Examples 7 in Ellipses

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Find the equation of the line tangent to an ellipse $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$ at a point $\boldsymbol{(s, t)}$.

Suggestions or Solutions
To the Problem in the Example

Find the equation of the line tangent to an ellipse $x^2/a^2 + y^2/b^2 = 1$ at a point (s, t).

Assuming first, the tangent line is $y = mx + n$, we get first, $x^2/a^2 + (mx + n)^2/b^2 = 1$.
And rearranging terms in the equation above, we get

$x^2/a^2 + (mx + n)^2/b^2 = 1 \Rightarrow b^2x^2 + a^2(mx + n)^2 = a^2b^2$

$\Rightarrow b^2x^2 + a^2(mx + n)^2 = b^2x^2 + a^2(m^2x^2 + 2mnx + n^2)$

$= (b^2 + a^2m^2)x^2 + 2mna^2x + a^2n^2 = a^2b^2 \Rightarrow (b^2 + a^2m^2)x^2 + 2mna^2x + a^2(n^2 - b^2) = 0$.

Next, since the line meets the ellipse at one point, the equation's discriminant has to be 0.

So we get $m^2n^2a^4 - (b^2 + a^2m^2)a^2(n^2 - b^2) = 0 \Rightarrow m^2n^2a^2 - (b^2 + a^2m^2)(n^2 - b^2) = 0$
$\Rightarrow b^4 - b^2n^2 + a^2b^2m^2 = 0 \Rightarrow b^2 - n^2 + a^2m^2 = 0$.

Next, we know the fact that the line passes through (s, t). So we get $t = ms + n$.
Thus, we get $n = t - ms$. So we get

$b^2 - n^2 + a^2m^2 = b^2 - (t - ms)^2 + a^2m^2 = 0$
$\Rightarrow b^2 - t^2 + 2stm - s^2m^2 + a^2m^2 = m^2(a^2 - s^2) + 2stm + b^2 - t^2 = 0$.

So we get $m = \dfrac{-st \pm \sqrt{s^2t^2 - (a^2 - s^2)(b^2 - t^2)}}{a^2 - s^2}$. We know however, m can have one value only. So what's inside the square root has to be 0. That is to say that the discriminant is 0.

So we get $m = \dfrac{-st}{a^2 - s^2}$. And since the discriminant is 0, we get

$s^2t^2 - (a^2 - s^2)(b^2 - t^2) = s^2b^2 + a^2t^2 - a^2b^2 = 0 \Rightarrow b^2(s^2 - a^2) = -a^2t^2 \Rightarrow a^2 - s^2 = \dfrac{a^2t^2}{b^2}$.

So we get $m = -\dfrac{sb^2}{ta^2}$. And we know that the line passes through (s, t).

So the tangent line is $y - t = -\dfrac{sb^2}{ta^2}(x - s)$. And it can be put this way, too: $\dfrac{sx}{a^2} + \dfrac{ty}{b^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose first, the tangent line is $\boldsymbol{y = mx + n}$, and is called $\boldsymbol{T}$, and the ellipse is $\boldsymbol{E}$.

Then, since the line $\boldsymbol{T}$ is tangent to the ellipse $\boldsymbol{E}$, we can say that the line $\boldsymbol{T}$ meets the ellipse $\boldsymbol{E}$ at one point.

And assuming we find the point where the line meets the ellipse, we get to solve a system of two equations, one is $\boldsymbol{y = mx + n}$, and the other is $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$.

And solving the system, since $\boldsymbol{y = mx + n}$, we can get first, $\boldsymbol{x^2/a^2 + (mx + n)^2/b^2 = 1}$, which is $\boldsymbol{b^2x^2 + a^2(mx + n)^2 = a^2b^2}$, which is a quadratic equation, which therefore, has to have a double root, because the line meets the ellipse at one point.

So the discriminant of the quadratic equation has to be 0.
By the way, if the coefficient of $\boldsymbol{x}$ is even, we can take a quarter of the discriminant, because the result will get simplified that way.

