

Examples 5 in Ellipses

Copyright © 2018 by Seong Ryeol Kim. All rights reserved

Click this to start.

Examples 5 in Ellipses

In each example below, the ellipse is in the x - y plane.

0. Find the ellipse that passes through a point $(1, \frac{2\sqrt{10}}{3})$, and has foci at $(0, 0)$ and $(4, 0)$.
1. Find the ellipse in which the four endpoints of the two main axes are $(1, 3)$, $(3, 4)$, $(5, 3)$, and $(3, 2)$.
2. Find the foci of the ellipse that passes through $(-1, 1)$, $(-2, -1)$, $(-3, 1)$, and $(-2, 3)$.
3. Find the ellipse of which the eccentricity is $3/5$, and a directrix is a line $x = 28/3$, and is corresponding to a focus at $(4, 2)$.
4. Suppose C is a point in a line segment \overline{AB} , and $\overline{AC} / \overline{CB} = 1/2$. Then, assuming the end point A is moving along the x -axis, and at the same time, the end point B is moving along the y -axis, find the curve that the point C makes.
5. Find the maximum area of a rectangle inscribed in an ellipse $x^2 + 9y^2 = 9$.

Suggestions or Solutions
To the Problem in the Example 0

Find the ellipse that passes through a point $(1, \frac{2\sqrt{10}}{3})$, and has foci at $(0, 0)$ and $(4, 0)$.

To begin with, the ellipse is horizontal, and the center is $(2, 0)$.

So next, assuming the ellipse is $\frac{(x-2)^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $a > b > 0$.

Next, assuming s is the sum of two distances from a point in the ellipse to the foci, we get $s = 2a$. And also, since the ellipse has the given point, we can get s the way below.

$$s = \sqrt{(1-0)^2 + (\frac{2\sqrt{10}}{3}-0)^2} + \sqrt{(1-4)^2 + (\frac{2\sqrt{10}}{3}-0)^2} = \sqrt{1 + \frac{40}{9}} + \sqrt{9 + \frac{40}{9}} = \frac{7}{3} + \frac{11}{3} = 6.$$

So we get $a = 3$.

Next, assuming c is the focal distance, we get $c = 2$, and $c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2 \Rightarrow b^2 = 9 - 4 = 5 \Rightarrow b = \sqrt{5}$.

So the ellipse is $\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$, often put this way, too: $\frac{(x-2)^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

First, the standard equation of an ellipse centered at (u, v) is $\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1$.

And if $a > b > 0$, the ellipse is horizontal, a is the major radius, and b is the minor radius. If however, $b > a > 0$, it is vertical, b is the major, and a is the minor.

And if horizontal, the center, foci, and vertices share the same y-coordinate. So since the foci given have the same y-coordinates, the ellipse we want to find is horizontal.

Next, the center is the midpoint between the foci. So the center is $\{(0+4)/2, (0+0)/2\}$, and thus, is $(2, 0)$.

Next, in an ellipse, the sum of two distances from a point to the two foci is constant, and is in fact, the length of the major axis.

So assuming s is the sum of the two distances, we get $s = 2a$.

And since the given point $(1, \frac{2\sqrt{10}}{3})$ is in the ellipse, the sum of the two distances from the point given to the two foci is $2a$, too. So finding the sum s , we get

$$s = \sqrt{(1-0)^2 + (\frac{2\sqrt{10}}{3}-0)^2} + \sqrt{(1-4)^2 + (\frac{2\sqrt{10}}{3}-0)^2} = \sqrt{1 + \frac{40}{9}} + \sqrt{9 + \frac{40}{9}} = \frac{7}{3} + \frac{11}{3} = 6.$$

