Examples 4 in Hyperbolas

Click this to start.

Examples 4 in Hyperbolas

Find if each hyperbola below is horizontal or vertical, the center, foci, vertices, transverse axis, conjugate axis, asymptotes, eccentricity, and directrices.

0. $\mathbf{25(y - 2)^2 - 16(x - 1)^2 = 400}$

1. $\dfrac{(x-1)^2}{4} - y^2 = 1$

2. $\mathbf{4(x - 2)^2 - (y - 1)^2 = -4}$

3. $\mathbf{1 + 16(y - 1)^2 - 25(x - 2)^2 = 0}$

Suggestions or Solutions
To the Problem in the Example 0

Find all the elements of the hyperbola as follows. $\mathbf{25(y-2)^2 - 16(x-1)^2 = 400}$.

To begin with, we can put the hyperbola given this way: $\dfrac{(y-2)^2}{4^2} - \dfrac{(x-1)^2}{5^2} = 1$.

So the hyperbola is vertical, the center is (1, 2), the transverse axis is 8, and the conjugate axis is 10.

Next, assuming c is the focal distance, b is half the transverse axis, and a is half the conjugate axis, we get $b = 4$, $a = 5$, and $c^2 = a^2 + b^2 \Rightarrow c^2 = 25 + 16 = 41 \Rightarrow c = \sqrt{41}$.

So the focal distance is $\sqrt{41}$, and the foci are $(1, 2+\sqrt{41})$ and $(1, 2-\sqrt{41})$.
And the vertices are (1, 6) and (1, -2).

Next, the asymptotes are $y - 2 = \pm(b/a)(x-1) \Rightarrow y - 2 = \pm\frac{4}{5}(x-1)$.

Next, assuming e is the eccentricity, we get $e = c/b = \frac{\sqrt{41}}{4}$.

And next, the directrices are $y = \pm b/e + 2 = \pm b^2/c + 2 = \pm\frac{16\sqrt{41}}{41} + 2$.

If not quite sure of the idea behind the solution above, follow the steps below.

To begin with, if a hyperbola is centered at (u, v), and is horizontal, the equation is $\dfrac{(x-u)^2}{a^2} - \dfrac{(y-v)^2}{b^2} = 1$, where a is half the transverse axis, and b is half the conjugate axis. And if vertical, its equation is $\dfrac{(y-v)^2}{b^2} - \dfrac{(x-u)^2}{a^2} = 1$, where b is half the transverse axis, and a is half the conjugate axis.

Next, we can put the hyperbola given the way below.

$$25(y-2)^2 - 16(x-1)^2 = 400 \Rightarrow \frac{25}{400}(y-2)^2 - \frac{16}{400}(x-1)^2 = 1$$

$$\Rightarrow \frac{5^2}{20^2}(y-2)^2 - \frac{4^2}{20^2}(x-1)^2 \Rightarrow \frac{1}{16}(y-2)^2 - \frac{1}{25}(x-1)^2 = 1 \Rightarrow \frac{(y-2)^2}{4^2} - \frac{(x-1)^2}{5^2} = 1.$$

So the hyperbola given is vertical, the center is (1, 2), the transverse axis is 8, and the conjugate axis is 10. What then, about the focal distance?

If $\boldsymbol{c}$ is the focal distance, $\mathbf{2}\boldsymbol{b}$ is the transverse axis, and $\mathbf{2}\boldsymbol{a}$ is the conjugate axis, we get $\boldsymbol{c^2 = a^2 + b^2}$. So we get $\boldsymbol{c^2 = 25 + 16 = 41 \Rightarrow c = \sqrt{41}}$.

Next, the center, foci, and vertices are all in the transverse axis. So if the hyperbola is <u>vertical</u>, the center, foci, and vertices share the <u>same $\boldsymbol{x}$-coordinate</u>.

The center is the midpoint between the foci, and the focal distance is the distance from the center to each focus, and is $\boldsymbol{c}$, which is $\boldsymbol{\sqrt{41}}$.

