Examples 8 in Ellipses

Click this to start.

Examples 8 in Ellipses

0. Suppose a point $\boldsymbol{P(s, t)}$ is outside an ellipse $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$, and two lines $\boldsymbol{Q}$ and $\boldsymbol{R}$ meet at the point $\boldsymbol{P}$ and are tangent to the ellipse. Find the slopes of the two tangent lines.

1. A point $\boldsymbol{P(s, t)}$ is in a circle $\boldsymbol{x^2 + y^2 = a^2 + b^2}$, an ellipse $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$ is inside the circle, and two lines $\boldsymbol{Q}$ and $\boldsymbol{R}$ meet at the point $\boldsymbol{P}$ and are tangent to the ellipse. Then, show that the two tangent lines $\boldsymbol{Q}$ and $\boldsymbol{R}$ are perpendicular to each other.

2. A point $\boldsymbol{P(s, t)}$ is outside an ellipse $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$, and two lines meet at the point $\boldsymbol{P}$, and are tangent to the ellipse at two points $\boldsymbol{Q}$ and $\boldsymbol{R}$. Then, show that the equation of the line connecting the two tangent points $\boldsymbol{Q}$ and $\boldsymbol{R}$ is $\boldsymbol{sx/a^2 + ty/b^2 = 1}$. (In this case, the point $\boldsymbol{P}$ is called the pole of the line connecting $\boldsymbol{Q}$ and $\boldsymbol{R}$, and the line is called the polar of the point $\boldsymbol{P}$.)

3. Two points $\boldsymbol{P(s, t)}$ and $\boldsymbol{Q(u, v)}$ are outside an ellipse $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$. Then, show that if $\boldsymbol{Q}$ is in the polar of $\boldsymbol{P}$, $\boldsymbol{P}$ is in the polar of $\boldsymbol{Q}$, too.

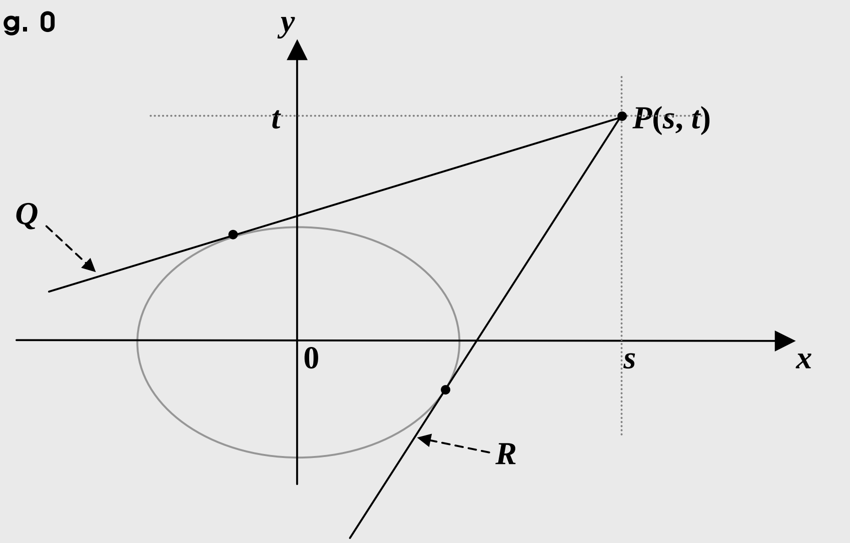
4. Suppose $\boldsymbol{P}$ is a point in an ellipse, $\boldsymbol{A}$ and $\boldsymbol{B}$ are the two foci, and $\boldsymbol{T}$ is a line tangent to the ellipse at the point $\boldsymbol{P}$. Suppose also, $\boldsymbol{m}$ and $\boldsymbol{n}$ are acute angles, $\boldsymbol{m}$ is an angle between the line segment $\boldsymbol{PA}$ and the tangent line $\boldsymbol{T}$, and $\boldsymbol{n}$ is an angle between the line segment $\boldsymbol{PB}$ and the line $\boldsymbol{T}$. Then, show that $\boldsymbol{m = n}$.

Suggestions or Solutions
To the Problem in the Example 0

Suppose a point $P(s, t)$ is outside an ellipse $x^2/a^2 + y^2/b^2 = 1$, and two lines Q and R meet at P and are tangent to the ellipse. Find the slopes of the two tangent lines.

Assuming first, $a > b$, and the ellipse is E, and putting in a graph the two tangent lines Q and R, the point P, and the ellipse E, we can put them the way below.

Fig. 0



So examining the graph above, we can notice that this example is quite close to the example in **Examples 7 in Ellipses**.

This is in fact, no other than the example 2, and the only difference is that the point the tangent line passes through is not a tangent point, but is outside the ellipse. So we get two slopes when we get the slope of the tangent line passing through the point $P(s, t)$.