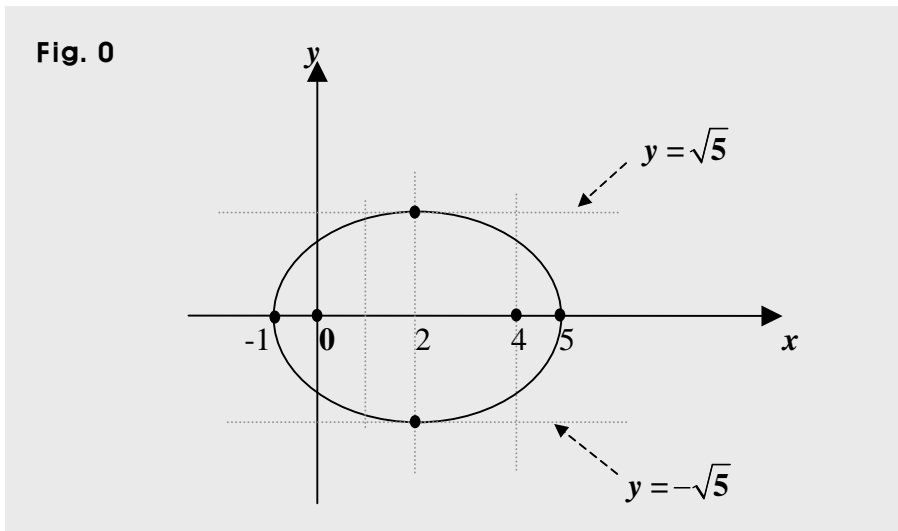
So we get $a = 3$. What then, about b ?

We have a connective equation $c^2 = a^2 - b^2$, where c is the focal distance, a is the major radius, and b is the minor radius. So we get $b^2 = a^2 - c^2$. What then, about c ?

The focal distance is the distance from the center to each focus. So since the center is $(2, 0)$, and a focus is $(0, 0)$, we get $c = 2$.

And thus, we get $b^2 = a^2 - c^2 = 9 - 4 = 5 \Rightarrow b = \sqrt{5}$.

So the ellipse is $\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$, often put this way, too: $\frac{(x-2)^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$.



Suggestions or Solutions

To the Problem in the Example 1

Find the ellipse where the four endpoints of the two main axes are (1, 3), (4, 1), (7, 3), and (4, 5).

To begin with, taking the distance from (1, 3) to (7, 3), we get 6.

Next, taking the distance from (4, 1) to (4, 5), we get 4.

So the major axis is 6, the minor axis is 4, and the ellipse is horizontal.

Next, the center is (4, 3).

So the ellipse is $\frac{(x-4)^2}{9} + \frac{(y-3)^2}{4} = 1$, often put this way, too: $\frac{(x-4)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

First, the standard equation of an ellipse centered at (u, v) is $\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1$.

And if $a > b > 0$, the ellipse is horizontal, a is the major radius, and b is the minor radius.

If however, $b > a > 0$, it is vertical, b is the major, and a is the minor.

Next, we can notice that the two points (4, 1) and (4, 5) share the same x -coordinate, so the two points are the two end points of the axis of symmetry parallel to the y -axis.

And taking the distance between the two, we get 4.

Next, we can also notice that the two points (1, 3) and (7, 3) share the same y -coordinate, so the two points are the two end points of the axis of symmetry parallel to the x -axis.

And taking the distance between the two, we get 6.

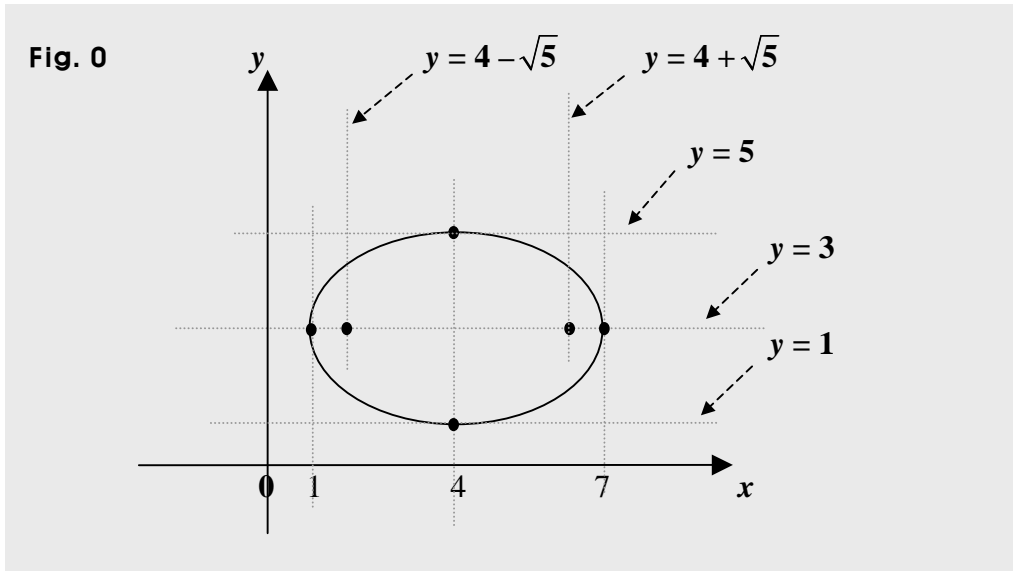
So the major axis is 6, and the minor axis is 4.

