Examples 9 in Ellipses

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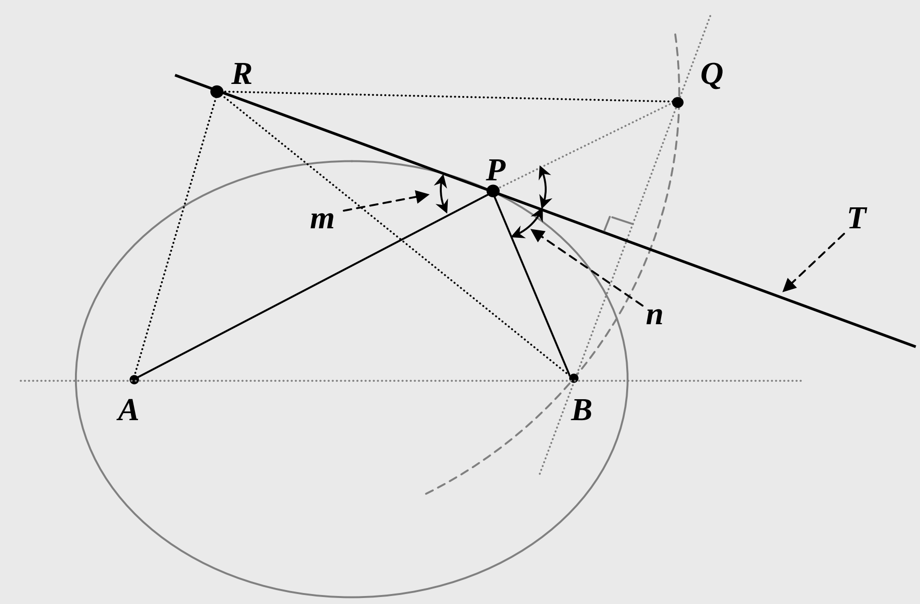
Suppose $\boldsymbol{P}$ is a point in an ellipse, $\boldsymbol{A}$ and $\boldsymbol{B}$ are the two foci, and $\boldsymbol{T}$ is a line tangent to the ellipse at the point $\boldsymbol{P}$. Suppose also, $\boldsymbol{m}$ and $\boldsymbol{n}$ are acute angles, $\boldsymbol{m}$ is an angle between the line segment $\boldsymbol{PA}$ and the tangent line $\boldsymbol{T}$, and $\boldsymbol{n}$ is an angle between the line segment $\boldsymbol{PB}$ and the line $\boldsymbol{T}$. Then, show that $\boldsymbol{m} = \boldsymbol{n}$.

Suggestions or Solutions
To the Problem in the Example

Suppose *P* is a point in an ellipse, *A* and *B* are the foci, and *T* is a line tangent to the ellipse at *P*. Suppose also, *m* and *n* are two acute angles, *m* is an angle between *PA* and *T*, and *n* is an angle between *PB* and *T*. Then, show that *m* = *n*.

As shown below, suppose ***Q*** is a point outside the ellipse, and ***R*** is a point moving along the tangent line ***T***. Suppose also, the sum of two distances ***RA*** and ***RB*** is always the same as the sum of two distances ***RA*** and ***RQ***. So though ***R*** is moving along the line ***T***, ***B*** and ***Q*** are always in a circle centered at ***R***.