So suppose again, the tangent line Q is $y = mx + n$.
Then, since the line Q is tangent to the ellipse E, we can say that the line Q meets the ellipse E at one point.

And assuming we find the point where the line meets the ellipse, we get to solve a system of two equations, one is $\boldsymbol{y = mx + n}$, and the other is $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$.

And solving the system, since $\boldsymbol{y = mx + n}$, we can get first, $\boldsymbol{x^2/a^2 + (mx + n)^2/b^2 = 1}$, which is $\boldsymbol{b^2x^2 + a^2(mx + n)^2 = a^2b^2}$, which is a quadratic equation, which therefore, has to have a double root, because the line meets the ellipse at one point.

So the discriminant of the quadratic equation has to be 0.

Rearranging terms in the equation first, we get

$$\boldsymbol{b^2x^2 + a^2(mx + n)^2 = b^2x^2 + a^2(m^2x^2 + 2mnx + n^2)}$$

$$\boldsymbol{= (b^2 + a^2m^2)x^2 + 2mna^2x + a^2n^2 = a^2b^2 \Rightarrow (b^2 + a^2m^2)x^2 + 2mna^2x + a^2(n^2 - b^2) = 0}.$$

Next, getting the quarter of the discriminant, we get

$$\boldsymbol{m^2n^2a^4 - (b^2 + a^2m^2)a^2(n^2 - b^2) = 0 \Rightarrow m^2n^2a^2 - (b^2 + a^2m^2)(n^2 - b^2) = 0}$$
$$\boldsymbol{\Rightarrow m^2n^2a^2 - b^2n^2 + b^4 - m^2n^2a^2 + a^2b^2m^2 = b^4 - b^2n^2 + a^2b^2m^2 = 0}$$
$$\boldsymbol{\Rightarrow b^2 - n^2 + a^2m^2 = 0}.$$

And we know the fact that $\boldsymbol{Q}$ passes through $\boldsymbol{(s, t)}$, and $\boldsymbol{Q}$ is $\boldsymbol{y = mx + n}$.
So we get $\boldsymbol{t = ms + n}$. Thus, we get $\boldsymbol{n = t - ms}$.

So we can now get $\boldsymbol{m}$ putting $\boldsymbol{n}$ into the other equation. Then, we get

$$\boldsymbol{b^2 - n^2 + a^2m^2 = b^2 - (t - ms)^2 + a^2m^2 = 0}$$
$$\boldsymbol{\Rightarrow b^2 - t^2 + 2stm - s^2m^2 + a^2m^2 = m^2(a^2 - s^2) + 2stm + b^2 - t^2 = 0}, \text{ which is quadratic.}$$

So using the quadratic formula, we get $\boldsymbol{m = \dfrac{-st \pm \sqrt{s^2b^2 + a^2t^2 - a^2b^2}}{a^2 - s^2}}$.

And thus, we get two slopes. One is the slope of $\boldsymbol{Q}$, and the other is the slope of $\boldsymbol{R}$.

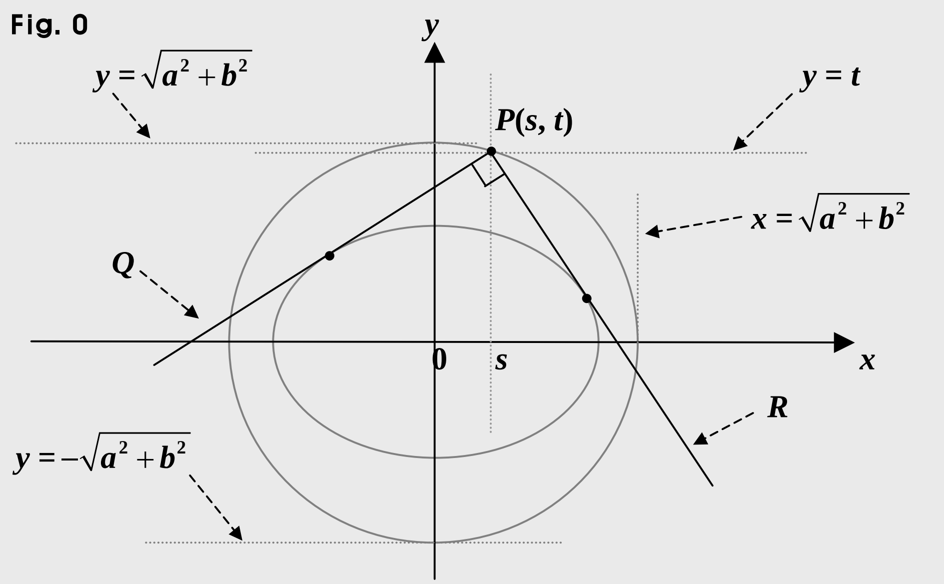
One is $\boldsymbol{\dfrac{-st + \sqrt{s^2b^2 + a^2t^2 - a^2b^2}}{a^2 - s^2}}$. And the other is $\boldsymbol{\dfrac{-st - \sqrt{s^2b^2 + a^2t^2 - a^2b^2}}{a^2 - s^2}}$.

