Examples 7 in Hyperbolas

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In each example below, the hyperbola is in the ***x-y*** plane.

0. Suppose ***A*** is a circle centered at (-2, 2), and ***B*** is a circle centered at (5, 2). The two circles are away from each other. Suppose now that the radii of ***A*** and ***B*** increase at the same rate. Then, the two circles meet each other for the first time at a line $\boldsymbol{x = 1}$. In other words, the line $\boldsymbol{x = 1}$ is the line tangent to both circles at the same point at the same time. Suppose further that the radii keep increasing at the same rate thereafter. Then, the two circles ***A*** and ***B*** keep meeting at two points, and the two points are in a curve, since the two circles are growing. Find the curve.

1. In the ***x-y*** plane, find the area where the points satisfy the inequality as follows.
$\boldsymbol{(x^2 + y^2 - 9)(4x^2 - y^2 - 16) \leq 0}$.

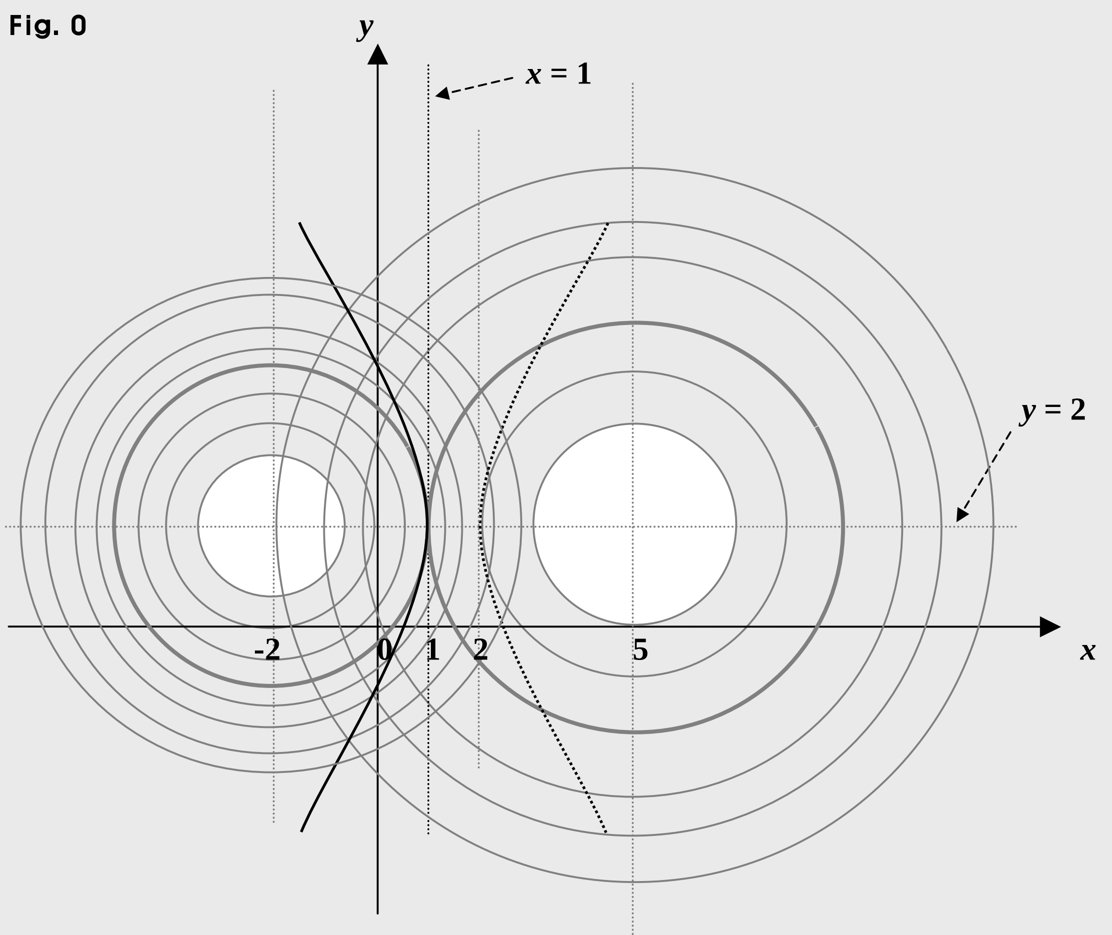
2. Assuming $\boldsymbol{9x^2 + ax - y^2 + by + c = 0}$ is a hyperbola, find the values of ***a***, ***b***, and ***c*** so that the hyperbola is tangent to the ***x***-axis at (1, 0), and passes through (3, 3).