If for instance, we have $\boldsymbol{Ax^2 + 2Bx + C = 0}$, the discriminant is $\boldsymbol{B^2 - AC}$.

So first, rearranging terms in the equation, we get

$$\boldsymbol{b^2x^2 + a^2(mx + n)^2 = b^2x^2 + a^2(m^2x^2 + 2mnx + n^2)}$$

$$\boldsymbol{= (b^2 + a^2m^2)x^2 + 2mna^2x + a^2n^2 = a^2b^2 \Rightarrow (b^2 + a^2m^2)x^2 + 2mna^2x + a^2(n^2 - b^2) = 0}.$$

So next, getting the quarter of the discriminant, we get

$$\boldsymbol{m^2n^2a^4 - (b^2 + a^2m^2)a^2(n^2 - b^2) = 0 \Rightarrow m^2n^2a^2 - (b^2 + a^2m^2)(n^2 - b^2) = 0}$$
$$\boldsymbol{\Rightarrow m^2n^2a^2 - b^2n^2 + b^4 - m^2n^2a^2 + a^2b^2m^2 = b^4 - b^2n^2 + a^2b^2m^2 = 0}$$
$$\boldsymbol{\Rightarrow b^2 - n^2 + a^2m^2 = 0}.$$

Next, knowing the slope and a point of a line, we can get the equation of the line.

We know the tangent line $\boldsymbol{T}$ passes through $\boldsymbol{(s, t)}$, and the slope is $\boldsymbol{m}$.
And also, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are all known values.

So finding $\boldsymbol{m}$, we can get the line $\boldsymbol{T}$. How then, can we find $\boldsymbol{m}$?

We now have two equations that have $\boldsymbol{m}$.

One is $\boldsymbol{b^2 - n^2 + a^2m^2 = 0}$. and the other is $\boldsymbol{y = mx + n}$.

And we know the fact that $\boldsymbol{T}$ passes through $\boldsymbol{(s, t)}$. So we get $\boldsymbol{t = ms + n}$.
Thus, we get $\boldsymbol{n = t - ms}$.

So we can now get $\boldsymbol{m}$ putting $\boldsymbol{n}$ into the other equation. Then, we get

$$\boldsymbol{b^2 - n^2 + a^2m^2 = b^2 - (t - ms)^2 + a^2m^2 = 0}$$
$$\Rightarrow \boldsymbol{b^2 - t^2 + 2stm - s^2m^2 + a^2m^2 = m^2(a^2 - s^2) + 2stm + b^2 - t^2 = 0}, \text{ which is quadratic.}$$

So using the quadratic formula, we get $\boldsymbol{m = \dfrac{-st \pm \sqrt{s^2t^2 - (a^2 - s^2)(b^2 - t^2)}}{a^2 - s^2}}$.

We know however, $\boldsymbol{m}$ can have one value only.

So what's inside the square root has to be 0. That is to say that the discriminant is 0.

So we get $\boldsymbol{m = \dfrac{-st}{a^2 - s^2} = \dfrac{st}{s^2 - a^2}}$.

And since the discriminant is 0, we get

$$\boldsymbol{s^2t^2 - (a^2 - s^2)(b^2 - t^2) = s^2t^2 - a^2b^2 + a^2t^2 + s^2b^2 - s^2t^2 = s^2b^2 + a^2t^2 - a^2b^2 = 0}$$
$$\Rightarrow \boldsymbol{s^2b^2 - a^2b^2 + a^2t^2 = 0} \Rightarrow \boldsymbol{b^2(s^2 - a^2) = -a^2t^2} \Rightarrow \boldsymbol{s^2 - a^2 = \frac{-a^2t^2}{b^2}}.$$

So we get $\boldsymbol{m = \dfrac{st}{s^2 - a^2} = st\dfrac{b^2}{-a^2t^2} = -\dfrac{sb^2}{ta^2}}$, which is the slope of the line $\boldsymbol{T}$.

