Examples 6 in Ellipses

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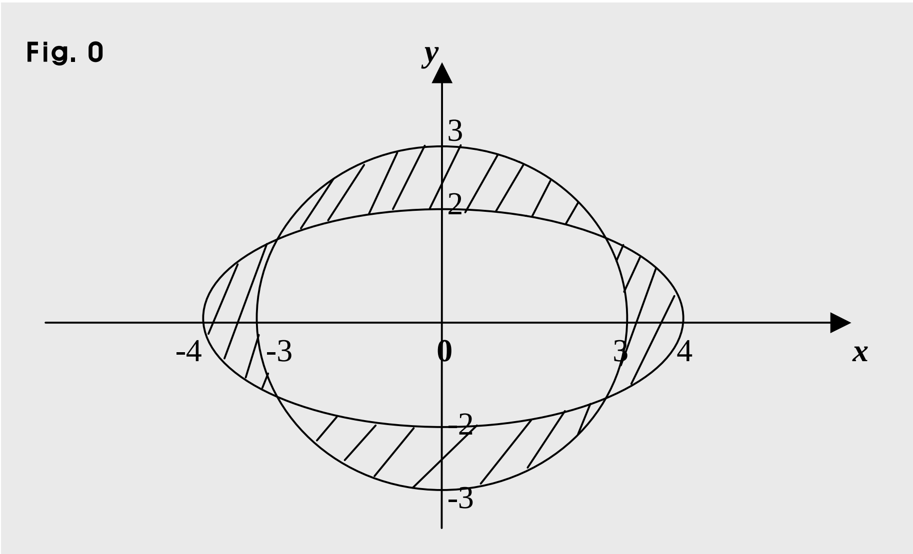
0. In the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, find the area where the points satisfy the inequality as follows.
$\mathbf{(x^2 + y^2 - 9)(x^2 + 4y^2 - 16) \le 0}$.

1. Assuming $\mathbf{9x^2 + ax + y^2 + by + c = 0}$ is an ellipse, find the values of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ that make the ellipse tangent to the $\boldsymbol{x}$-axis at a point (1, 0) and pass through a point (3, 3).

Suggestions or Solutions
To the Problem in the Example 0

In the *x*-*y* plane, find the area where the points satisfy the inequality as follows.
$(x^2 + y^2 - 9)(x^2 + 4y^2 - 16) \le 0$**.**

In the graph below, the area we want is the area with slash marks, and the area includes the circle and ellipse themselves, too.



If not quite sure of the idea behind the processes above, follow the steps below.

Assuming first, $AB \le 0$, we get two cases.
One is $A \le 0$ and $B \ge 0$, and the other is $A \ge 0$ and $B \le 0$.

So if $(x^2 + y^2 - 9)(x^2 + 4y^2 - 16) \le 0$, we get two cases, too.

One is $x^2 + y^2 - 9 \le 0$ and $x^2 + 4y^2 - 16 \ge 0$.
And the other is $x^2 + y^2 - 9 \ge 0$, and $x^2 + 4y^2 - 16 \le 0$.

And we have $x^2 + y^2 \le 9 = 3^2$, and $x^2 + 4y^2 \ge 16 \Rightarrow x^2/16 + y^2/4 \ge 1 \Rightarrow x^2/4^2 + y^2/2^2 \ge 1$.

So in one case, we have $x^2 + y^2 \le 3^2$, and $x^2/4^2 + y^2/2^2 \ge 1$.

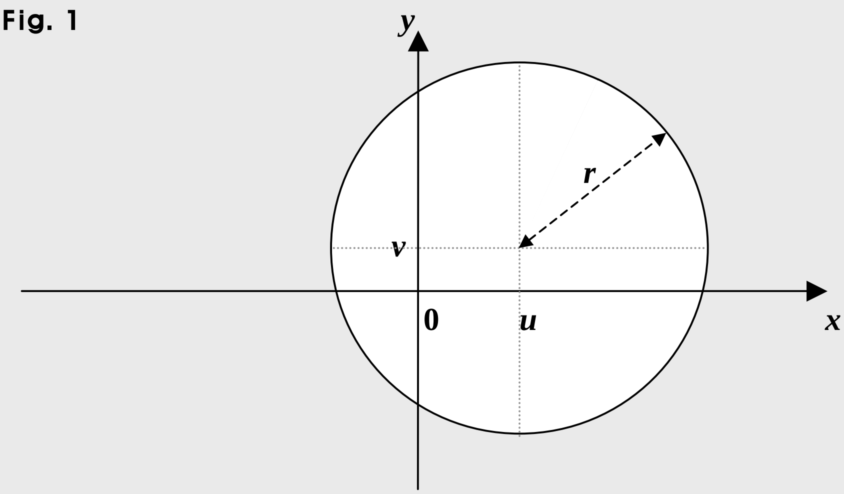
And in the other case, we have $x^2 + y^2 \ge 3^2$, and $x^2/4^2 + y^2/2^2 \le 1$.