Suggestions or Solutions
To the Problem in the Example 0

Suppose ***A*** is a circle centered at (-2, 2), and ***B*** is a circle centered at (5, 2). The two circles are away from each other. Suppose now that the radii of ***A*** and ***B*** increase at the same rate. Then, the two circles meet each other for the first time at a line ***x* = 1**. In other words, the line ***x* = 1** is the line tangent to both circles at the same point at the same time. Suppose further that the radii keep increasing at the same rate thereafter. Then, the two circles ***A*** and ***B*** keep meeting at two points, and the two points are in a curve, since the two circles are growing. Find the curve.

To begin with, putting the problem in a graph, we can put it the way below.

Fig. 0



Then, we can see that the curve seems to be a hyperbola.

We want to make sure though, it is the case, and more importantly, we want to get the equation of the curve.

The two radii are growing at the same rate. So the difference between the two radii is constant. So assuming P is one of the two points where the two circles meet, we can say that the difference between the two distances form P to the centers of the two circles is constant. So we can say that the centers of the two circles are the foci of a hyperbola.

Next, the center of a hyperbola is the midpoint between the foci. So since the foci are the centers of the two circles, the center of the hyperbola is {(-2 + 5)/2, (2 + 2)/2} = (3/2, 2).

So since the hyperbola is horizontal, we can put, for now, the equation of the hyperbola the way as follows. $\frac{(x-\frac{3}{2})^2}{a^2}-\frac{(y-2)^2}{b^2}=1$. How then, can we get a and b?

The value of a is half the transverse axis, and is the distance from a vertex to the center of the hyperbola. And in this case, one of the two vertices is (1, 2), which is the vertex of the left branch, and is the point where the two circles meet for the first time.

So since the center of the hyperbola is (3/2, 2), the distance from the vertex to the center is 1/2. And thus, we get $a = 1/2$. What then, about b?

Assuming c is the focal distance, we can set $c^2 = a^2 + b^2$. And we have $a = 1/2$.
And the focal distance is the distance from a focus to the center of the hyperbola.
So since (-2, 2) is one of the two foci, and the center is (3/2, 2), we get $c = 7/2$.

So we get $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 = 49/4 - 1/4 = 48/4 = 12 \Rightarrow b = 2\sqrt{3}$.

And thus, the hyperbola is $\frac{(x-\frac{3}{2})^2}{(\frac{1}{2})^2}-\frac{(y-2)^2}{(2\sqrt{3})^2}=1$, which can be put the way below, too.

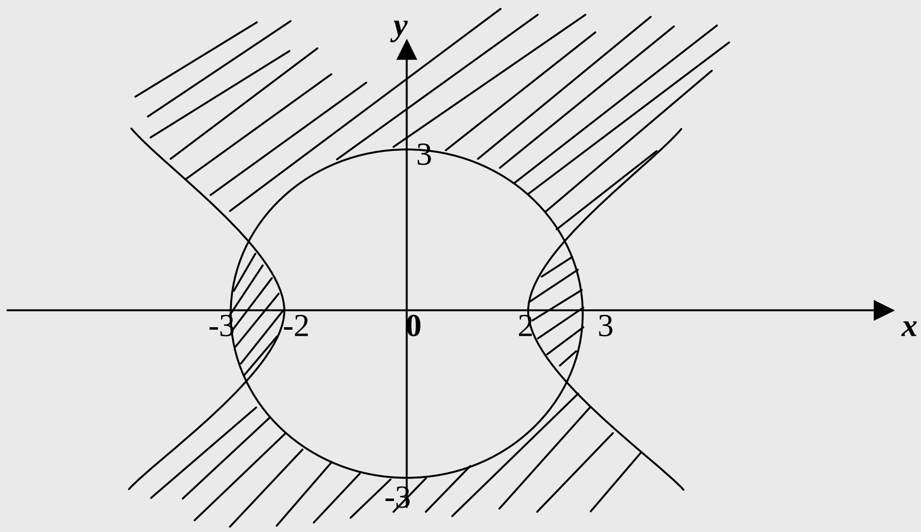
$4(x-\frac{3}{2})^2-\frac{(y-2)^2}{12}=1$.