Suggestions or Solutions
To the Problem in the Example 1

A point $P(s, t)$ is in a circle $x^2 + y^2 = a^2 + b^2$, an ellipse $x^2/a^2 + y^2/b^2 = 1$ is inside the circle, and two lines Q and R meet at the point P and are tangent to the ellipse. Then, show that the two tangent lines Q and R are perpendicular to each other.

Assuming first, $a > b$, and putting in a graph the two tangent lines, the point P, the circle, and the ellipse, we can put them the way below.

Fig. 0



And we have a fact that if the product of the slopes of two lines is –1, the two lines are perpendicular to each other. So we can use the fact above.

And in fact, in the example 0 above, we have found the slopes of the two lines Q and R, since the two lines pass through the point P**(s, t)** outside the ellipse, and are tangent to the ellipse.

And the equation for the two slopes is $m^2(a^2 - s^2) + 2stm + b^2 - t^2 = 0$.

And we can put it this way: $m^2 + \dfrac{2stm}{a^2 - s^2} + \dfrac{b^2 - t^2}{a^2 - s^2} = 0$.

So showing $\dfrac{b^2 - t^2}{a^2 - s^2} = -1$, we show that the product of the two slopes is –1. Why?

We have $(x - u)(y - v) = 0 \Rightarrow x^2 - (u + v)x + uv = 0$. And the two roots are u and v.

So assuming $ax^2 + bx + c = 0$, we get $x^2 + (b/a)x + c/a = 0$, and thus, the sum of the two roots is $-b/a$, and the product of the two roots is c/a.

How then, can we show this: $\dfrac{b^2 - t^2}{a^2 - s^2} = -1$?

We know that the point $P(s, t)$ is in the circle $x^2 + y^2 = a^2 + b^2$.

So we get $s^2 + t^2 = a^2 + b^2 \Rightarrow b^2 - t^2 = s^2 - a^2 = -(a^2 - s^2) \Rightarrow b^2 - t^2 = -(a^2 - s^2)$.

Thus, we get $\dfrac{b^2 - t^2}{a^2 - s^2} = -1$.

Let's this time though, actually take the product of the slopes, and see if it is –1.

We know that the slopes are the two roots of the equation as follows.
$m^2(a^2 - s^2) + 2stm + b^2 - t^2 = 0$.

And using the quadratic formula, we get $m = \dfrac{-st \pm \sqrt{s^2b^2 + a^2t^2 - a^2b^2}}{a^2 - s^2}$.

To begin with, we have an identity, $(A + B)(A - B) = A^2 - B^2$.
So taking the product, we get

$$\frac{-st + \sqrt{s^2b^2 + a^2t^2 - a^2b^2}}{a^2 - s^2} \cdot \frac{-st - \sqrt{s^2b^2 + a^2t^2 - a^2b^2}}{a^2 - s^2} = \frac{s^2t^2 - (s^2b^2 + a^2t^2 - a^2b^2)}{(a^2 - s^2)^2}$$

Meanwhile, $s^2t^2 - (s^2b^2 + a^2t^2 - a^2b^2) = s^2t^2 - s^2b^2 - a^2t^2 + a^2b^2$
$= a^2(b^2 - t^2) - s^2(b^2 - t^2) = (b^2 - t^2)(a^2 - s^2)$.

So we get $\dfrac{s^2t^2-(s^2b^2+a^2t^2-a^2b^2)}{(a^2-s^2)^2}=\dfrac{(b^2-t^2)(a^2-s^2)}{(a^2-s^2)^2}=\dfrac{b^2-t^2}{a^2-s^2}$.