Next, $\boldsymbol{x^2 + y^2 = 9 = 3^2}$ is the equation of a circle centered at the origin with a radius of 3.

And $\boldsymbol{x^2/4^2 + y^2/2^2 = 1}$ is the equation of a horizontal ellipse centered at the origin with a semi major axis of 4 and a semi minor axis of 2.

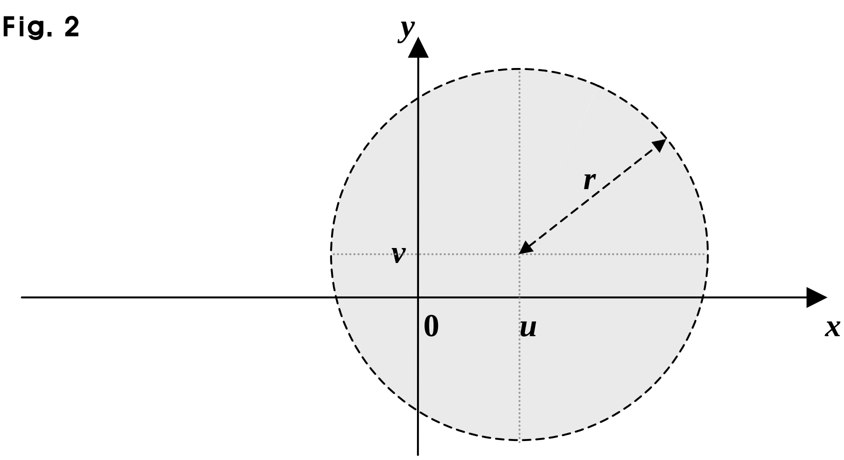
Next, if $\boldsymbol{(x - u)^2 + (y - v)^2 \geq r^2}$, the inequality means the set of all the points in the area <u>outside</u> a circle centered at $\boldsymbol{(u, v)}$ with a radius of $\boldsymbol{r}$, and also, the area includes the circle itself, too, because of the equal sign.

Fig. 1



So if $\boldsymbol{(x - u)^2 + (y - v)^2 < r^2}$, the inequality means the set of all the points in only the area inside the circle, so the area does not include the circle itself.

Fig. 2



And the same is true for an ellipse, too.

So if $\boldsymbol{(x-u)^2/a^2 + (y-v)^2/b^2 \geq 1}$, the inequality means the set of all the points in the area outside an ellipse centered at $\boldsymbol{(u, v)}$ with the main axes of $\boldsymbol{2a}$ and $\boldsymbol{2b}$, and also, the area includes the ellipse itself, too. And if $\boldsymbol{a > b > 0}$, we get

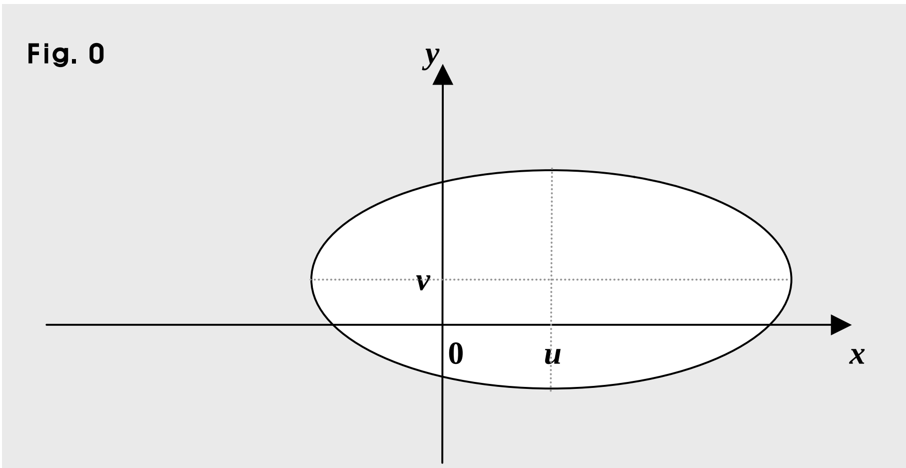


Fig. 0

And if $\boldsymbol{(x-u)^2/a^2 + (y-v)^2/b^2 < 1}$, the inequality means the set of all the points in only the area insides the ellipse, so the area does not include the ellipse itself.