Fig. 0

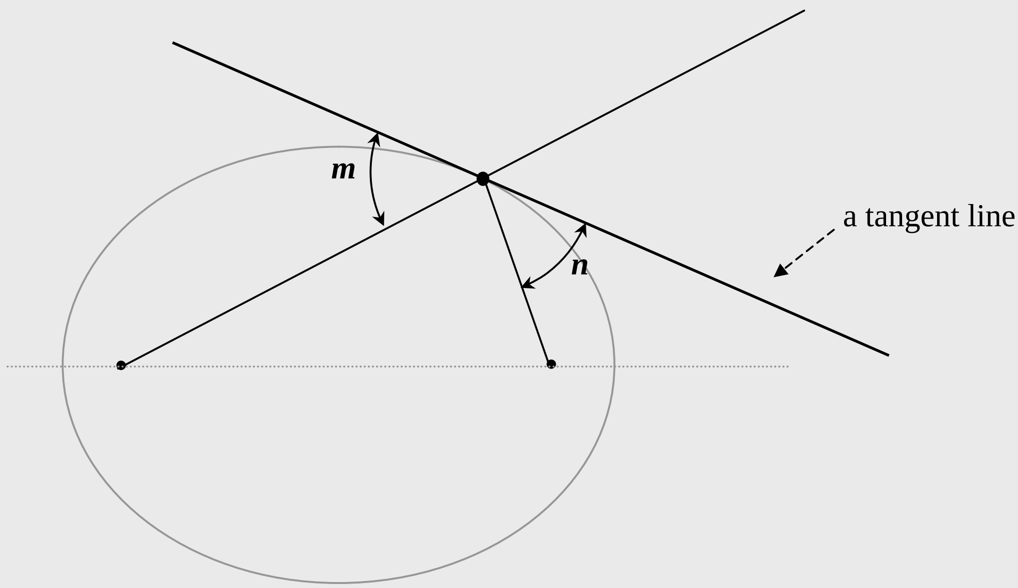


Then, ***A***, ***P***, and ***Q*** are in a line, and ***B*** and ***Q*** are symmetric about the tangent line ***T***, which is thus, perpendicular to ***BQ***, and bisects the angle ***BPQ***. So the triangle ***BPQ*** is an isosceles triangle.

Next, the angle ***m*** and the angle between ***PQ*** and ***T*** are opposite angles, so both angles are the same. And we know that the angle ***n*** is the same as the angle between ***PQ*** and ***T***. So the angle ***m*** is the same as the angle ***n***. The solution above is quite succinct, and is said to have been produced by Akiyama Hitoshi, a Japanese mathematician.

And if ***m*** is called an angle of incidence, ***n*** can be called the angle of refraction.
So in an ellipse, the angle of inclination is the same as the angle of refraction.

Fig. 1

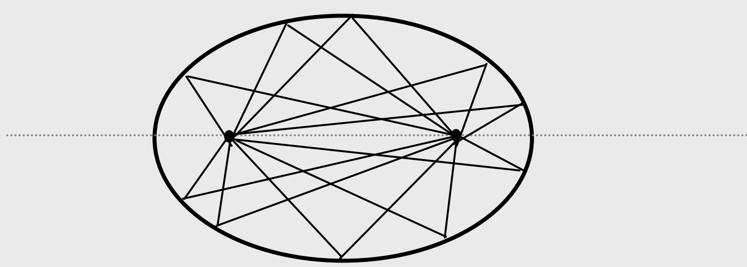


So an ellipse has a physical property as follows.

If for instance, a light beam comes from one focus, and reflects off the ellipse, it goes to the other focus. So all the light beams coming from one focus and reflecting off the ellipse gather at the other focus.

So we can use such a property constructing a reflecting telescope.

Fig. 2



And also, there can be different ways to the solution, of course. If you like complex calculations, you might like to refer to the method as follows. The method uses a trig-ratio, **tan**, and uses trigonometric identities as follows.

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B},\quad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B},\quad \text{and}\quad \tan(\pi-\theta)=-\tan\theta.$$

Basically, $\tan\theta$ is a ratio of the opposite to the adjacent in a right triangle if the angle between the hypotenuse and the adjacent is θ.

So it is quite useful when we indicate a slope of a line. We can indicate the slope with the tangent of the angle the line makes against the x-axis.

And in the x-y plane, the difference between the slopes of two lines can be indicated by the absolute value of the tangent of the angle between the two lines.

If two lines are parallel to each other, the tangent of the angle is 0, because the angle between the two lines is 0, and the tangent of 0 is 0.
If however, the angle between two lines is not 0, the angle is the difference between the angles that the two lines make respectively against the x-axis.