And also, the center is the midpoint between the <u>vertices</u>, too, which are the <u>endpoints</u> of the <u>transverse axis</u>, which is twice the distance from the center to each vertex, and the distance in this case, is $\boldsymbol{b}$, which is 4.

So since the center is (1, 2), and the hyperbola is vertical, the foci are $\mathbf{(0+1, \sqrt{41}+2)}$ and $\mathbf{(0+1, 2-\sqrt{41})}$, that is, $\mathbf{(1, \sqrt{41}+2)}$ and $\mathbf{(1, 2-\sqrt{41})}$, and the vertices are (0 + 1, 4 + 2) and (0 + 1, -4 + 2), that is, **(1, 6)** and **(1, -2)**.

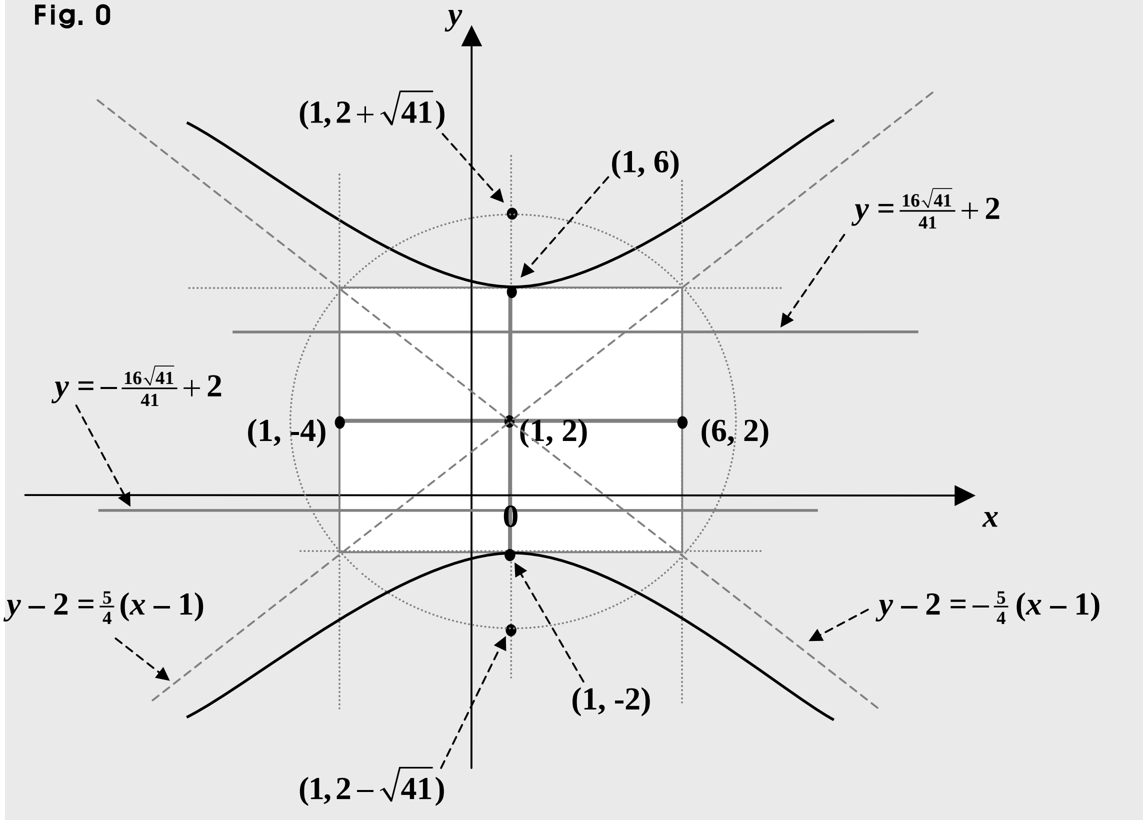
Next, a hyperbola has two lines called the <u>asymptotes</u>, the slope of one is $\boldsymbol{b/a}$, and the other is $\boldsymbol{-b/a}$. And the asymptotes pass through the center, which is (1, 2) in this case. So the asymptotes are $\boldsymbol{y - 2 = \pm(b/a)(x - 1) \Rightarrow y - 2 = \pm\frac{5}{4}(x - 1)}$.

