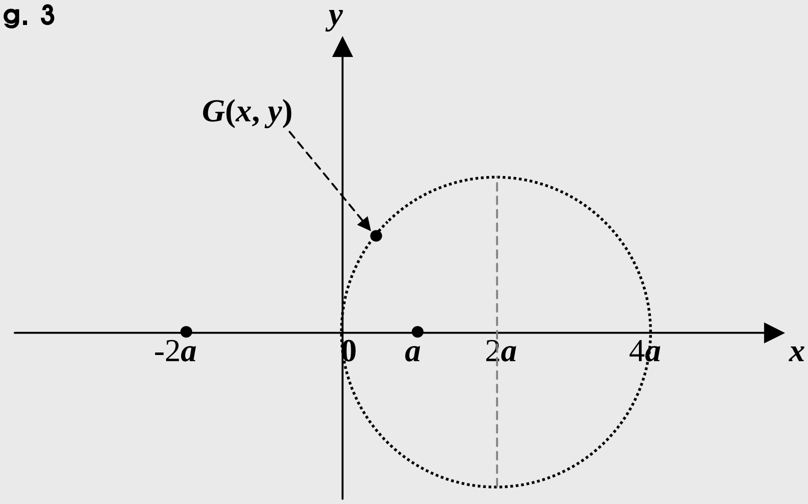
$$\Rightarrow x^2-4ax+4a^2-4a^2+y^2=(x-2a)^2+y^2-4a^2=0\Rightarrow(x-2a)^2+y^2=4a^2,$$

which is the connective expression between the coordinates of the point G.

The connective expression indicates a circle of radius of $2a$ centered at $(2a, 0)$.

Therefore, the curve C is $(x-2a)^2+y^2=4a^2$, where a is a nonzero constant.

Fig. 3



Such a circle as above is called Apollonius Circle.

Note:
If a problem says simply a point in a plane, we can put the point anywhere in the plane as long as we maintain its relative location to the other objects in the problem, and the location of the point should be a location where we can make the problem simpler. So we may want to begin such a problem with trying some several locations of such a point.

In short:

Suppose $\boldsymbol{a}$ is a nonzero constant, $\boldsymbol{A}$ is at $\mathbf{(-2}\boldsymbol{a}\mathbf{, 0)}$, $\boldsymbol{B}$ is at $(\boldsymbol{a}, \mathbf{0})$, and $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y})$ is the arbitrary point in the curve $\boldsymbol{C}$.
Then, since we have $\frac{\overline{AG}}{\overline{BG}} = 2$, by means of the distance formula between points, we get

$$\sqrt{\{x-(-2a)\}^2+(y-0)^2} = 2\sqrt{(x-a)^2+(y-0)^2}.$$

So we get $\sqrt{(x+2a)^2+y^2} = 2\sqrt{(x-a)^2+y^2} \Rightarrow (x+2a)^2+y^2 = 4\{(x-a)^2+y^2\}$

$$\Rightarrow 4\{(x-a)^2+y^2\}-(x+2a)^2-y^2 = 4(x^2-2ax+a^2+y^2)-x^2-4ax-4a^2-y^2$$
$$= 3x^2-12ax+3y^2 = 0 \Rightarrow x^2-4ax+y^2 = 0$$
$$\Rightarrow x^2-4ax+4a^2-4a^2+y^2 = (x-2a)^2+y^2-4a^2 = 0 \Rightarrow (x-2a)^2+y^2 = 4a^2.$$

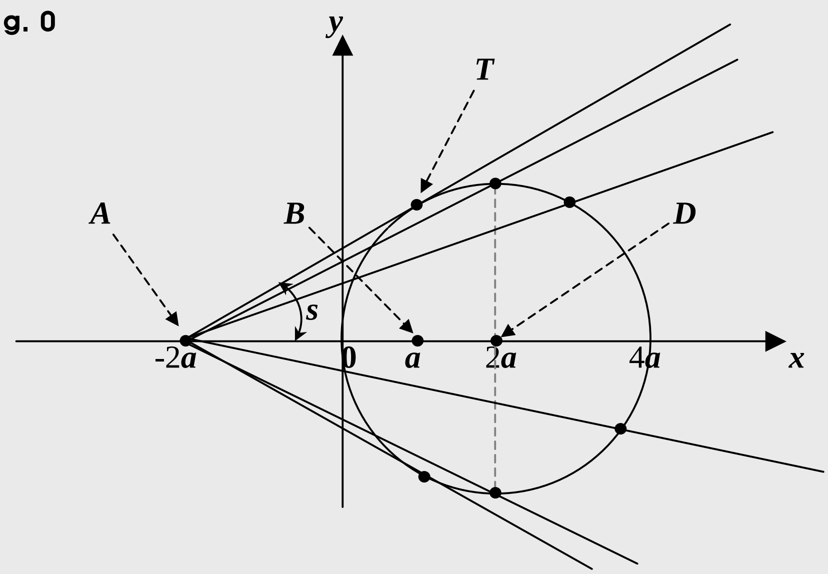
Therefore, the equation of $\boldsymbol{C}$ is $(x-2a)^2+y^2 = 4a^2$, where $\boldsymbol{a}$ is a nonzero constant.