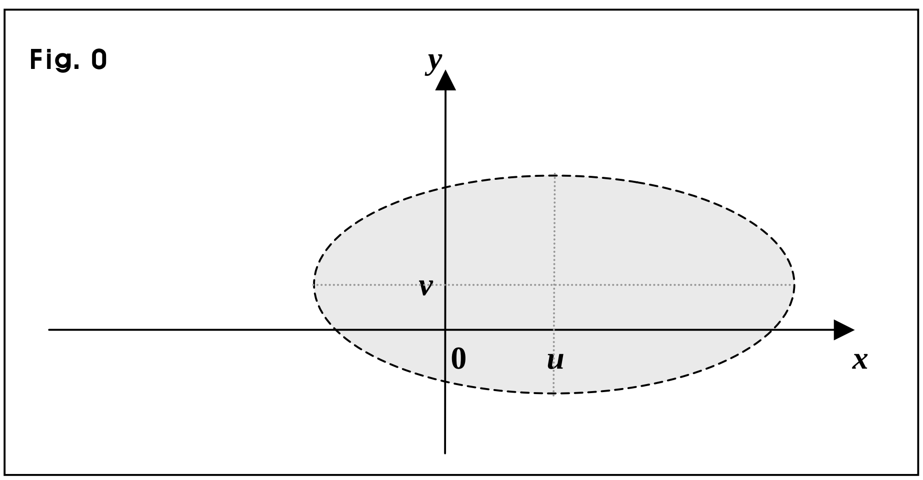


Fig. 0

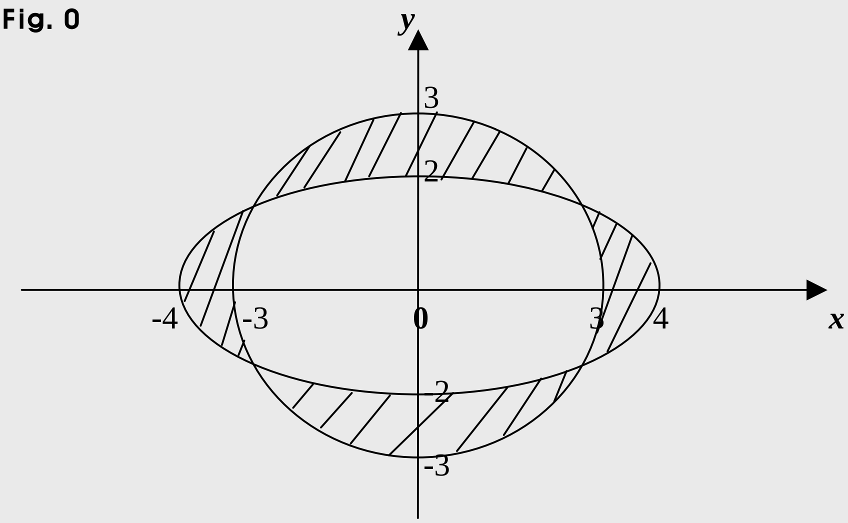
Now, we have two cases.

One is $\boldsymbol{x^2 + y^2 \leq 3^2}$, and $\boldsymbol{x^2/4^2 + y^2/2^2 \geq 1}$.

And the other is $\boldsymbol{x^2 + y^2 \geq 3^2}$, and $\boldsymbol{x^2/4^2 + y^2/2^2 \leq 1}$.

So in the graph below, the area we want is the area with slash marks, and the area includes the circle and ellipse themselves, too.

Fig. 0



What if we have $\mathbf{(x^2 + y^2 - 9)(x^2 + 4y^2 - 16) > 0}$?

Then, we get two cases, too, but the area is the opposite of the area with slash marks above.

So one is $\mathbf{x^2 + y^2 > 3^2}$, and $\mathbf{x^2/4^2 + y^2/2^2 > 1}$.
And the other is $\mathbf{x^2 + y^2 < 3^2}$, and $\mathbf{x^2/4^2 + y^2/2^2 < 1}$.

So in the graph above, the area is the area without slash marks, and the area does not include the circle and ellipse themselves.