Next, the <u>eccentricity</u> of a hyperbola is a ratio, <u>the focal distance over half the transverse axis</u>. So assuming $\boldsymbol{e}$ is the eccentricity, we get $\boldsymbol{e = c/b = \frac{\sqrt{41}}{4}}$.

And next, a hyperbola has two lines called the <u>directrices</u>, and the distance from each to the center is a ratio, <u>half the transverse over the eccentricity</u>.
So since the center is (1, 2), and the hyperbola is <u>vertical</u>, the directrices are as follows.

$$\boldsymbol{y = \pm b/e + 2 = \pm b^2/c + 2 = \pm\frac{16}{\sqrt{41}} + 2 = \pm\frac{16\sqrt{41}}{41} + 2}.$$

Fig. 0



The hyperbola is $\frac{(y-2)^2}{16} - \frac{(x-1)^2}{25} = 1$, often put this way: $\frac{(y-2)^2}{4^2} - \frac{(x-1)^2}{5^2} = 1$.

Note that half the transverse axis is called the semi major axis, too, and half the conjugate axis is called the semi minor axis.

Suggestions or Solutions
To the Problem in the Example 1

Find all the elements of the hyperbola as follows. $\frac{(x-1)^2}{4} - y^2 = 1.$

To begin with, the hyperbola is horizontal, the center is (1, 0), the transverse axis is 4, and the conjugate axis is 2.

Next, assuming c is the focal distance, a is half the transverse axis, and b is half the conjugate axis, we get $a = 2$, $b = 1$, and $c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 1 = 5 \Rightarrow c = \sqrt{5}$.

So the focal distance is $\sqrt{5}$, and the foci are $(1-\sqrt{5}, 0)$ and $(1+\sqrt{5}, 0)$.
And the vertices are (-1, 0) and (3, 0).

Next, the asymptotes are $y = \pm(b/a)(x - 1) = \pm\frac{1}{2}(x - 1)$.

Next, assuming e is the eccentricity, we get $e = c/a = \frac{\sqrt{5}}{2}$.

And next, the directrices are $x = \pm a/e + 1 = \pm a^2/c + 1 = \pm\frac{4\sqrt{5}}{5} + 1$.

If not quite sure of the idea behind the solution above, follow the steps below.

To begin with, if a hyperbola is centered at $(u, 0)$, and is horizontal, the equation is $\frac{(x-u)^2}{a^2} - \frac{y^2}{b^2} = 1$, where a is half the transverse axis, and b is half the conjugate axis.

And if vertical, its equation is $\frac{y^2}{b^2} - \frac{(x-u)^2}{a^2} = 1$, where b is half the transverse axis, and a is half the conjugate axis.

Next, we can put the hyperbola given this way: $\frac{(x-1)^2}{2^2} - \frac{y^2}{1^2} = 1.$

So the hyperbola given is horizontal, the center is (1, 0), the transverse axis is 4, and the conjugate axis is 2. What then, about the focal distance?

If c is the focal distance, $2a$ is the transverse axis, and $2b$ is the conjugate axis, we get $c^2 = a^2 + b^2$. So we get $c^2 = 4 + 1 = 5 \Rightarrow c = \sqrt{5}$.

Next, the center, foci, and vertices are all in the transverse axis. So if the hyperbola is horizontal, the center, foci, and vertices share the same ***y***-coordinate.

The center is the midpoint between the foci, and the focal distance is the distance from the center to each focus, and is ***c***, which is $\sqrt{5}$.

And also, the center is the midpoint between the vertices, too, which are the endpoints of the transverse axis, which is twice the distance from the center to each vertex, and the distance in this case, is ***a***, which is 2.

So since the center is (1, 0), and the hyperbola is horizontal, the foci are $(1-\sqrt{5}, 0)$ and $(1+\sqrt{5}, 0)$, and the vertices are (-2 + 1, 0) and (2 + 1, 0), that is, **(-1, 0)** and **(3, 0)**.