Suggestions or Solutions
To the Problem in the Example 1

Suppose *A* and *B* are two points in the *x-y* plane, and *G* is an arbitrary point in a curve *C* in the plane. Suppose also, that the distance between *G* and *A* is twice the distance between *G* and *B*. Then, find the maximum magnitude of the slope of a line passing through a point in the curve *C* and the point *A*.

Putting some rays emitting from the point ***A*** and passing through the circle ***C***, we get

Fig. 0



Then, when a ray emitting from the point ***A*** is tangent to the circle, the ray makes the largest angle ***s*** against the ***x***-axis, and thus, has the maximum slope.

Suppose the center of ***C*** is ***D***, and a ray from ***A*** is tangent to the circle ***C*** at a point ***T***. Then, ***D*** is **(2*a*, 0)**, and $\overline{TD}$ is perpendicular to $\overline{TA}$, for $\overline{TA}$ is a part of the ray tangent to ***C***, and $\overline{TD}$ is a part of the line passing through the center and the tangent point ***T***.

We have $\overline{TD} = |2a|$, and $\overline{AD} = |2a-(-2a)| = |4a|$.

So from the distance formula, we get

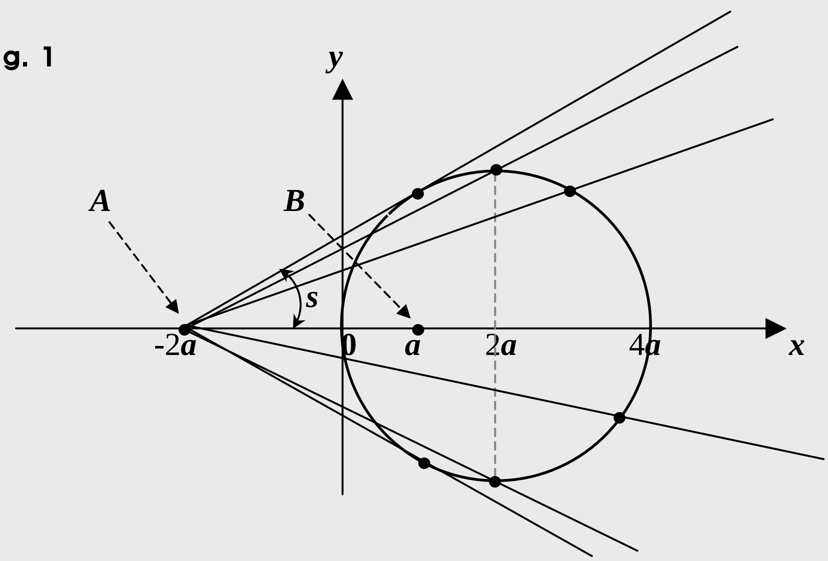
$$(\overline{AD})^2 = (\overline{TA})^2 + (\overline{TD})^2 \Rightarrow 16a^2 = (\overline{TA})^2 + 4a^2 \Rightarrow (\overline{TA})^2 = 12a^2 \Rightarrow \overline{TA} = |2a\sqrt{3}|.$$

Therefore, the maximum magnitude of the slope is $\frac{\overline{TD}}{\overline{TA}} = \frac{|2a|}{|2a\sqrt{3}|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

If not quite sure of the idea behind the processes above, follow the steps below.

In the previous problem, we've found that the curve ***C*** is a circle. So to begin with, let's put in a graph the circle ***C*** and the two points ***A*** and ***B***. Then, putting some rays emitting from the point ***A*** and passing through the circle ***C***, we can get a graph as below.

Fig. 1



Then, we can see that when a ray emitting from the point ***A*** is tangent to the circle ***C***, the ray makes the largest angle against the ***x***-axis, and therefore, has the maximum slope.