Fig. 2

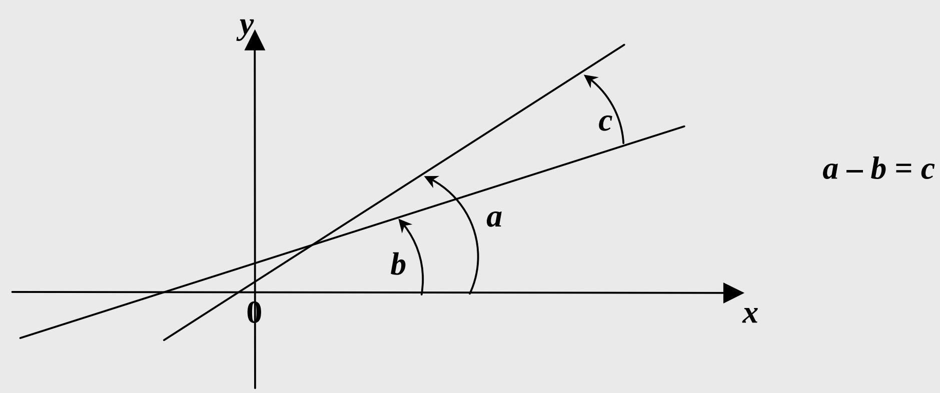
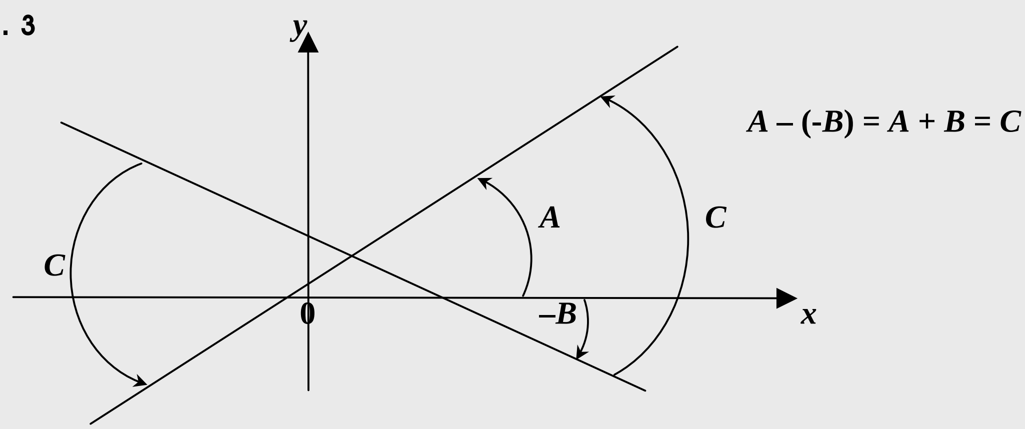
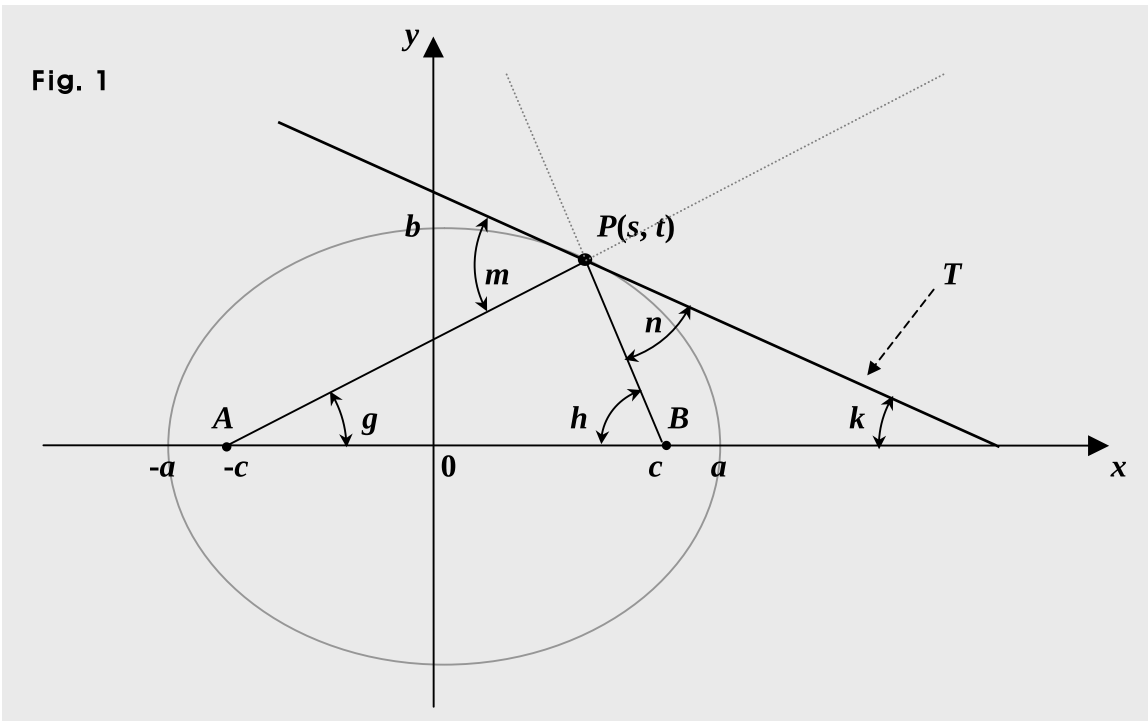