Suggestions or Solutions
To the Problem in the Example 1

In the ***x-y*** plane, find the area where the points satisfy the inequality as follows.
$(x^2 + y^2 - 9)(4x^2 - y^2 - 16) \le 0$.

In the graph below, the area we want is the area with slash marks, and the area includes the circle and hyperbola themselves, too.

Fig. 0



If not quite sure of the idea behind the processes above, follow the steps below.

Assuming first, $AB \le 0$, we get two cases.
One is $A \le 0$ and $B \ge 0$, and the other is $A \ge 0$ and $B \le 0$.

So if $(x^2 + y^2 - 9)(4x^2 - y^2 - 16) \le 0$, we get two cases, too.

One is $x^2 + y^2 - 9 \le 0$ and $4x^2 - y^2 - 16 \ge 0$.
And the other is $x^2 + y^2 - 9 \ge 0$, and $4x^2 - y^2 - 16 \le 0$.

And we have $x^2 + y^2 \le 9 = 3^2$, and $4x^2 - y^2 \ge 16 \Rightarrow x^2/4 - y^2/16 \ge 1 \Rightarrow x^2/2^2 - y^2/4^2 \ge 1$.

So in one case, we have $x^2 + y^2 \le 3^2$, and $x^2/2^2 - y^2/4^2 \ge 1$.

And in the other case, we have $x^2 + y^2 \ge 3^2$, and $x^2/2^2 - y^2/4^2 \le 1$.

Next, $\boldsymbol{x^2 + y^2 = 3^2}$ is the equation of a circle centered at the origin with a radius of 3.

And $\boldsymbol{x^2/2^2 - y^2/4^2 = 1}$ is the equation of a horizontal hyperbola centered at the origin with a transverse axis of 4 and a conjugate axis of 8.

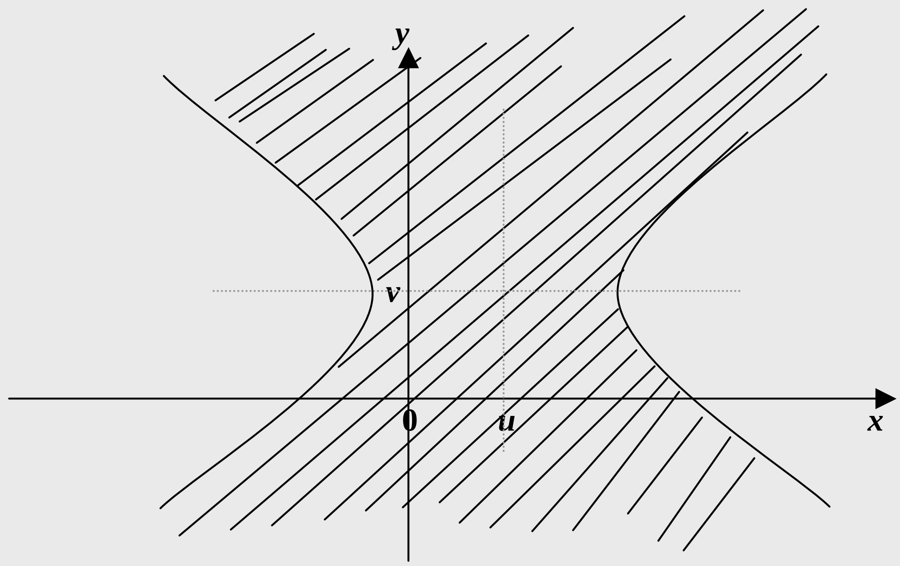
Next, if $\boldsymbol{(x - u)^2 + (y - v)^2 \geq r^2}$, the inequality means the set of all the points in the area outside a circle centered at $\boldsymbol{(u, v)}$ with a radius of $\boldsymbol{r}$, and also, the area includes the circle itself, too.

So if $\boldsymbol{(x - u)^2 + (y - v)^2 < r^2}$, the inequality means the set of all the points in only the area inside the circle, so the area does not include the circle itself.

