Examples 6 in Hyperbolas

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In each example below, the hyperbola is in the ***x-y*** plane.

0. Find the hyperbola where the foci and vertices are (-1, 3), (4, 3), (-2, 3), and (3, 3).

1. Find the hyperbola that has a point at (-3, 9/4) and has the foci at (-3, 0) and (7, 0).

2. Find the hyperbola that has a point at $\mathbf{(1, 3-2\sqrt{2})}$, and has the vertices at (4, 5) and (4, 1).

3. Find the hyperbola of which the eccentricity is 5/3 and a directrix is a line $\boldsymbol{x} = \mathbf{14/5}$, which is corresponding a focus at (6, 2).

Suggestions or Solutions
To the Problem in the Example 0

Find the hyperbola where the foci and vertices are (-1, 3), (4, 3), (-2, 3), and (3, 3).

To begin with, the hyperbola is horizontal, the vertices are (-1, 3) and (3, 3), the foci are (4, 3) and (-2, 3), and the center is (1, 3).

So next, assuming $\boldsymbol{a}$ is half the transverse axis, and $\boldsymbol{c}$ is the focal distance, we get $\boldsymbol{a} = \mathbf{2}$, and $\boldsymbol{c} = \mathbf{3}$.

Thus, next, assuming $\boldsymbol{b}$ is half the conjugate axis, we get
$\boldsymbol{c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 = 9 - 4 = 5 \Rightarrow b = \sqrt{5}}$.

So the hyperbola is $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{5} = 1$, often put this way: $\dfrac{(x-1)^2}{2^2} - \dfrac{(y-3)^2}{(\sqrt{5})^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, if a hyperbola is centered at $(\boldsymbol{u}, \boldsymbol{v})$, and is horizontal, the equation is $\dfrac{(x-u)^2}{a^2} - \dfrac{(y-v)^2}{b^2} = 1$, where $\boldsymbol{a}$ is half the transverse axis, and $\boldsymbol{b}$ is half the conjugate axis. If vertical though, its equation is $\dfrac{(y-v)^2}{b^2} - \dfrac{(x-u)^2}{a^2} = 1$, where $\boldsymbol{b}$ is half the transverse axis, and $\boldsymbol{a}$ is half the conjugate axis.

And if horizontal, the center, foci, and vertices share the same $\boldsymbol{y}$-coordinate.

Next, we can notice that all the points given have the same $\boldsymbol{y}$-coordinate.
So the hyperbola we want to find is horizontal.

Next, the vertices are between the foci.
So the vertices are (-1, 3) and (3, 3), and the foci are (4, 3) and (-2, 3).
And the transverse axis is the distance between the vertices, and thus, is 4.

What then about the conjugate axis?

Assuming c is the focal distance, $2a$ is the transverse axis, $2b$ is the conjugate axis, we get $c^2 = a^2 + b^2$. And we know $2a = 4$. So we get $a = 2$, and $b^2 = c^2 - a^2 = c^2 - 4$. What then, about c?

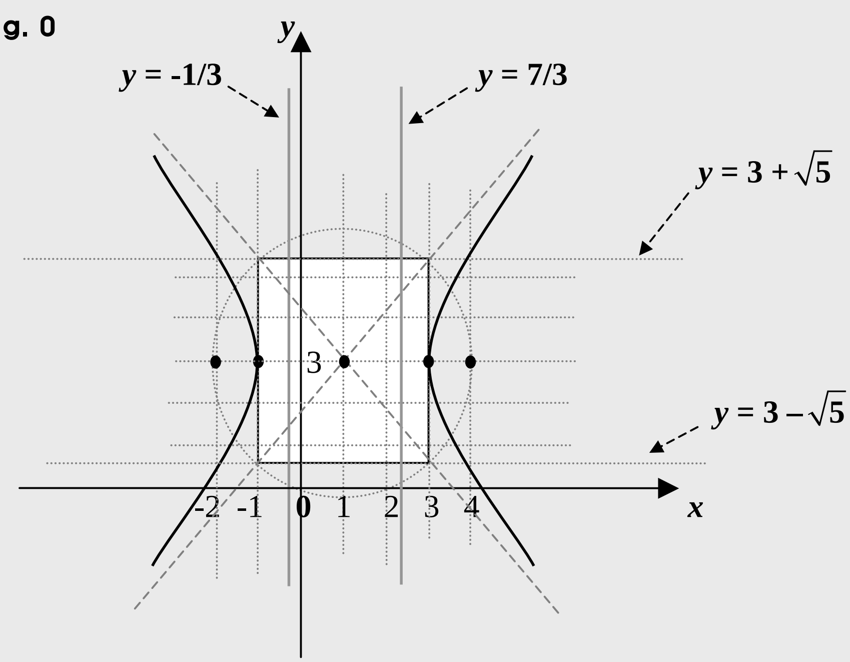
The focal distance is the distance from the center to a focus, which is in this case, (4, 3) or (-2, 3). So we want to get the center.