Fig. 3



Let's see now, how we can get $\boldsymbol{m = n}$ in this example, using the idea of the ratio, called the tangent.

So suppose first, the ellipse is $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$, which is centered at the origin. And assuming next, $\boldsymbol{g}$ is the angle ***PAB***, $\boldsymbol{h}$ is the angle ***PBA***, $\boldsymbol{k}$ is the angle between $\boldsymbol{T}$ and the $\boldsymbol{x}$-axis, and the two foci are $\boldsymbol{A(-c, 0)}$ and $\boldsymbol{B(c, 0)}$, we can put the problem in a graph the way as follows.



Then, we get $\mathbf{tan}\ \boldsymbol{m} = \mathbf{tan}(\boldsymbol{g + k})$ and $\mathbf{tan}\ \boldsymbol{n} = \mathbf{tan}(\boldsymbol{h - k})$. And by the trigonometric identities, we get

$$\tan m = \tan(g+k) = \frac{\tan g + \tan k}{1 - \tan g \tan k} \quad \text{and} \quad \tan n = \tan(h-k) = \frac{\tan h - \tan k}{1 + \tan h \tan k}$$

Assuming next, $\boldsymbol{u}$ is the slope of ***AP***, $\boldsymbol{v}$ is that of ***BP***, and $\boldsymbol{w}$ is that of $\boldsymbol{T}$, we can get

$$u = \frac{t-0}{s-(-c)} = \frac{t}{s+c}, \quad v = \frac{t-0}{s-c} = \frac{t}{s-c}, \quad \text{and} \quad w = -\frac{sb^2}{ta^2},$$ which was found in the example in **Examples 7 in Ellipses**.

And we have $\tan(\pi - \theta) = -\tan\theta$.

So we get $\tan g = u$, $\tan h = -v$, and $\tan k = -w$.

That's because $\tan h = -\tan(\pi - h) = -v$, and $\tan k = -\tan(\pi - k) = -w$.

Thus, we get $\tan m = \dfrac{u-w}{1+uw}$ and $\tan n = \dfrac{-v+w}{1+vw}$.

So next, beginning with $\tan m$, we can get first,

$$u - w = \frac{t}{s+c} - \left(-\frac{sb^2}{ta^2}\right) = \frac{t}{s+c} + \frac{sb^2}{ta^2} = \frac{t^2a^2 + sb^2(s+c)}{ta^2(s+c)}.$$

$$1 - uw = 1 + \frac{t}{s+c}\cdot\frac{sb^2}{ta^2} = 1 + \frac{sb^2}{a^2(s+c)} = \frac{a^2s + a^2c - sb^2}{a^2(s+c)} = \frac{a^2c + s(a^2 - b^2)}{a^2(s+c)} = \frac{a^2c + sc^2}{a^2(s+c)},$$

because we have $c^2 = a^2 - b^2$, since c is the focal distance.

So we get $\tan m = \dfrac{u-w}{1+uw} = \dfrac{t^2a^2 + sb^2(s+c)}{ta^2(s+c)}\cdot\dfrac{a^2(s+c)}{a^2c + sc^2} = \dfrac{t^2a^2 + s^2b^2 + csb^2}{t(a^2c + sc^2)}$.

Next, we know that $P(s, t)$ is in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

So we get $s^2/a^2 + t^2/b^2 = 1 \Rightarrow s^2b^2 + t^2a^2 = a^2b^2 \Rightarrow s^2b^2 + t^2a^2 + csb^2 = a^2b^2 + csb^2$.

Thus, we get $\tan m = \dfrac{t^2a^2 + s^2b^2 + csb^2}{t(a^2c + sc^2)} = \dfrac{a^2b^2 + csb^2}{t(a^2c + sc^2)} = \dfrac{b^2(a^2 + cs)}{ct(a^2 + cs)} = \dfrac{b^2}{ct}$.