And the same is true for a hyperbola, too.

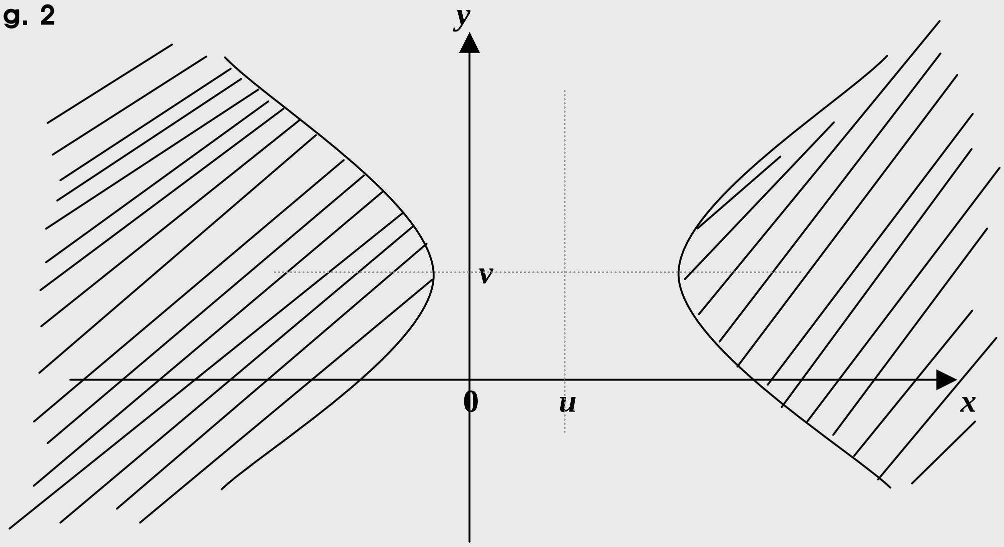
So if $\boldsymbol{(x - u)^2/a^2 - (y - v)^2/b^2 < 1}$, the inequality means the set of all the points in only the area insides the hyperbola, so the area does not include the hyperbola itself.
The area inside a hyperbola means the area between the braches.

Fig. 1



And if $\boldsymbol{(x - u)^2/a^2 - (y - v)^2/b^2 \geq 1}$, the inequality means the set of all the points in the area outside a hyperbola centered at $\boldsymbol{(u, v)}$ with the main axes of $\boldsymbol{2a}$ and $\boldsymbol{2b}$, and also, the area includes the hyperbola itself, too.

Fig. 2



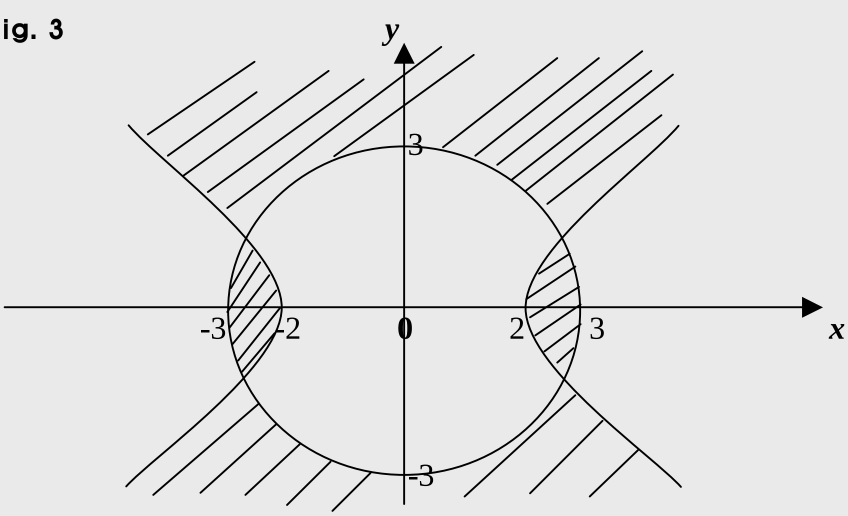
Now, we have two cases.

One is $\boldsymbol{x^2 + y^2 \leq 3^2}$, and $\boldsymbol{x^2/2^2 - y^2/4^2 \geq 1}$.