And the center is the midpoint between the foci.
So the center is {(-2 + 4)/2, (3 + 3)/2}, and thus, is (1, 3). So the focal distance c is 3.

And thus, we get $b^2 = c^2 - 4 = 9 - 4 = 5 \Rightarrow b = \sqrt{5}$.

So the hyperbola is $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{5} = 1$, often put this way: $\dfrac{(x-1)^2}{2^2} - \dfrac{(y-3)^2}{(\sqrt{5})^2} = 1$.

Fig. 0



Suggestions or Solutions
To the Problem in the Example 1

Find the hyperbola that has a point at (-3, 9/4) and has the foci at (-3, 0) and (7, 0).

To begin with, the hyperbola is horizontal, and the center is (2, 0).

So next, assuming the hyperbola is $\frac{(x-u)^2}{a^2} - \frac{(y-v)^2}{b^2} = 1$, we have $u = 2$, and $v = 0$.

Next, assuming d is the difference between two distances from a point in the hyperbola to the foci, we get $d = 2a$. And since the hyperbola has the given point, we can get d the way as follows. $d = \sqrt{(7+3)^2 + (0-\frac{9}{4})^2} - \frac{9}{4} = \sqrt{\frac{1681}{16}} - \frac{9}{4} = \frac{41}{4} - \frac{9}{4} = 8$.

So we get $a = 4$.

Next, assuming c is the focal distance, we get $c = 5$, and $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$
$\Rightarrow b^2 = 25 - 16 = 9 \Rightarrow b = 3$.

So the hyperbola is $\frac{(x-2)^2}{16} - \frac{y^2}{9} = 1$, often put this way, too: $\frac{(x-2)^2}{4^2} + \frac{y^2}{3^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, if a hyperbola is centered at (u, v), and is horizontal, the equation is
$\frac{(x-u)^2}{a^2} - \frac{(y-v)^2}{b^2} = 1$, where a is half the transverse axis, and b is half the conjugate.

And if it's <u>horizontal</u>, the center, foci, and vertices share the <u>same y-coordinate</u>. So since the foci given have the same y-coordinates, the hyperbola we want to find is horizontal.

Next, the center is the midpoint between the foci. So the center is {(0 + 4)/2, (0 + 0)/2}, and thus, is (2, 0).

Next, in a hyperbola, the difference between two distances from a point to the two foci is constant, and is in fact, the length of the transverse axis.

So assuming d is the difference, we get $d = 2a$.

And since the given point (-3, 9/4) is in the hyperbola, the difference between the two distances from the given point to the two foci is **2*a***, too. And the distance from (-3, 9/4) to (-3, 0) is simply 9/4. So finding the difference ***d***, we get

$$\boldsymbol{d} = \sqrt{(7+3)^2 + (0-\tfrac{9}{4})^2} - \tfrac{9}{4} = \sqrt{\tfrac{1681}{16}} - \tfrac{9}{4} = \tfrac{41}{4} - \tfrac{9}{4} = \mathbf{8}.$$

So we get ***a* = 4**. What then, about ***b***?