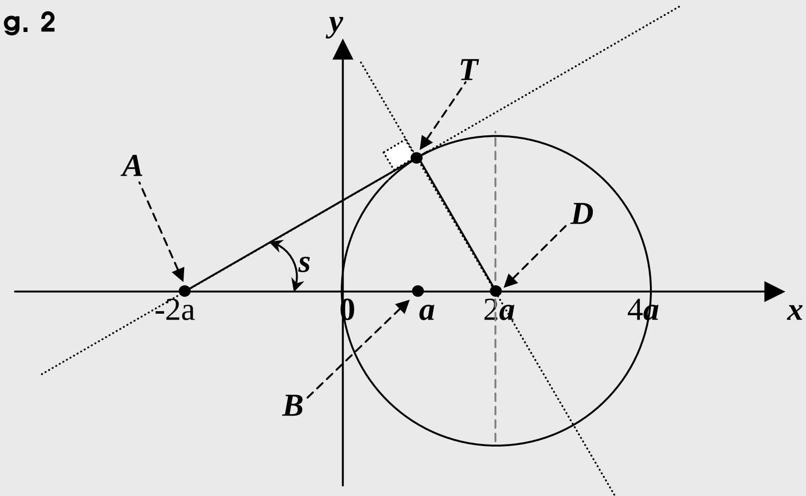
Also, we can see a right triangle in the graph above because of the fact as follows.

• Suppose ***L*** is a line tangent to a circle at a point ***P***. Then, the line ***L*** is perpendicular to the line passing through the center of the circle and the tangent point ***P***.

Suppose now, that the center of the circle ***C*** is ***D***, and that a ray emitting from the point ***A*** is tangent to the circle ***C*** at a point ***T*** as shown in the figure below.

Then, the center ***D*** is **(2*a*, 0)**, and a line segment $\overline{TD}$ is perpendicular to a line segment $\overline{TA}$, because the segment $\overline{TA}$ is a part of the ray tangent to the circle ***C***, and the segment $\overline{TD}$ is a part of the line passing through the center and the tangent point ***T***.

Fig. 2

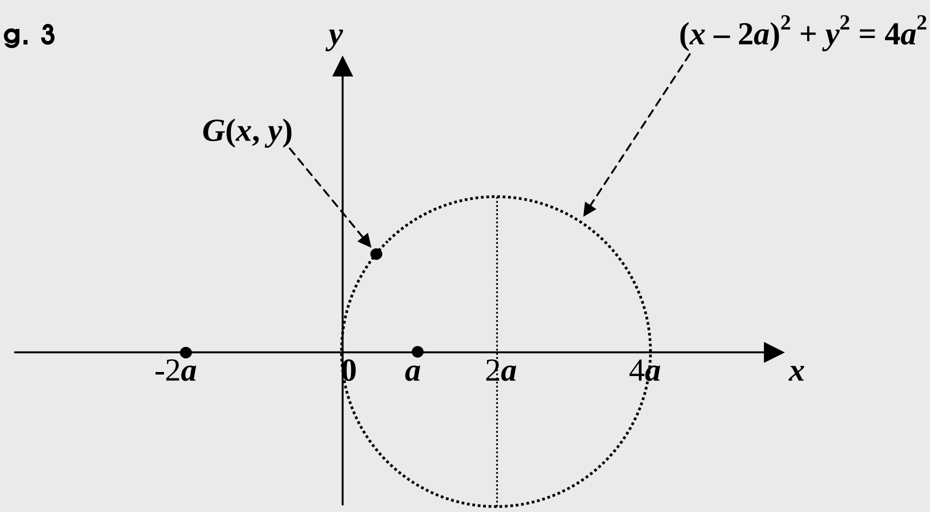


Therefore, we can see that the maximum slope is the slope of the line segment $\overline{TA}$, so we want to find the ratio $\frac{\overline{TD}}{\overline{TA}}$, which is the magnitude of the slope of a line, which includes the line segment $\overline{TA}$. To begin with, the line segment $\overline{TD}$ is the radius of the circle. So we have $\overline{TD} = |2a|$. Why not just $\overline{TD} = 2a$, though?

That's because the line segment $\overline{TD}$ is a length, which cannot be negative, but a is just nonzero, and thus, can be negative.