And we know that the line $\boldsymbol{T}$ passes through $\boldsymbol{(s, t)}$.
If a line has a slope of $\boldsymbol{k}$, and has a point $\boldsymbol{(p, q)}$, the line is $\boldsymbol{y - q = k(x - p)}$.

So the line $\boldsymbol{T}$ is $\boldsymbol{y - t = -\dfrac{sb^2}{ta^2}(x - s)}$, which looks a bit complicated.

So simplifying the equation above, we can get

$$y - t = -\frac{sb^2}{ta^2}(x - s) \Rightarrow ta^2(y - t) = -sb^2(x - s) \Rightarrow tya^2 - t^2a^2 = -sxb^2 + s^2b^2$$

$$\Rightarrow sxb^2 + tya^2 = s^2b^2 + t^2a^2 \Rightarrow sx + \frac{tya^2}{b^2} = s^2 + \frac{t^2a^2}{b^2} \Rightarrow \frac{sx}{a^2} + \frac{ty}{b^2} = \frac{s^2}{a^2} + \frac{t^2}{b^2}.$$

And we know that the ellipse $\boldsymbol{E}$ passes through the point (s, t), and $\boldsymbol{E}$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

So we get $\frac{s^2}{a^2} + \frac{t^2}{b^2} = 1$, and thus, the tangent line is $\frac{sx}{a^2} + \frac{ty}{b^2} = 1$.

So for instance, finding the line tangent to $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ at $(1, \frac{4\sqrt{2}}{3})$, we get

$\frac{x}{3^2} + \frac{4\sqrt{2}y}{3 \cdot 2^2} = \frac{x}{9} + \frac{\sqrt{2}y}{3} = 1$, which is the tangent line.

And of course, we can put it this way, too: $x + 3\sqrt{2}y - 9 = 0$.

What if we are given the slope of the tangent line and are asked to find the tangent line?

Assuming the slope given is m, we can assume that the tangent line is $y = mx + n$.

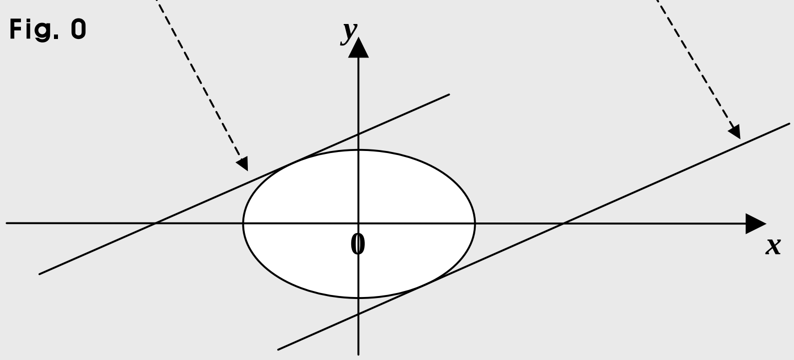
Then, finding n, we get the line tangent to the ellipse.

Finding n, we just solve $b^2 - n^2 + a^2m^2 = 0$ for n. And we get the equation taking the discriminant of $x^2/a^2 + (mx + n)^2/b^2 = 1$, which is $b^2x^2 + a^2(mx + n)^2 = a^2b^2$.

So solving it for n, we get $n = \pm\sqrt{b^2 + a^2m^2}$. And thus, we get two tangent lines.

One is $y = mx + \sqrt{b^2 + a^2m^2}$, and the other is $y = mx - \sqrt{b^2 + a^2m^2}$.

Fig. 0



So for instance, finding the lines that have a slope of 2, and are tangent to $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, we get $y = 2x \pm \sqrt{2^2 + 3^2 2^2}$. So we get two tangent lines.

One is $y = 2x + 2\sqrt{10}$, and the other is $y = 2x - 2\sqrt{10}$.

What if this time, we want to find the line tangent to an ellipse $\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1$ at a point (s, t)?