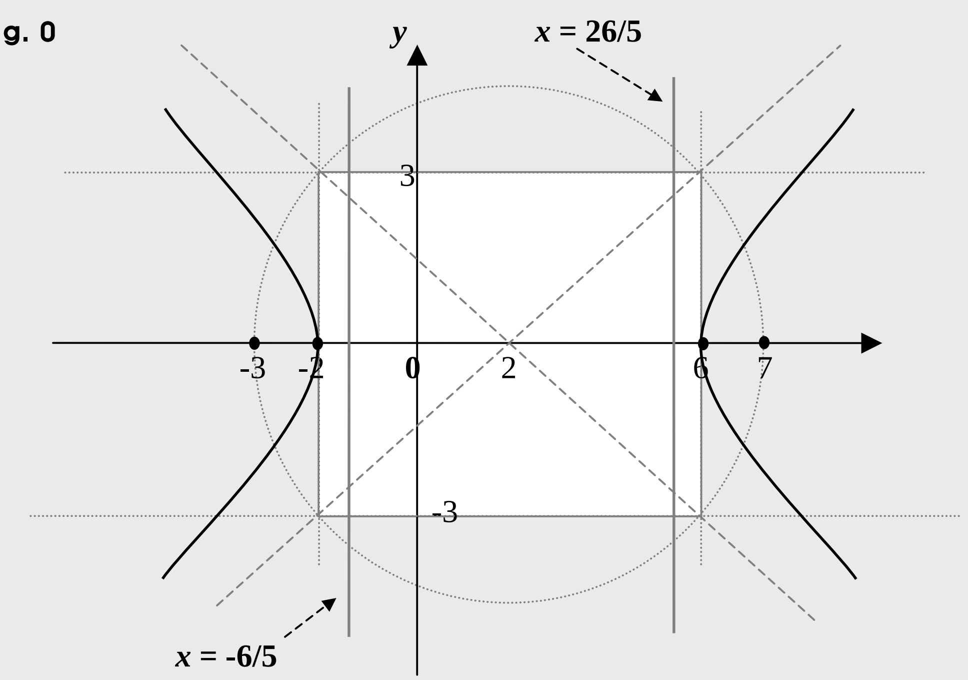
We have a connective equation $c^2 = a^2 + b^2$, where ***c*** is the focal distance, **2*a*** is the transverse axis, and **2*b*** is the conjugate axis. So we get $b^2 = c^2 - a^2$. What then, about ***c***?

The focal distance is the distance from the center to each focus.
So since the center is (2, 0), and a focus is (7, 0), we get ***c* = 5**.

And thus, we get $b^2 = c^2 - a^2 = 25 - 16 = 9 \Rightarrow b = 3$.

So the hyperbola is $\dfrac{(x-2)^2}{16} - \dfrac{y^2}{9} = 1,$ often put this way, too: $\dfrac{(x-2)^2}{4^2} + \dfrac{y^2}{3^2} = 1.$

Fig. 0



Suggestions or Solutions
To the Problem in the Example 2

Find the hyperbola that has a point at $(1, 3-2\sqrt{2})$, and has the vertices at (4, 5) and (4, 1).

To begin with, the hyperbola is vertical, and the center is (4, 3).

So next, assuming the hyperbola is $\dfrac{(y-v)^2}{b^2}-\dfrac{(x-u)^2}{a^2}=1$, we have $u = 4$, and $v = 1$.

Next, assuming d is the difference between two distances from a point in the hyperbola to the foci, we get $s = 2a$. So we get $2a = 4 \Rightarrow a = 2$.

Next, if c is the focal distance, the foci are $(4, 3 + c)$ and $(4, 3 - c)$.
So next, since the hyperbola has the given point, we can get d the way as follows.

$$d=\sqrt{(1-4)^2+(3-2\sqrt{2}-3-c)^2}-\sqrt{(1-4)^2+(3-2\sqrt{2}-3+c)^2}$$

$$=\sqrt{9+(2\sqrt{2}+c)^2}-\sqrt{9+(2\sqrt{2}-c)^2}$$

$$9+(2\sqrt{2}+c)^2=9+8+4c\sqrt{2}+c^2=17+4\sqrt{2}c+c^2$$

$$9+(2\sqrt{2}-c)^2=9+8-4c\sqrt{2}+c^2=17-4\sqrt{2}c+c^2$$

And we know $d = 4$. So setting $s = 17 + c^2$, and $t = 4\sqrt{2}c$, we get

$$\sqrt{s+t}-\sqrt{s-t}=4 \Rightarrow \sqrt{s+t}=4+\sqrt{s-t} \Rightarrow s+t=16+8\sqrt{s-t}+s-t$$

$$\Rightarrow 8\sqrt{s-t}=2t-16 \Rightarrow 4\sqrt{s-t}=t-8 \Rightarrow 16(s-t)=t^2-16t+64 \Rightarrow 16s=t^2+64.$$

So we get $16(17 + c^2) = 32c^2 + 64 \Rightarrow 16c^2 = 208 \Rightarrow c^2 = 13 \Rightarrow c = \sqrt{13}$.