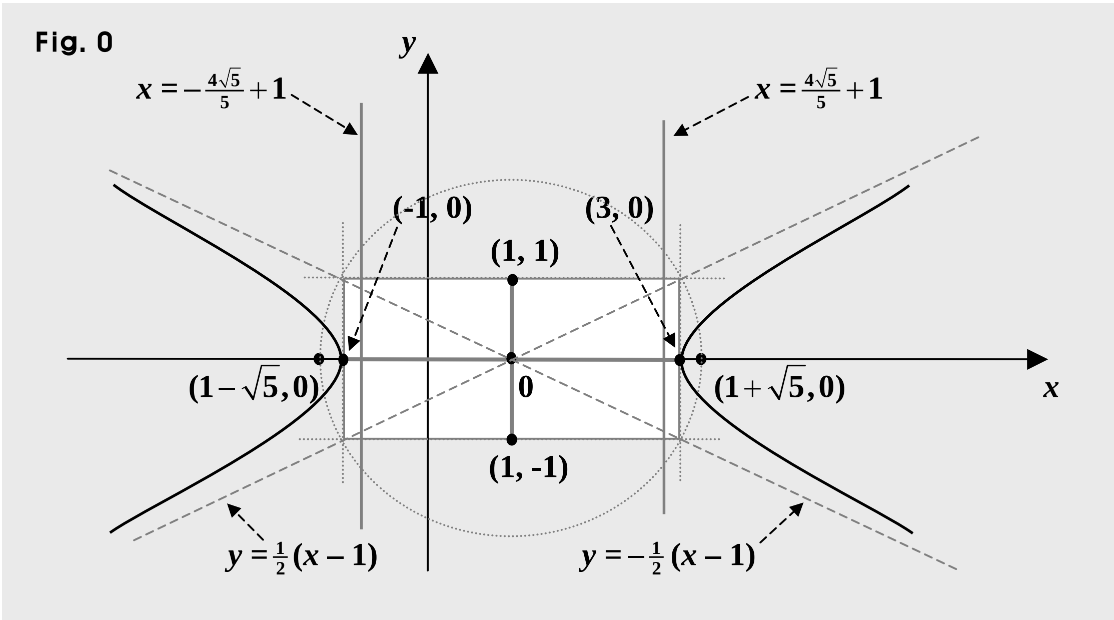
Next, a hyperbola has two lines called the asymptotes, the slope of one is b/a, and the other is $-b/a$. And the asymptotes pass through the center, which is (1, 0) in this case. So the asymptotes are $y = \pm(b/a)(x - 1) \Rightarrow y = \pm\frac{1}{2}(x - 1)$.

Next, the eccentricity of a hyperbola is a ratio, the focal distance over half the transverse axis. So assuming ***e*** is the eccentricity, we get $e = c/a = \frac{\sqrt{5}}{2}$.

And next, a hyperbola has two lines called the directrices, and the distance from each to the center is a ratio, half the transverse over the eccentricity. So since the center is (1, 0), and the hyperbola is horizontal, the directrices are as follows.

$$x = \pm a/e + 1 = \pm a^2/c + 1 = \pm\frac{4}{\sqrt{5}} + 1 = \pm\frac{4\sqrt{5}}{5} + 1.$$

Fig. 0



The hyperbola is $\dfrac{(x-1)^2}{4} - y^2 = 1$, often put this way: $\dfrac{(x-1)^2}{2^2} - \dfrac{y^2}{1^2} = 1$.

Note that half the transverse axis is called the semi major axis, too, and half the conjugate axis is called the semi minor axis.

Suggestions or Solutions
To the Problem in the Example 2

Find all the elements of the hyperbola as follows. $\mathbf{4(x-2)^2-(y-1)^2=-4}$.

To begin with, we can put the hyperbola given this way: $\frac{(y-1)^2}{2^2}-\frac{(x-2)^2}{1^2}=1$.

So the hyperbola is vertical, the center is (2, 1), the transverse axis is 4, and the conjugate axis is 2.