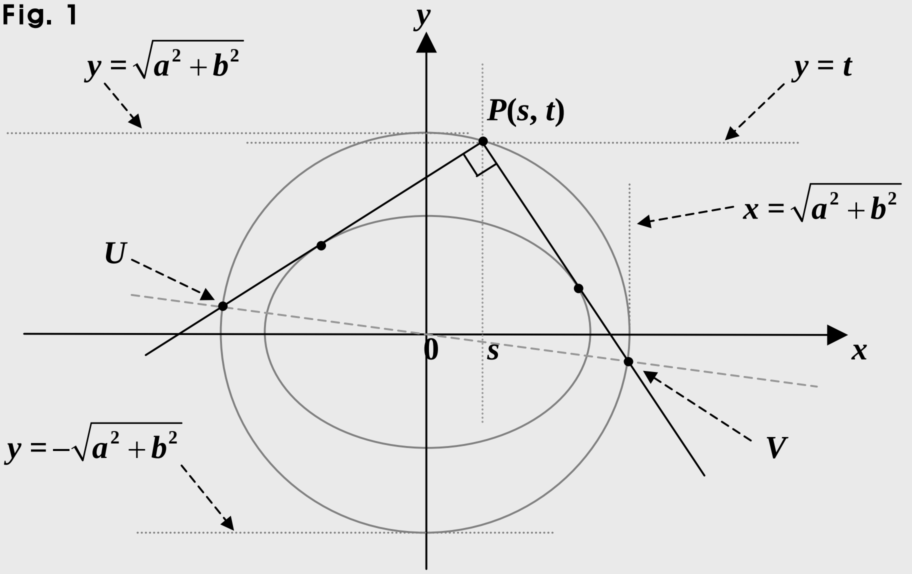
And since the point $P(s, t)$ is in the circle $x^2 + y^2 = a^2 + b^2$, we get $\dfrac{b^2-t^2}{a^2-s^2}=-1$.

That's because $s^2 + t^2 = a^2 + b^2 \Rightarrow b^2 - t^2 = s^2 - a^2 = -(a^2 - s^2) \Rightarrow b^2 - t^2 = -(a^2 - s^2)$.

Consequently, the two tangent lines Q and R are perpendicular to each other.

Suppose now, that U and V are two points where the two tangent lines meet the circle again respectively as shown below.

Fig. 1



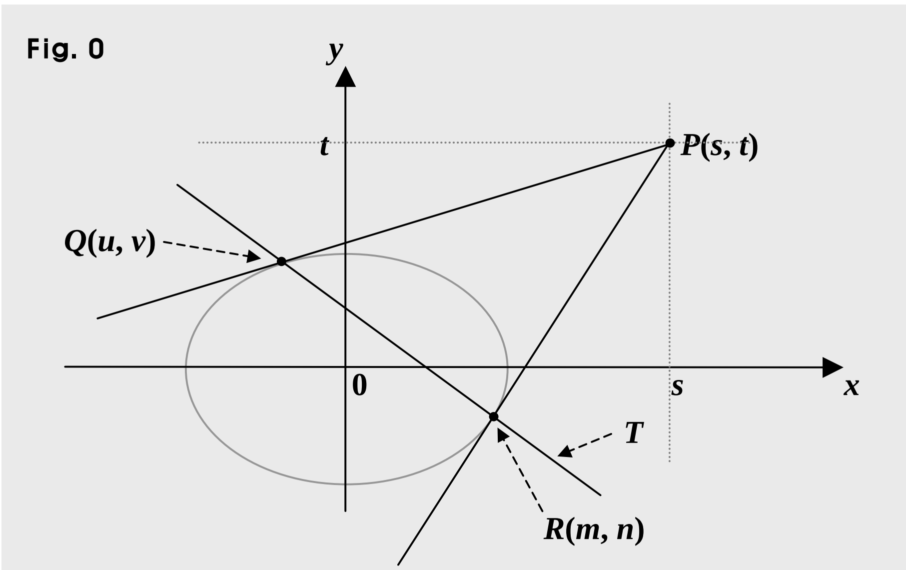
Then, if the line segment UV passes through the origin, the line segment is the diameter of the circle.
A point in a circle and two endpoints of a line segment that is the diameter of the circle constitute a right triangle.

And since the line segment UV is a diameter and passes through the origin, U and V are symmetric about the center, which is the origin.
So the sum of the x-coordinates at U and V is 0, and so is the sum of the y-coordinates.

Suggestions or Solutions
To the Problem in the Example 2

Assuming a point $P(s, t)$ is outside an ellipse $x^2/a^2 + y^2/b^2 = 1$, and two lines meet at the point P, and are tangent to the ellipse at two points Q and R, show that the equation of the line connecting the two tangent points Q and R is $sx/a^2 + ty/b^2 = 1$. (In this case, the point P is called the pole of the line connecting Q and R, and the line is called the polar of the point P.)

Assuming first, $a > b$, and putting in a graph the two tangent lines, the points P, Q and R, and the ellipse, we can put them the way below.



Assuming next, Q is at (u, v), and R is at (m, n), we can say that
the line tangent to the ellipse at Q is $ux/a^2 + vy/b^2 = 1$.
The line tangent to the ellipse at R is $mx/a^2 + ny/b^2 = 1$.

Next, the two lines above pass through the point $P(s, t)$.
So we get $us/a^2 + vt/b^2 = 1$, and $ms/a^2 + nt/b^2 = 1$.

Assuming thus, T is the line connecting Q and R, we can say that T is $sx/a^2 + ty/b^2 = 1$, which is the polar of the point P, and passes through the two tangent points Q and R. That's because there is only one line that can pass through two particular points.