And two points (1, 3) and (7, 3) are the two vertices. And if the ellipse is horizontal, the vertices share the same y -coordinate. So the ellipse we want to find is horizontal.

Next, the center is the midpoint between the vertices.

So the center is $\{(1 + 7)/2, (3 + 3)/2\}$, and thus, is $(4, 3)$.

So the ellipse is $\frac{(x-4)^2}{9} + \frac{(y-3)^2}{4} = 1$, often put this way, too: $\frac{(x-4)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$.



Suggestions or Solutions

To the Problem in the Example 2

Find the foci of the ellipse that passes through $(-1, 1)$, $(-2, -1)$, $(-3, 1)$, and $(-2, 3)$.

To begin with, taking the midpoint of $(-1, 1)$ and $(-3, 1)$, we get $(-2, 1)$.

And taking the midpoint of $(-2, -1)$ and $(-2, 3)$, we get $(-2, 1)$, too.

So we can notice that the four points are the four endpoints of the two main axes.

Thus next, taking the distance from $(-1, 1)$ to $(-3, 1)$, we get 2.

Next, taking the distance from $(-2, -1)$ to $(-2, 3)$, we get 4.

So the major axis is 4, the minor axis is 2, and the ellipse is vertical.

And next, the center is $(-2, 1)$.

So the ellipse is $(x + 2)^2 + \frac{(y - 1)^2}{4} = 1$, often put this way, too: $\frac{(x + 2)^2}{1^2} + \frac{(y - 1)^2}{2^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

First, the standard equation of an ellipse centered at (u, v) is $\frac{(x - u)^2}{a^2} + \frac{(y - v)^2}{b^2} = 1$.

And if $b > a > 0$, the ellipse is vertical, b is the major radius, and a is the minor radius. If however, $a > b > 0$, it is horizontal, a is the major, and b is the minor.

Next, we can notice that the two points $(-1, 1)$ to $(-3, 1)$ share the same y -coordinate, so we can expect that the two points are the two endpoints of the axis of symmetry parallel to the x -axis. And the axis of symmetry is one of the two main axes.

And also, we can notice that the other two points $(-2, -1)$ and $(-2, 3)$ share the same x -coordinate, so we can also expect that the two points are the two endpoints of the axis of symmetry parallel to the y -axis.

How then, can we check to see if the four points are the four endpoints of the main axes?

We know that the center is the midpoint of the vertices, which are the two endpoints of the major axis, and also, that the center is the midpoint of the two endpoints of the minor axis, too.