And the other is $\boldsymbol{x^2 + y^2 \geq 3^2}$, and $\boldsymbol{x^2/2^2 - y^2/4^2 \leq 1}$.

So in the graph below, the area we want is the area with slash marks, and the area includes the circle and hyperbola themselves, too.

Fig. 3



What if we have $(x^2 + y^2 - 9)(x^2 + 4y^2 - 16) > 0$?

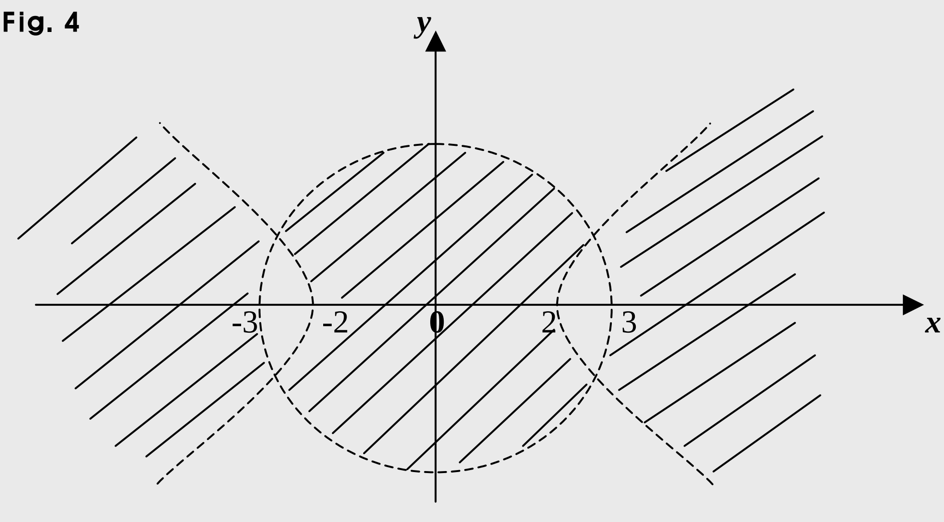
Then, we get two cases, too, but the area is the opposite of the area with slash marks above.

So one is $x^2 + y^2 > 3^2$, and $x^2/2^2 - y^2/4^2 > 1$.
And the other is $x^2 + y^2 < 3^2$, and $x^2/2^2 - y^2/4^2 < 1$.

So in the graph above, the area is the area without slash marks, and the area does not include the circle and hyperbola themselves.

In other words, the area is the area with slash marks in the graph below, and the area does not include the circle and hyperbola themselves.

Fig. 4



Suggestions or Solutions
To the Problem in the Example 2

Assuming $9x^2 + ax - y^2 + by + c = 0$ is a hyperbola, find the values of a, b, and c so that the hyperbola is tangent to the x-axis at (1, 0), and passes through (3, 3).

First, if $\boldsymbol{H}$ is the hyperbola, since $\boldsymbol{H}$ has the point (3, 3), we can get
$\mathbf{9(3^2) + 3a - 3^2 + 3b + c = 0 \Rightarrow 3a + 3b + c + 72 = 0}$.

Next, since $\boldsymbol{H}$ is tangent to the $\boldsymbol{x}$-axis at (1, 0), it has (1, 0), too, so we get $\mathbf{9 + a + c = 0}$.

And next, since the $\boldsymbol{x}$-axis is tangent to $\boldsymbol{H}$, we can say that a line $\boldsymbol{y = 0}$ meets the hyperbola $\boldsymbol{H}$ at one point.
And assuming we find the point where the line meets the hyperbola, we get to solve a system of two equations, one is $\boldsymbol{y = 0}$, and the other is $\mathbf{9x^2 + ax - y^2 + by + c = 0}$.

And solving the system, since $\boldsymbol{y = 0}$, we can get first, $\mathbf{9x^2 + ax - 0^2 + b \cdot 0 + c = 0}$, which is $\mathbf{9x^2 + ax + c = 0}$, which is a quadratic equation, which therefore, has to have a double root, because the line meets the hyperbola at one point.

So the discriminant of the quadratic equation has to be 0.
Thus, we get $\mathbf{a^2 - 4 \cdot 9c = a^2 - 36c = 0}$.