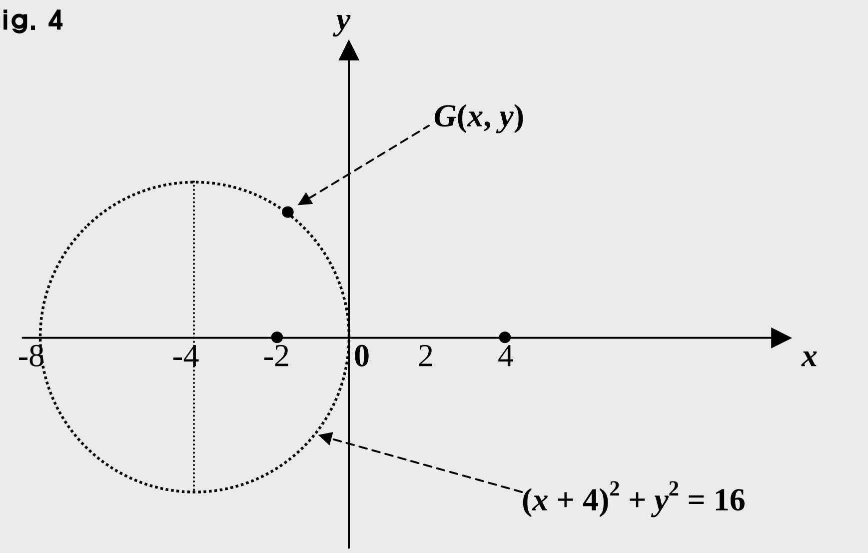
Now, in the example 0, we found the Apollonius Circle as shown below.

Fig. 3



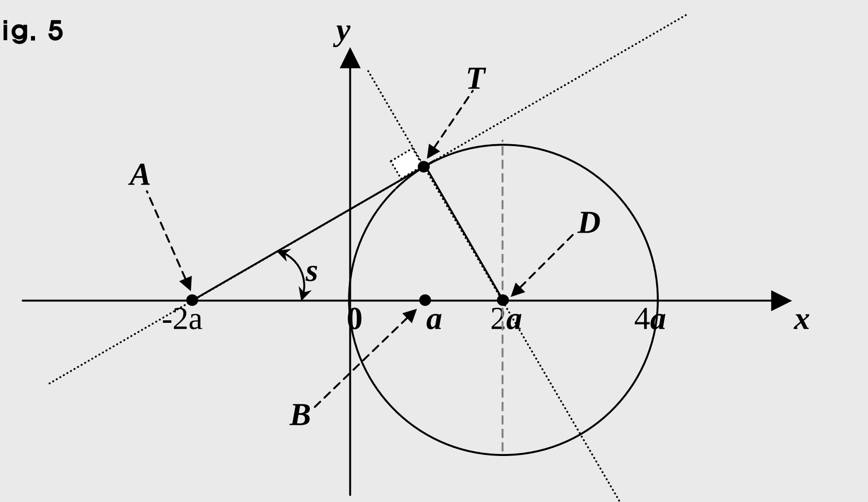
For instance, we can set $\boldsymbol{a}$ = -2. Then, we get another Apollonius Circle as shown below.

Fig. 4



Now, getting back to the point where we left off, we get

Fig. 5



We can see that from the graph above, the line segment $\overline{AD} = \ |2a-(-2a)|\ = \ |4a|$.

Next, let's find the length of the segment $\overline{TA}$. Then, the distance formula (often called Pythagorean Theorem) comes in to play its part. Thus, we get

$$(\overline{AD})^2 = (\overline{TA})^2 + (\overline{TD})^2 \Rightarrow 16a^2 = (\overline{TA})^2 + 4a^2 \Rightarrow (\overline{TA})^2 = 12a^2 \Rightarrow \overline{TA} = \ |2a\sqrt{3}|.$$

So we get $\frac{\overline{TD}}{\overline{TA}} = \frac{|2a|}{|2a\sqrt{3}|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, which is the maximum magnitude of the slope.

It seems that the point ***T*** is right above the point **(*a*, 0)** when the slope is the maximum.

In fact, it is the case, because we get $\boldsymbol{y = a\sqrt{3}}$ when $\boldsymbol{x = a}$ in the equation of the circle above, and the equation is $\boldsymbol{(x - 2a)^2 + y^2 = 4a^2}$. Thus, we get

$(\overline{AT})^2 = 12a^2 = (\overline{AB})^2 + (\overline{TB})^2 = 9a^2 + 3a^2$, so the triangle ***ABT*** is a right triangle.

Of course, if familiar with trigonometry, you can see at once, that the angle ***s*** is 30^{o} because the triangle ***ATD*** is a right triangle and $\frac{\overline{TD}}{\overline{AD}} = \frac{|2a|}{|4a|} = \frac{1}{2}$, which is $\mathbf{sin\ 30^o}$.