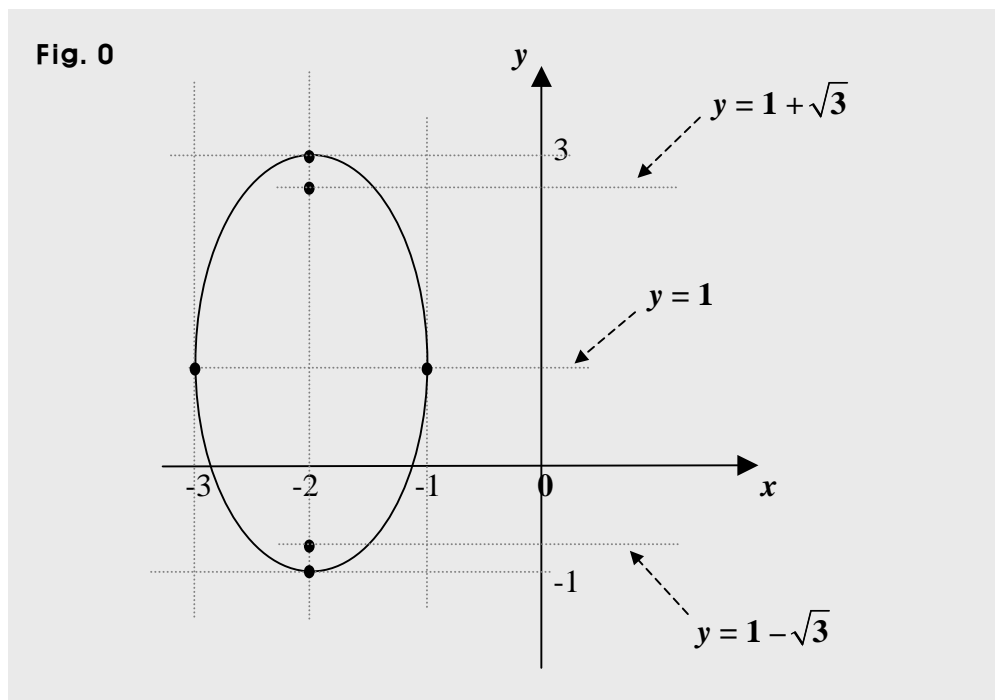
So we can check to see if the midpoint of the two points $(-1, 1)$ to $(-3, 1)$ is the same as the midpoint of the other two points $(-2, -1)$ and $(-2, 3)$.

Taking thus, the midpoint of $(-1, 1)$ to $(-3, 1)$, we get $(-2, 1)$. And taking the midpoint of $(-2, -1)$ and $(-2, 3)$, we get $(-2, 1)$, too. So the four points are the four endpoints of the main axes, and the center is $(-2, 1)$.

So next, taking the distance between $(-1, 1)$ to $(-3, 1)$, we get 2. And taking the distance between $(-2, -1)$ and $(-2, 3)$, we get 4. So the major axis is 4, and the minor axis is 2.

And the vertices are the endpoints of the major axis, so two points $(-2, -1)$ and $(-2, 3)$ are the two vertices. And if the ellipse is vertical, the vertices share the same x -coordinate. So the ellipse we want to find is vertical.

So the ellipse is $(x + 2)^2 + \frac{(y - 1)^2}{4} = 1$, often put this way, too: $\frac{(x + 2)^2}{1^2} + \frac{(y - 1)^2}{2^2} = 1$.



Suggestions or Solutions
To the Problem in the Example 3

Find the ellipse of which the eccentricity is $3/5$, and a directrix is a line $x = 28/3$, and is corresponding to a focus at $(4, 2)$.

Suppose first, $T(x, y)$ is an arbitrary point of an ellipse, D is the distance from T to a directrix, F is the distance from T to the focus corresponding to the directrix, and e is the eccentricity. Then, we get $F = eD$, that is, $e = F/D$. So in this case, we get $F/D = 3/5$.

So next, taking the distances, we get $D^2 = (x - 28/3)^2$, and $F^2 = (x - 4)^2 + (y - 2)^2$.

$$\begin{aligned} \text{Then, we get } F/D = 3/5 &\Rightarrow (F/D)^2 = F^2/D^2 = 9/25 = \frac{(x-4)^2 + (y-2)^2}{(x - \frac{28}{3})^2} \\ &\Rightarrow 9(x - 28/3)^2 = 25\{(x - 4)^2 + (y - 2)^2\} \Rightarrow \underline{25(x - 4)^2 - 9(x - 28/3)^2} + 25(y - 2)^2 = 0. \end{aligned}$$

And we have an identity, $A - B = (A + B)(A - B)$. So we get

$$\begin{aligned} \underline{25(x - 4)^2 - 9(x - 28/3)^2} &= \{5(x - 4) + 3(x - 28/3)\}\{5(x - 4) - 3(x - 28/3)\} \\ &= (5x - 20 + 3x - 28)(5x - 20 - 3x + 28) = (8x - 48)(2x + 8) = 16(x - 6)(x + 4) \\ &= 16(x^2 - 2x - 24) = 16(x^2 - 2x + 1 - 25) = 16(x - 1)^2 - 16 \cdot 25 = 16(x - 1)^2 - 20^2. \end{aligned}$$

So we get $16(x - 1)^2 - 20^2 + 25(y - 2)^2 = 0 \Rightarrow 16(x - 1)^2 + 25(y - 2)^2 = 20^2$

$$\Rightarrow 4^2(x - 1)^2 + 5^2(y - 2)^2 = 4^2 5^2 \Rightarrow \frac{(x-1)^2}{5^2} + \frac{(y-2)^2}{4^2} = 1, \text{ which is the ellipse we want.}$$

Suggestions or Solutions
To the Problem in the Example 4

Suppose C is a point in a line segment \overline{AB} , and $\overline{AC} / \overline{CB} = 1/2$. Then, assuming the end point A is moving along the x -axis, and at the same time, the end point B is moving along the y -axis, find the curve that the point C makes.