So next, we get $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b^2 = 13 - 4 = 9 \Rightarrow b = 3$.

So the hyperbola is $\dfrac{(y-3)^2}{9}-\dfrac{(x-4)^2}{16}=1$, often put this way: $\dfrac{(y-3)^2}{3^2}+\dfrac{(x-4)^2}{4^2}=1$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, if a hyperbola is centered at **(*u*, *v*)**, and is vertical, the equation is $\dfrac{(y-v)^2}{b^2}-\dfrac{(x-u)^2}{a^2}=1$, where b is half the transverse axis, and a is half the conjugate.

And if it's <u>vertical</u>, the center, foci, and vertices share the <u>same *x*-coordinate</u>. So since the vertices given have the same *x*-coordinates, the hyperbola we want to find is vertical.

Next, the center is the midpoint between the vertices.
So the center is {(4 + 4)/2, (5 + 1)/2}, and thus, is **(4, 3)**. So we get $u = 4$, and $v = 3$.

Next, in a hyperbola, the difference between two distances from a point to the two foci is constant, and is in fact, the length of the transverse axis.

So assuming d is the difference between the two distances, we get $d = 2a$, which is the distance between the vertices. And the distance is 4. So we get $2a = 4 \Rightarrow a = 2$.
What then, about b?

We have a connective equation $c^2 = a^2 + b^2$, where c is the focal distance, $2a$ is the transverse axis, and $2b$ is the conjugate axis. So we get $b^2 = c^2 - a^2$. What then, about c?

Since c is the focal distance, and the center is (4, 3), the foci are $(4, 3 + c)$ and $(4, 3 - c)$. So next, since the hyperbola has the given point $(1, 3-2\sqrt{2})$, we can put d the way as follows.

$$d=\sqrt{(1-4)^2+(3-2\sqrt{2}-3-c)^2}-\sqrt{(1-4)^2+(3-2\sqrt{2}-3+c)^2}$$

$$=\sqrt{9+(2\sqrt{2}+c)^2}-\sqrt{9+(2\sqrt{2}-c)^2}$$

$$9+(2\sqrt{2}+c)^2=9+8+4c\sqrt{2}+c^2=17+4\sqrt{2}c+c^2$$

$$9+(2\sqrt{2}-c)^2=9+8-4c\sqrt{2}+c^2=17-4\sqrt{2}c+c^2$$

And we know $d = 4$. So setting $s = 17 + c^2$, and $t = 4\sqrt{2}c$, we get

$$\sqrt{s+t}-\sqrt{s-t}=4\Rightarrow\sqrt{s+t}=4+\sqrt{s-t}\Rightarrow s+t=16+8\sqrt{s-t}+s-t$$

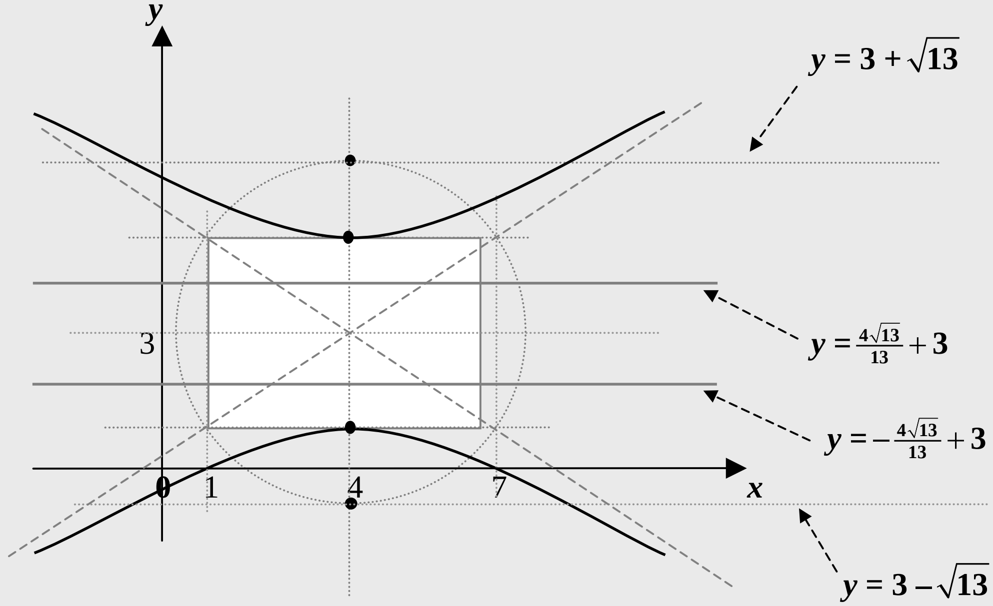
$$\Rightarrow 8\sqrt{s-t}=2t-16\Rightarrow 4\sqrt{s-t}=t-8\Rightarrow 16(s-t)=t^2-16t+64\Rightarrow 16s=t^2+64.$$