So we get $\tan m = \dfrac{b^2}{ct}$.

And next, moving on to **tan *n***, we have $\mathbf{tan}\,\boldsymbol{n} = \dfrac{-v+w}{1+vw}$. Then, we can get first,

$$-v+w = -(\frac{t}{s-c} + \frac{sb^2}{ta^2}) = -\frac{t^2a^2 + sb^2(s-c)}{ta^2(s-c)}.$$

$$1+vw = 1-\frac{t}{s-c}\cdot\frac{sb^2}{ta^2} = 1-\frac{sb^2}{a^2(s-c)} = \frac{a^2s - a^2c - sb^2}{a^2(s-c)} = \frac{s(a^2-b^2)-a^2c}{a^2(s-c)} = \frac{sc^2-a^2c}{a^2(s-c)},$$

since we have $\boldsymbol{c^2 = a^2 - b^2}$, where $\boldsymbol{c}$ is the focal distance.

So we get

$$\tan m = \frac{-v+w}{1+vw} = -\frac{t^2a^2+sb^2(s-c)}{ta^2(s-c)}\cdot\frac{a^2(s-c)}{sc^2-a^2c} = -\frac{t^2a^2+s^2b^2-csb^2}{t(sc^2-a^2c)} = \frac{t^2a^2+s^2b^2-csb^2}{t(a^2c-sc^2)}.$$

Next, we know that ***P*(s, t)** is in the ellipse $\boldsymbol{x^2/a^2 + y^2/b^2 = 1}$.

So we get $\boldsymbol{s^2/a^2 + t^2/b^2 = 1 \Rightarrow s^2b^2 + t^2a^2 = a^2b^2 \Rightarrow s^2b^2 + t^2a^2 - csb^2 = a^2b^2 - csb^2}$.

Thus, we get $\tan n = \dfrac{t^2a^2+s^2b^2-csb^2}{t(a^2c-sc^2)} = \dfrac{a^2b^2-csb^2}{t(a^2c-sc^2)} = \dfrac{b^2(a^2-cs)}{ct(a^2-cs)} = \dfrac{b^2}{ct}$.

So we get $\boldsymbol{\tan m = \tan n \Rightarrow m = n}$.

And in fact, we get $\boldsymbol{\tan m = \tan n = |b^2/ct|}$, because ***m*** and ***n*** both are acute angles.

What if however, the tangent point ***P*(*s*, *t*)** is right above or below the focus?

Then, the line segment connecting the focus and ***P*** is perpendicular to the ***x***-axis, and therefore, does not have a slope.
That's because a vertical line has no slope, because a slope cannot be defined for a vertical line, because a slopes is rise over run, but in this case, the run is 0, which is the denominator. No denominator can be 0.

So in the cases where $s = \pm c$, we cannot directly apply the calculations above as is. What then, can we do?

In such a case, we can find **tan** m and **tan** n finding the actual values of t when $s = \pm c$, and using the values in the final expression above. So let's now, find first, the values of t.

We know $P(s, t)$ is in the ellipse, and if P is right above or below the foci, we get $s = \pm c$.

So we get $x^2/a^2 + y^2/b^2 = 1 \Rightarrow s^2/a^2 + t^2/b^2 = 1 \Rightarrow c^2/a^2 + t^2/b^2 = 1 \Rightarrow c^2b^2 + t^2a^2 = a^2b^2$

$\Rightarrow t^2a^2 = a^2b^2 - c^2b^2 = b^2(a^2 - c^2) = b^4$, since $c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$.

So we get $t^2a^2 = b^4 \Rightarrow t^2 = b^4/a^2 \Rightarrow t = \pm b^2/a$.

And thus, when P is right above or below the focus, we get $P(-c, \pm b^2/a)$.

Now, we have $\tan m = \tan n = |b^2/ct|$.   So we get

$|b^2/(ct)| = |b^2/(\pm cb^2/a)| = |1/(\pm c/a)| = |\pm a/c| = |a/c| = a/c$, since a and c are lengths.

Therefore, if P is $P(\pm c, \pm b^2/\mathrm{a})$, we get $\tan m = \tan n = a/c$.

And in fact, the value above is the reciprocal of the eccentricity, because we have $e = c/a$, where e is the eccentricity.