Setting first, $C = (x, y)$, $A = (u, 0)$, $B = (0, v)$, and $\overline{AB} = k$, we get $x = 2u/3$ and $y = v/3$.

Then, by the distance formula, we get $(\overline{AB})^2 = (u - 0)^2 + (0 - v)^2 = k^2 \Rightarrow u^2 + v^2 = k^2$.

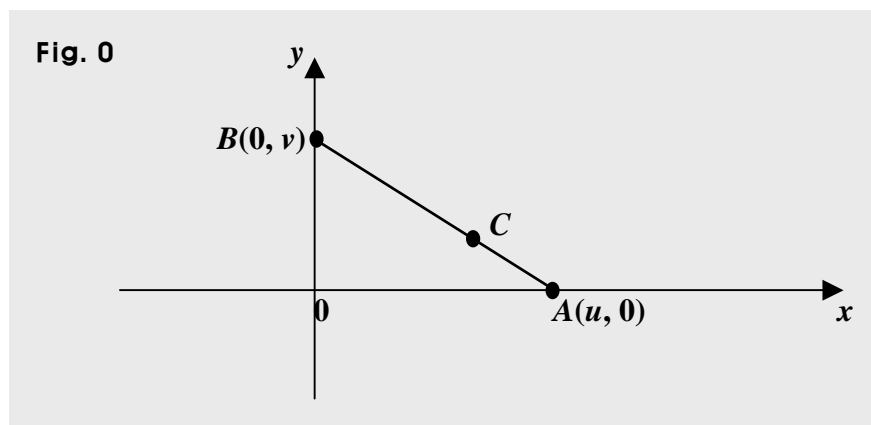
And we get $x = 2u/3 \Rightarrow u = 3x/2$, and $y = v/3 \Rightarrow v = 3y$.

So next, we get $u^2 + v^2 = k^2 \Rightarrow (3x/2)^2 + (3y)^2 = \frac{x^2}{(\frac{2}{3})^2} + \frac{y^2}{(\frac{1}{3})^2} = k^2 \Rightarrow \frac{x^2}{(\frac{2}{3}k)^2} + \frac{y^2}{(\frac{1}{3}k)^2} = 1$,

which is a horizontal ellipse where the center is $(0, 0)$, the major radius is $2k/3$, and the minor radius is $k/3$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $A = (u, 0)$, and $B = (0, v)$, we can put the point C and the line segment \overline{AB} in a graph the way below.

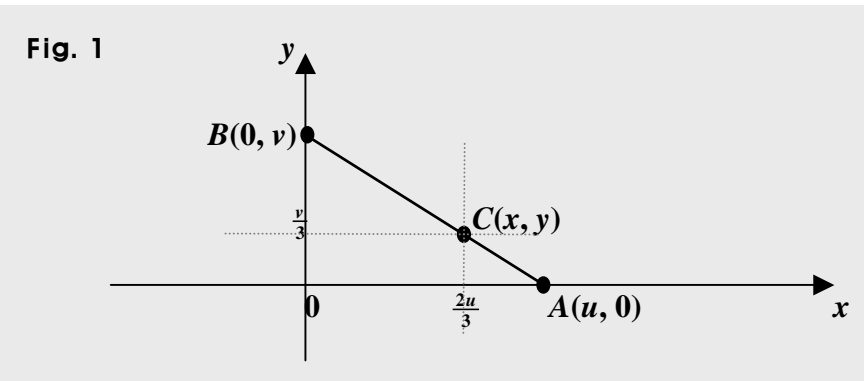


Next, we know that the point C moves in the x - y plane as the two points A and B , and makes a curve. So we can take the point C as the arbitrary point in the curve, and thus, can set $C = (x, y)$.