So we get $\mathbf{16(17 + c^2) = 32c^2 + 64 \Rightarrow 16c^2 = 208 \Rightarrow c^2 = 13 \Rightarrow c = \sqrt{13}}$.

So next, we get $\mathbf{c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b^2 = 13 - 4 = 9 \Rightarrow b = 3}$.

So the hyperbola is $\mathbf{\frac{(y-3)^2}{9} - \frac{(x-4)^2}{16} = 1}$, often put this way: $\mathbf{\frac{(y-3)^2}{3^2} + \frac{(x-4)^2}{4^2} = 1}$.

Fig. 0



Suggestions or Solutions
To the Problem in the Example 3

Find the hyperbola of which the eccentricity is 5/3 and a directrix is a line $x = 14/5$, which is corresponding a focus at (6, 2).

Suppose first, $\boldsymbol{T(x, y)}$ is an arbitrary point of a hyperbola, $\boldsymbol{D}$ is the distance from $\boldsymbol{T}$ to a directrix, $\boldsymbol{F}$ is the distance from $\boldsymbol{T}$ to the focus corresponding to the directrix, and $\boldsymbol{e}$ is the eccentricity. Then, we get $\boldsymbol{F = eD}$, that is, $\boldsymbol{e = F/D}$. So in this case, we get $\boldsymbol{F/D = 5/3}$.

So next, taking the distances, we get $\boldsymbol{D^2 = (x - 14/5)^2}$, and $\boldsymbol{F^2 = (x - 6)^2 + (y - 2)^2}$.

Then, we get $\boldsymbol{F/D = 5/3 \Rightarrow (F/D)^2 = F^2/D^2 = 25/9 = \dfrac{(x-6)^2+(y-2)^2}{(x-\frac{14}{5})^2}}$

$\boldsymbol{\Rightarrow 25(x - 14/5)^2 = 9\{(x - 6)^2 + (y - 2)^2\} \Rightarrow \underline{9(x - 6)^2 - 25(x - 14/5)^2} + 9(y - 2)^2 = 0}$.

And we have an identity, $\boldsymbol{A - B = (A + B)(A - B)}$. So we get

$\boldsymbol{\underline{9(x - 6)^2 - 25(x - 14/5)^2} = \{3(x - 6) + 5(x - 14/5)\}\{3(x - 6) - 5(x - 14/5)\}}$

$\boldsymbol{= (3x - 18 + 5x - 14)(3x - 18 - 5x + 14) = (8x - 32)(-2x - 4) = -16(x - 4)(x + 2)}$

$\boldsymbol{= -16(x^2 - 2x - 8) = -16(x^2 - 2x + 1 - 9) = -16(x - 1)^2 + 16 \cdot 9 = -16(x - 1)^2 + 12^2}$.

So we get $\boldsymbol{-16(x - 1)^2 + 12^2 + 9(y - 2)^2 = 0 \Rightarrow 16(x - 1)^2 - 9(y - 2)^2 = 12^2}$

$\boldsymbol{\Rightarrow 4^2(x - 1)^2 - 3^2(y - 2)^2 = 3^2 4^2 \Rightarrow \dfrac{(x-1)^2}{3^2} - \dfrac{(y-2)^2}{4^2} = 1}$, which is the hyperbola we want.