Next, assuming c is the focal distance, b is half the transverse axis, and a is half the conjugate axis, we get $b=2$, $a=1$, and $c^2=a^2+b^2\Rightarrow c^2=1+4=5\Rightarrow c=\sqrt{5}$.

So the focal distance is $\sqrt{5}$, and the foci are $(2,1-\sqrt{5})$ and $(2,1+\sqrt{5})$.
And the vertices are (2, 3) and (2, -1).

Next, the asymptotes are $y-1=\pm(b/a)(x-2)\Rightarrow y-1=\pm2(x-2)$.

Next, assuming e is the eccentricity, we get $e=c/b=\frac{\sqrt{5}}{2}$.

And next, the directrices are $y=\pm b/e+1=\pm b^2/c+1=\pm\frac{4\sqrt{5}}{5}+1$.

If not quite sure of the idea behind the solution above, follow the steps below.

To begin with, if a hyperbola is centered at (u, v), and is horizontal, the equation is $\frac{(x-u)^2}{a^2}-\frac{(y-v)^2}{b^2}=1$, where a is half the transverse axis, and b is half the conjugate axis. And if vertical, its equation is $\frac{(y-v)^2}{b^2}-\frac{(x-u)^2}{a^2}=1$, where b is half the transverse axis, and a is half the conjugate axis.

Next, we can put the hyperbola given the way below.

$$4(x-2)^2-(y-1)^2=-4\Rightarrow \frac{1}{4}(y-1)^2-(x-2)^2=1\Rightarrow\frac{(y-1)^2}{2^2}-\frac{(x-2)^2}{1^2}=1.$$

So the hyperbola given is vertical, the center is (2, 1), the transverse axis is 4, and the conjugate axis is 2. What then, about the focal distance?

If $\boldsymbol{c}$ is the focal distance, $\mathbf{2}\boldsymbol{b}$ is the transverse axis, and $\mathbf{2}\boldsymbol{a}$ is the conjugate axis, we get $\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2$. So we get $\boldsymbol{c}^2 = \mathbf{1 + 4 = 5} \Rightarrow \boldsymbol{c} = \sqrt{\mathbf{5}}$.

Next, the center, foci, and vertices are all in the transverse axis. So if the hyperbola is vertical, the center, foci, and vertices share the same $\boldsymbol{x}$-coordinate.

The center is the midpoint between the foci, and the focal distance is the distance from the center to each focus, and is $\boldsymbol{c}$, which is $\sqrt{\mathbf{5}}$.

And also, the center is the midpoint between the vertices, too, which are the endpoints of the transverse axis, which is twice the distance from the center to each vertex, and the distance in this case, is $\boldsymbol{b}$, which is 2.

So since the center is (2, 1), and the hyperbola is vertical, the foci are $\mathbf{(0+2, 1-\sqrt{5})}$ and $\mathbf{(0+2, 1+\sqrt{5})}$, that is, $\mathbf{(2, 1-\sqrt{5})}$ and $\mathbf{(2, 1+\sqrt{5})}$, and the vertices are (0 + 2, -2 + 1) and (0 + 2, 2 + 1), that is, **(2, -1)** and **(2, 3)**.