Next, assuming $\overline{AB} = k$, that is, the length of \overline{AB} is k , by the distance formula, we get $(\overline{AB})^2 = (u - 0)^2 + (0 - v)^2 = k^2 \Rightarrow u^2 + v^2 = k^2$.

Next, we know $\overline{AC} / \overline{CB} = 1 / 2$. So we get $x = 2u/3$ and $y = v/3$.

In other words, the point C is at the third of the line segment from the point A . It's because \overline{CB} is twice \overline{AC} .



Then, we get $x = 2u/3 \Rightarrow u = 3x/2$, and $y = v/3 \Rightarrow v = 3y$.

So next, getting the connective equation between x and y , we get the equation of the curve, and getting the equation, we get

$$u^2 + v^2 = k^2 \Rightarrow \left(\frac{3}{2}x\right)^2 + (3y)^2 = k^2 \Rightarrow \left(\frac{x}{\frac{2}{3}}\right)^2 + \left(\frac{y}{\frac{1}{3}}\right)^2 = k^2 \Rightarrow \frac{x^2}{\left(\frac{2}{3}\right)^2} + \frac{y^2}{\left(\frac{1}{3}\right)^2} = k^2$$

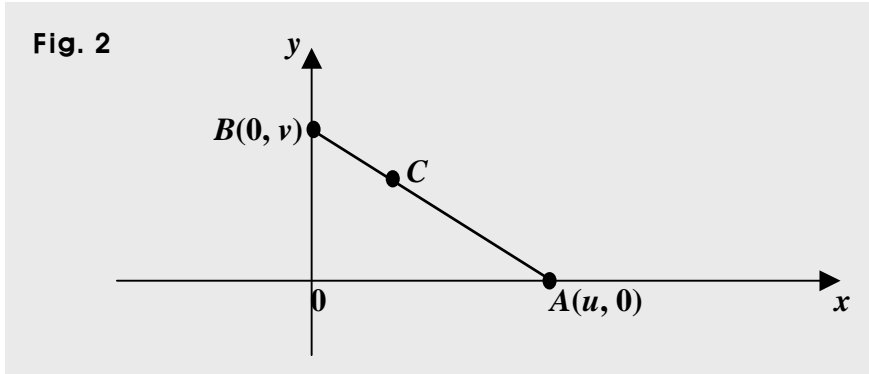
$\Rightarrow \frac{x^2}{\left(\frac{2}{3}k\right)^2} + \frac{y^2}{\left(\frac{1}{3}k\right)^2} = 1$, which is a horizontal ellipse where the center is $(0, 0)$, the major

radius is $2k/3$, and the minor radius is $k/3$.

In other words, the curve is an ellipse horizontal where the center is $(0, 0)$, the major axis is $4k/3$, and the minor axis is $2k/3$.

What then, about the case where $\overline{BC} / \overline{CA} = 1 / 2$?

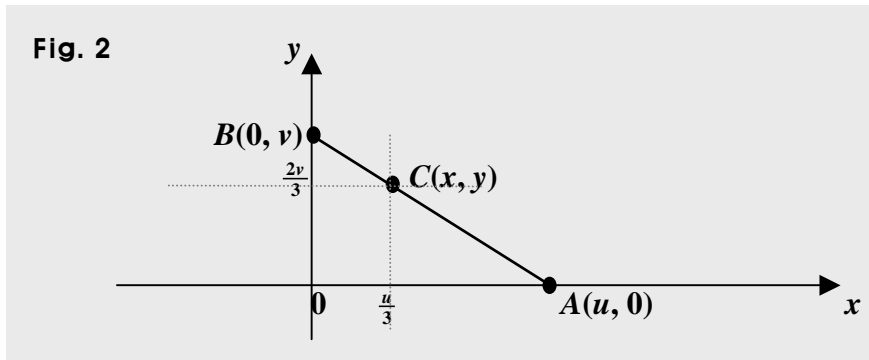
Setting again, $A = (u, 0)$, and $B = (0, v)$, we can put the point C and the line segment \overline{AB} in a graph the way below.



And taking the point C as the arbitrary point in the curve, we can set $C = (x, y)$.
 Next, assuming again, $\overline{AB} = k$, we get $(\overline{AB})^2 = (u - 0)^2 + (0 - v)^2 = k^2 \Rightarrow u^2 + v^2 = k^2$.