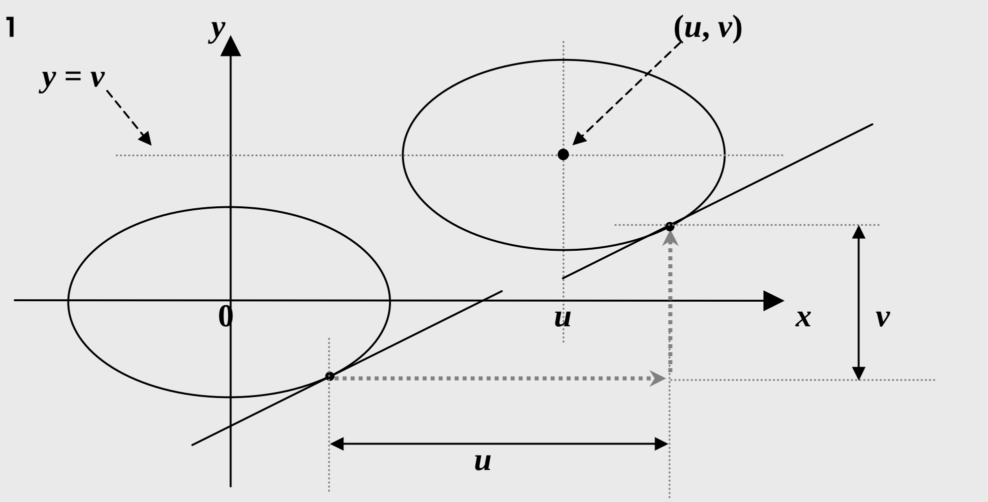
Translating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the amount of u along the x-axis, and in the amount of v along the y-axis, we get the ellipse $\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1$.

So finding the line tangent to $\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1$ at the point (s, t), we find first the line tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(s - u, t - v)$. Then, we get $\frac{(s-u)x}{a^2} + \frac{(t-v)y}{b^2} = 1$.

Next, translating the line above in the amount of $\boldsymbol{u}$ along the $\boldsymbol{x}$-axis, and in the amount of $\boldsymbol{v}$ along the $\boldsymbol{y}$-axis, we get the line tangent to $\frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}=1$ at the point (s, t).

Then, we get $\frac{(s-u)(x-u)}{a^2}+\frac{(t-v)(y-v)}{b^2}=1$, which is the line tangent to $\frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}=1$ at the point (s, t).

Fig. 1



So for instance, finding the line tangent to $\frac{(x-1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$ at $(-1, \frac{2\sqrt{5}}{3}+2)$, we can get first, the line tangent to $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$ at $(-1-1, \frac{2\sqrt{5}}{3}+2-2)$, that is, $(-2, \frac{2\sqrt{5}}{3})$.

Then, the line is $\frac{-2x}{3^2}+\frac{\frac{2\sqrt{5}}{3}y}{2^2}=1$.

Next, translating the line above in the amount of 1 along the $\boldsymbol{x}$-axis, and in the amount of 2 along the $\boldsymbol{y}$-axis, we get the line tangent to $\frac{(x-1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$ at $(-1, \frac{2\sqrt{5}}{3}+2)$.

Then, we get $\frac{-2(x-1)}{3^2}+\frac{\frac{2\sqrt{5}}{3}(y-2)}{2^2}=1$, which is $\frac{-2(x-1)}{9}+\frac{\sqrt{5}(y-2)}{6}=1$, which is the line tangent to $\frac{(x-1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$ at $(-1, \frac{2\sqrt{5}}{3}+2)$.

And of course, we can put the line the way below, too.

$\frac{-2(x-1)}{9}+\frac{\sqrt{5}(y-2)}{6}=1 \Rightarrow -4(x-1)+3\sqrt{5}(y-2)=18$.

$\Rightarrow 4x-3\sqrt{5}y+14+6\sqrt{5}=0$.

And the same is true, too, for the tangent line with the slope given.

So the lines that are tangent to $\frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}=1$, and have a slope of m are as follows. $y-v=m(x-u)\pm\sqrt{b^2+a^2m^2}$.