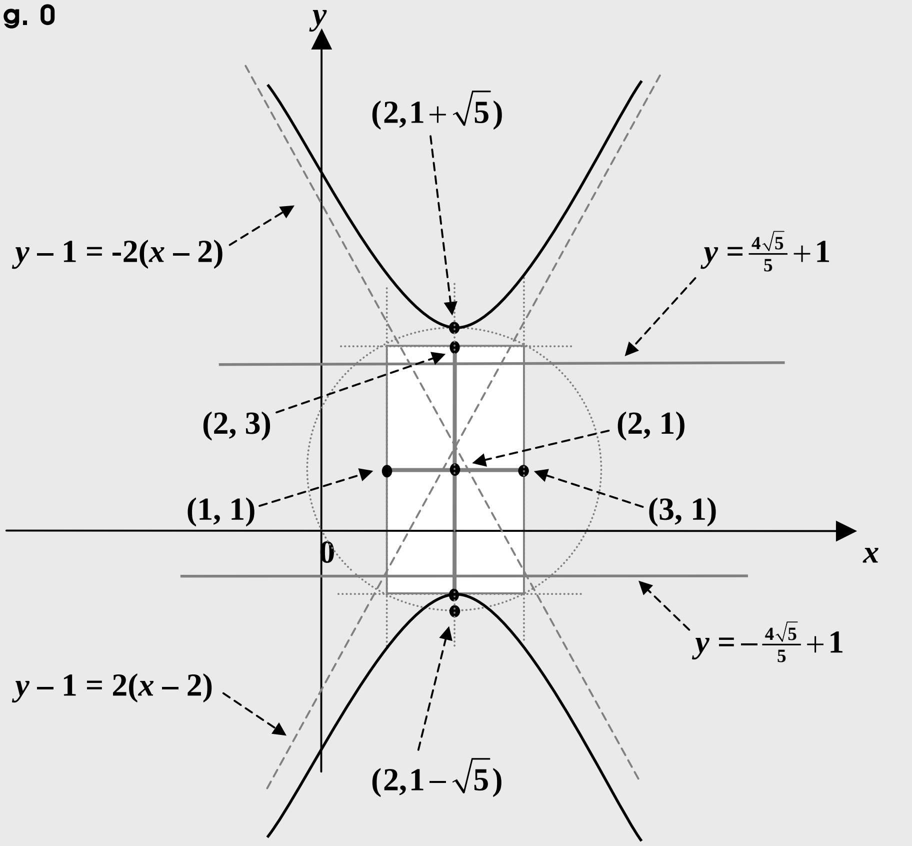
Next, a hyperbola has two lines called the asymptotes, the slope of one is $\boldsymbol{b/a}$, and the other is $\boldsymbol{-b/a}$. And the asymptotes pass through the center, which is (2, 1) in this case. So the asymptotes are $\boldsymbol{y} - \mathbf{1} = \pm(\boldsymbol{b/a})(\boldsymbol{x} - \mathbf{2}) \Rightarrow \boldsymbol{y} - \mathbf{1} = \pm\mathbf{2}(\boldsymbol{x} - \mathbf{2})$.

Next, the eccentricity of a hyperbola is a ratio, the focal distance over half the transverse axis. So assuming $\boldsymbol{e}$ is the eccentricity, we get $\boldsymbol{e} = \boldsymbol{c/b} = \frac{\sqrt{5}}{2}$.

And next, a hyperbola has two lines called the directrices, and the distance from each to the center is a ratio, half the transverse over the eccentricity. So since the center is (2, 1), and the hyperbola is vertical, the directrices are as follows.

$$\boldsymbol{y} = \pm\boldsymbol{b/e} + \mathbf{1} = \pm\boldsymbol{b}^2/\boldsymbol{c} + \mathbf{1} = \pm\frac{4}{\sqrt{5}} + \mathbf{1} = \pm\frac{4\sqrt{5}}{5} + \mathbf{1}.$$

Fig. 0



The hyperbola is $\frac{(y-1)^2}{4} - (x-2)^2 = 1$, often put this way: $\frac{(y-1)^2}{2^2} - \frac{(x-2)^2}{1^2} = 1$.

Note that half the transverse axis is called the semi major axis, too, and half the conjugate axis is called the semi minor axis.

Suggestions or Solutions
To the Problem in the Example 3

Find all the elements of the hyperbola as follows. $\mathbf{1 + 16(y - 1)^2 - 25(x - 2)^2 = 0}$.

To begin with, we can put the hyperbola given this way: $\dfrac{(x-2)^2}{(\frac{1}{5})^2} - \dfrac{(y-1)^2}{(\frac{1}{4})^2} = 1$.

So the hyperbola is horizontal, the center is (2, 1), the transverse axis is $\frac{2}{5}$, and the conjugate axis is $\frac{1}{2}$.