Next, we know $\overline{BC} / \overline{CA} = 1 / 2$. So we get $x = 2u/3$ and $y = v/3$.

In other words, the point C is at the third of the line segment from the point A . It's because \overline{CB} is twice \overline{AC} .



Then, we get $x = u/3 \Rightarrow u = 3x$, and $y = 2v/3 \Rightarrow v = 3y/2$.

So next, getting the connective equation between x and y , we get

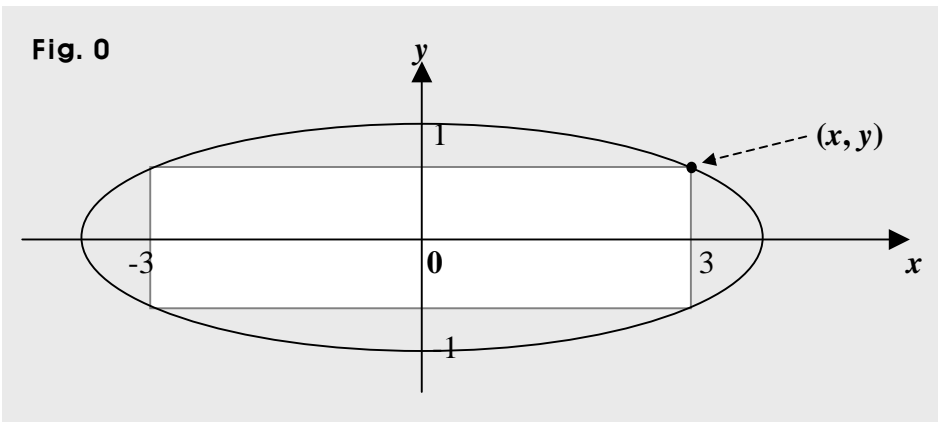
$$u^2 + v^2 = k^2 \Rightarrow (3x)^2 + \left(\frac{3}{2}y\right)^2 = k^2 \Rightarrow \left(\frac{x}{\frac{1}{3}}\right)^2 + \left(\frac{y}{\frac{2}{3}}\right)^2 = k^2 \Rightarrow \frac{x^2}{\left(\frac{1}{3}\right)^2} + \frac{y^2}{\left(\frac{2}{3}\right)^2} = k^2$$

$\Rightarrow \frac{x^2}{\left(\frac{1}{3}k\right)^2} + \frac{y^2}{\left(\frac{2}{3}k\right)^2} = 1$, which is a vertical ellipse where the center is $(0, 0)$, the major axis is $4k/3$, so the major radius is $2k/3$, and the minor axis is $2k/3$, so the minor radius is $k/3$.

Suggestions or Solutions
To the Problem in the Example 5

Find the maximum area of a rectangle inscribed in an ellipse $x^2 + 9y^2 = 9$.

First, we can put the equation this way, too: $x^2/3^2 + y^2/1^2 = 1$. So assuming (x, y) is an arbitrary point in the ellipse, we can put the rectangle in the ellipse the way below.



So next, assuming A is the area of the rectangle, we can set $A = 4|xy|$.

Next, we can get $x^2 + 9y^2 = 9 \Rightarrow x = \pm\sqrt{9 - 9y^2} = \pm 3\sqrt{1 - y^2}$.

So we get $A = 4|xy| = 12|y\sqrt{1 - y^2}| = 12\sqrt{y^2 - y^4}$.

Next, setting $s = y^2$, we get $-1 \leq y \leq 1 \Rightarrow 0 \leq y^2 \leq 1 \Rightarrow 0 \leq s \leq 1$.

So next, setting $t = y^2 - y^4$, we get $t = s - s^2$ for $0 \leq s \leq 1$.

And we get $t = -(s^2 - s) = -(s^2 - s + 1/4 - 1/4) = -(s - 1/2)^2 + 1/4$.

So when $s = 1/2$, t gets its maximum value, which is $1/4$.

That is, when $y^2 = 1/2$, $(y^2 - y^4)$ gets its maximum value, which is $1/4$.

So when $y = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$, $A = 12\sqrt{t}$ gets its maximum value, which is $12\sqrt{\frac{1}{4}} = 6$.