Suggestions or Solutions
To the Problem in the Example 1

Assuming $9x^2 + ax + y^2 + by + c = 0$ is an ellipse, find the values of *a*, *b*, and *c* that make the ellipse tangent to the *x*-axis at (1, 0) and pass through a point (3, 3).

First, if $\boldsymbol{E}$ is the ellipse, since $\boldsymbol{E}$ has the point (3, 3), we can get
$\mathbf{9(3^2) + 3a + 3^2 + 3b + c = 0 \Rightarrow 3a + 3b + c + 90 = 0}$.

Next, since $\boldsymbol{E}$ is tangent to the $\boldsymbol{x}$-axis at (1, 0), it has (1, 0), too, so we get $\mathbf{9 + a + c = 0}$.

And next, since the $\boldsymbol{x}$-axis is tangent to $\boldsymbol{E}$, we can say that a line $\boldsymbol{y = 0}$ meets the ellipse $\boldsymbol{E}$ at one point. And assuming we find the point where the line meets the ellipse, we get to solve a system of two equations, one is $\boldsymbol{y = 0}$, and the other is $\mathbf{9x^2 + ax + y^2 + by + c = 0}$.

And solving the system, since $\boldsymbol{y = 0}$, we can get first, $\mathbf{9x^2 + ax + 0^2 + b \cdot 0 + c = 0}$, which is $\mathbf{9x^2 + ax + c = 0}$, which is a quadratic equation, which therefore, has to have a double root, because the line meets the ellipse at one point.

So the discriminant of the quadratic equation has to be 0.
Thus, we get $\mathbf{a^2 - 4 \cdot 9c = a^2 - 36c = 0}$.

So we get a system of three equations as follows.
$\mathbf{3a + 3b + c + 90 = 0}$, $\mathbf{9 + a + c = 0}$, and $\mathbf{a^2 - 36c = 0}$.

So solving the system, we can get first, $\mathbf{9 + a + c = 0 \Rightarrow c = -a - 9}$.

So next, we can get $\mathbf{a^2 - 36c = 0 \Rightarrow a^2 - 36(-a - 9) = 0 \Rightarrow a^2 + 36a + 36 \cdot 9 = 0}$.

Meanwhile, $36 \cdot 9 = 4 \cdot 9 \cdot 9 = 2 \cdot 9 \cdot 2 \cdot 9 = 18^2$, and $36 = 2 \cdot 18$.

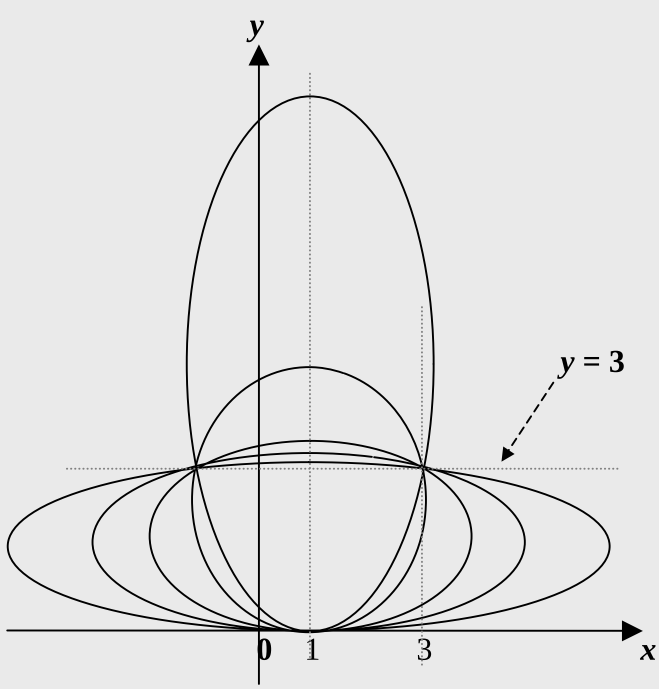
So we get $\mathbf{a^2 + 36a + 36 \cdot 9 = a^2 + 2 \cdot 18a + 18^2 = (a + 18)^2 = 0 \Rightarrow a = -18}$.

Thus next, we can get $\mathbf{c = -a - 9 = 18 - 9 = 9 \Rightarrow c = 9}$.
And next, $\mathbf{3a + 3b + c + 90 = 0 \Rightarrow 3b = -3a - c - 90 = 54 - 9 - 90 = -45 \Rightarrow b = -15}$.