Next, assuming c is the focal distance, a is half the transverse axis, and b is half the conjugate axis, we get $a = \frac{1}{5}$, $b = \frac{1}{4}$, and $c^2 = a^2 + b^2 \Rightarrow c^2 = (\frac{1}{5})^2 + (\frac{1}{4})^2 = \frac{41}{400} \Rightarrow c = \frac{\sqrt{41}}{20}$.

So the focal distance is $\frac{\sqrt{41}}{20}$, and the foci are $(2 + \frac{\sqrt{41}}{20}, 1)$ and $(2 - \frac{\sqrt{41}}{20}, 1)$.

And the vertices are $(\frac{11}{5}, 1)$ and $(\frac{9}{5}, 1)$.

Next, the asymptotes are $y - 2 = \pm(b/a)(x - 1) \Rightarrow y - 2 = \pm\frac{5}{4}(x - 1)$.

Next, assuming e is the eccentricity, we get $e = c/a = \frac{\sqrt{41}}{4}$.

And next, the directrices are $x = \pm a/e + 2 = \pm a^2/c + 2 = \pm\frac{4\sqrt{41}}{205} + 2$.

If not quite sure of the idea behind the solution above, follow the steps below.

To begin with, if a hyperbola is centered at (u, v), and is horizontal, the equation is $\dfrac{(x-u)^2}{a^2} - \dfrac{(y-v)^2}{b^2} = 1$, where a is half the transverse axis, and b is half the conjugate axis. And if vertical, its equation is $\dfrac{(y-v)^2}{b^2} - \dfrac{(x-u)^2}{a^2} = 1$, where b is half the transverse axis, and a is half the conjugate axis.

Next, we can put the hyperbola given the way below.

$$1 + 16(y - 1)^2 - 25(x - 2)^2 = 0 \Rightarrow 25(x - 2)^2 - 16(y - 1)^2 = 1 \Rightarrow \frac{(x-2)^2}{(\frac{1}{5})^2} - \frac{(y-1)^2}{(\frac{1}{4})^2} = 1.$$

So the hyperbola is horizontal, the center is (2, 1), the transverse axis is $\frac{2}{5}$, and the conjugate axis is $\frac{1}{2}$.   What then, about the focal distance?

If ***c*** is the focal distance, **2*a*** is the transverse axis, and **2*b*** is the conjugate axis, we get $\boldsymbol{c^2 = a^2 + b^2}$.   So we get $\boldsymbol{c^2 = (\frac{1}{5})^2 + (\frac{1}{4})^2 = \frac{1}{25} + \frac{1}{16} = \frac{41}{400} \Rightarrow c = \frac{\sqrt{41}}{20}}$.

Next, the center, foci, and vertices are all in the transverse axis. So if the hyperbola is <u>horizontal</u>, the center, foci, and vertices share the <u>same ***y***-coordinate</u>.

The center is the midpoint between the foci, and the focal distance is the distance from the center to each focus, and is ***c***, which is $\frac{\sqrt{41}}{20}$.

And also, the center is the midpoint between the <u>vertices</u>, too, which are the <u>endpoints</u> of the <u>transverse axis</u>, which is twice the distance from the center to each vertex, and the distance in this case, is ***a***, which is $\frac{1}{5}$.

So since the center is (2, 1), and the hyperbola is horizontal, the foci are $\boldsymbol{(\frac{\sqrt{41}}{20} + 2, 0 + 1)}$ and $\boldsymbol{(2 - \frac{\sqrt{41}}{20}, 0 + 1)}$, that is, $\boldsymbol{(\frac{\sqrt{41}}{20} + 2, 1)}$ and $\boldsymbol{(2 - \frac{\sqrt{41}}{20}, 1)}$, and the vertices are $\boldsymbol{(\frac{1}{5} + 2, 0 + 1)}$ and $\boldsymbol{(2 - \frac{1}{5}, 0 + 1)}$, that is, $\boldsymbol{(\frac{11}{5}, 1)}$ and $\boldsymbol{(\frac{9}{5}, 1)}$.