So we get a system of three equations as follows.
$\mathbf{3a + 3b + c + 72 = 0}$, $\mathbf{9 + a + c = 0}$, and $\mathbf{a^2 - 36c = 0}$.

So solving the system, we can get first, $\mathbf{9 + a + c = 0 \Rightarrow c = -a - 9}$.

So next, we can get $\mathbf{a^2 - 36c = 0 \Rightarrow a^2 - 36(-a - 9) = 0 \Rightarrow a^2 + 36a + 36 \cdot 9 = 0}$.

Meanwhile, $36 \cdot 9 = 4 \cdot 9 \cdot 9 = 2 \cdot 9 \cdot 2 \cdot 9 = 18^2$, and $36 = 2 \cdot 18$.

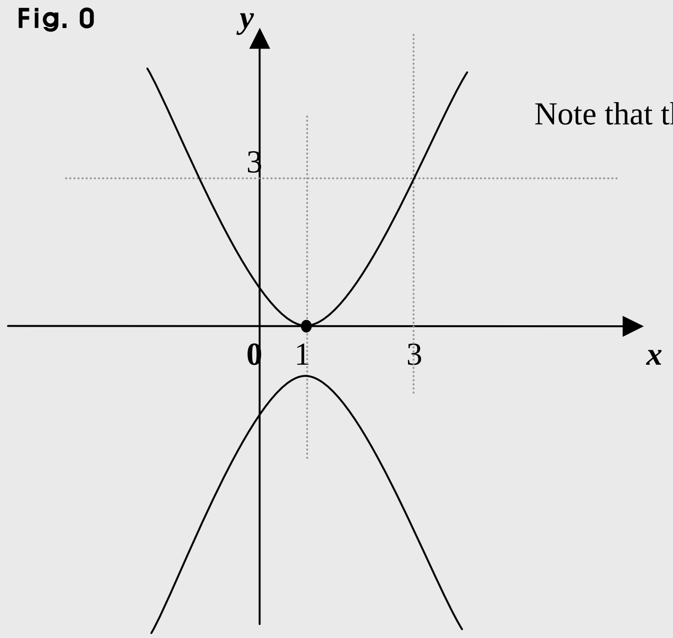
So we get $\mathbf{a^2 + 36a + 36 \cdot 9 = a^2 + 2 \cdot 18a + 18^2 = (a + 18)^2 = 0 \Rightarrow a = -18}$.

Thus next, we can get $\mathbf{c = -a - 9 = 18 - 9 = 9 \Rightarrow c = 9}$.
And next, $\mathbf{3a + 3b + c + 72 = 0 \Rightarrow 3b = -3a - c - 72 = 54 - 9 - 72 = -27 \Rightarrow b = -9}$.

What hyperbola then, is it?

Unlike the case of ellipses, there can be only one hyperbola that can be tangent to the ***x***-axis at (1, 0), and pass through (3, 3). And the schematic drawing is as follows.

Fig. 0



Note that the curves are not parabolas.

Next, putting values of ***a***, ***b***, and ***c*** into the equation $\mathbf{9x^2 + ax - y^2 + by + c = 0}$, we get

$$\mathbf{9x^2 - 18x - y^2 - 9y + 9 = 9(x^2 - 2x + 1 - 1) - \{y^2 + 9y + (\tfrac{9}{2})^2 - (\tfrac{9}{2})^2\} + 9}$$

$$\mathbf{= 9(x-1)^2 - 9 - (y + \tfrac{9}{2})^2 + (\tfrac{9}{2})^2 + 9 = 9(x-1)^2 - (y + \tfrac{9}{2})^2 + (\tfrac{9}{2})^2 = 0}$$

$$\Rightarrow \frac{\mathbf{9(x-1)^2}}{(\frac{9}{2})^2} - \frac{\mathbf{(y+\frac{9}{2})^2}}{(\frac{9}{2})^2} + \mathbf{1} = \mathbf{0} \Rightarrow \frac{\mathbf{(y+\frac{9}{2})^2}}{(\frac{9}{2})^2} - \frac{\mathbf{(x-1)^2}}{(\frac{3}{2})^2} = \mathbf{1}.$$

So the hyperbola ***H*** is a vertical hyperbola centered at $(1, -\frac{9}{2})$ with the transverse axes of 9 and the conjugate axis of 3. And we can put it in a graph the way below.