What ellipse then, is it?

There can be infinitely many ellipses that can be tangent to the ***x***-axis at (1, 0), and pass through (3, 3).

Fig. 0



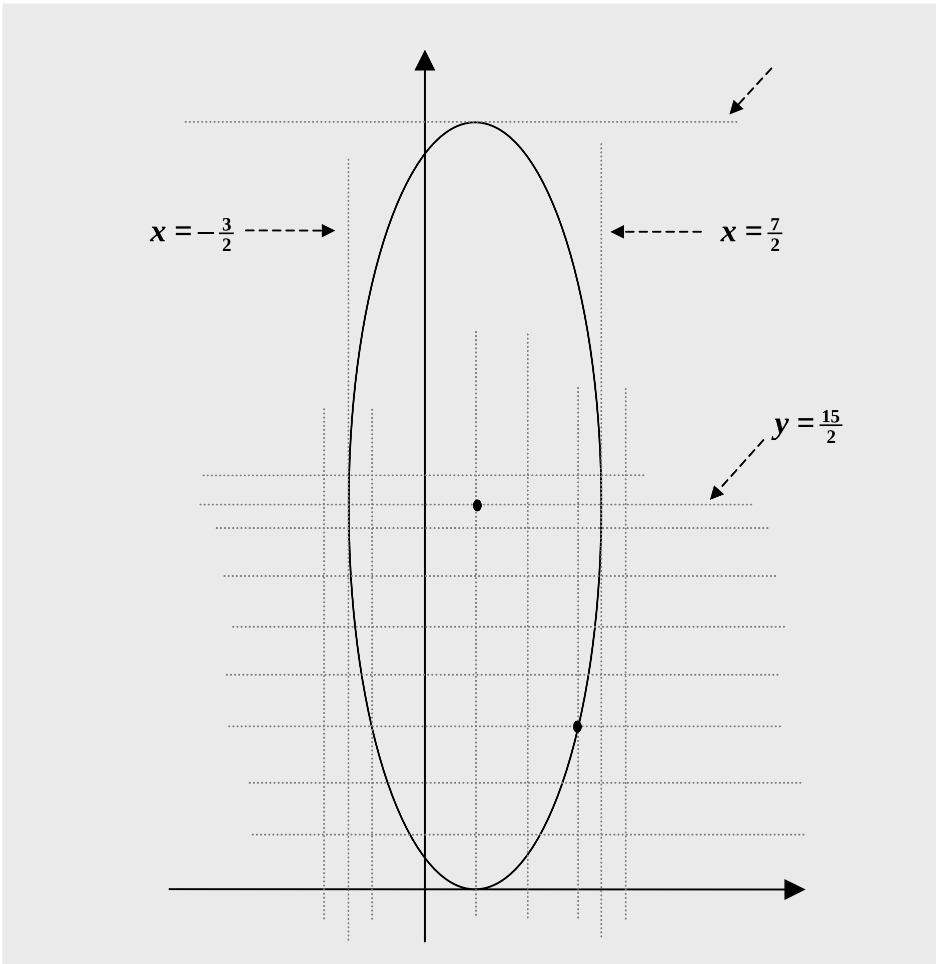
Next, putting values of ***a***, ***b***, and ***c*** into the equation $\mathbf{9x^2 + ax + y^2 + by + c = 0}$, we get

$$9x^2 - 18x + y^2 - 15y + 9 = 9(x^2 - 2x + 1 - 1) + y^2 - 15y + (\tfrac{15}{2})^2 - (\tfrac{15}{2})^2 + 9$$

$$= 9(x-1)^2 - 9 + (y - \tfrac{15}{2})^2 - (\tfrac{15}{2})^2 + 9 = 9(x-1)^2 + (y - \tfrac{15}{2})^2 - (\tfrac{15}{2})^2 = 0$$

$$\Rightarrow \frac{9(x-1)^2}{(\frac{15}{2})^2} + \frac{(y-\frac{15}{2})^2}{(\frac{15}{2})^2} - 1 = 0 \Rightarrow \frac{(x-1)^2}{(\frac{5}{2})^2} + \frac{(y-\frac{15}{2})^2}{(\frac{15}{2})^2} = 0.$$

So the ellipse ***E*** is a vertical ellipse centered at $(1, \frac{15}{2})$ with the main axes of 5 and 15.


$x = -\frac{3}{2}$
$x = \frac{7}{2}$
$y = \frac{15}{2}$