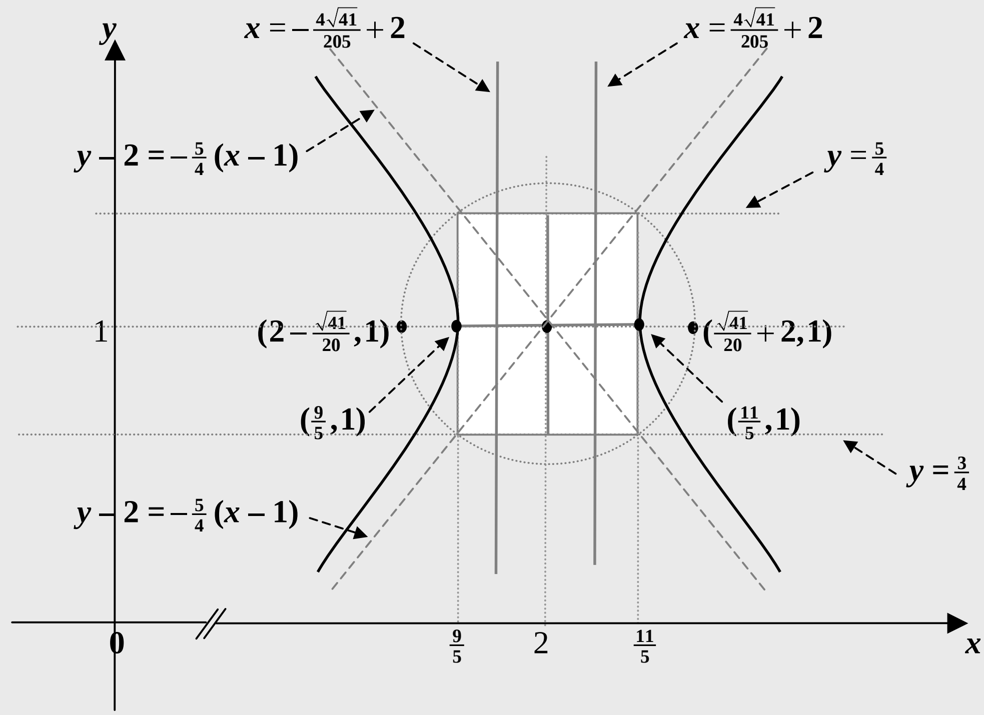
Next, a hyperbola has two lines called the <u>asymptotes</u>, the slope of one is ***b*/*a***, and the other is ***–b*/*a***. And the asymptotes pass through the center, which is (1, 2) in this case. So the asymptotes are $\boldsymbol{y - 2 = \pm(b/a)(x - 1) \Rightarrow y - 2 = \pm\frac{5}{4}(x - 1)}$.

Next, the <u>eccentricity</u> of a hyperbola is a ratio, <u>the focal distance over half the transverse axis</u>. So assuming ***e*** is the eccentricity, we get $\boldsymbol{e = c/a = \frac{\sqrt{41}}{20} \cdot 5 = \frac{\sqrt{41}}{4}}$.

And next, a hyperbola has two lines called the <u>directrices</u>, and the distance from each to the center is a ratio, <u>half the transverse over the eccentricity</u>. So since the center is (2, 1), and the hyperbola is <u>horizontal</u>, the directrices are as follows.

$$\boldsymbol{x = \pm a/e + 2 = \pm a^2/c + 2 = \pm\frac{20}{25\sqrt{41}} + 2 = \pm\frac{4}{5\sqrt{41}} + 2 = \pm\frac{4\sqrt{41}}{5 \cdot 41} + 2 = \pm\frac{4\sqrt{41}}{205} + 2}.$$

Fig. 0



The hyperbola is $\mathbf{25(x-2)^2 - 16(y-1)^2 = 1}$, often put this way: $\dfrac{(x-2)^2}{(\frac{1}{5})^2} - \dfrac{(y-1)^2}{(\frac{1}{4})^2} = \mathbf{1}$.

Note that half the transverse axis is called the semi major axis, too, and half the conjugate axis is called the semi minor axis.