Examples 8 in Hyperbolas

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0. Find the equation of the line tangent to a hyperbola $\boldsymbol{x^2/a^2 - y^2/b^2 = 1}$ at a point $\boldsymbol{(s, t)}$.

1. Suppose that $\boldsymbol{P}$ is a point in a hyperbola, and that $\boldsymbol{A}$ and $\boldsymbol{B}$ are the two foci of the hyperbola. Suppose also, $\boldsymbol{T}$ is the line tangent to the hyperbola at $\boldsymbol{P}$, $\boldsymbol{M}$ is the angle between the line segment $\boldsymbol{PA}$ and the line $\boldsymbol{T}$, $\boldsymbol{N}$ is the angle between the line segment $\boldsymbol{PB}$ and the line $\boldsymbol{T}$, and $\boldsymbol{M}$ and $\boldsymbol{N}$ both are acute angles. Show that $\boldsymbol{M = N}$.

Suggestions or Solutions
To the Problem in the Example 0

Find the equation of the line tangent to a hyperbola $x^2/a^2 - y^2/b^2 = 1$ at a point (s, t).

Assuming first, the tangent line is $y = mx + n$, we get first, $x^2/a^2 - (mx + n)^2/b^2 = 1$.
And rearranging terms in the equation above, we get

$x^2/a^2 - (mx + n)^2/b^2 = 1 \Rightarrow b^2x^2 - a^2(mx + n)^2 = a^2b^2$

$\Rightarrow b^2x^2 - a^2(mx + n)^2 = b^2x^2 - a^2(m^2x^2 + 2mnx + n^2)$

$= (b^2 - a^2m^2)x^2 - 2mna^2x - a^2n^2 = a^2b^2 \Rightarrow (b^2 - a^2m^2)x^2 - 2mna^2x - a^2(n^2 + b^2) = 0$.

Next, the line meets the hyperbola at one point, so the equation's discriminant is 0.

Thus, we get $m^2n^2a^4 + (b^2 - a^2m^2)a^2(n^2 + b^2) = 0 \Rightarrow m^2n^2a^2 + (b^2 - a^2m^2)(n^2 + b^2) = 0$
$\Rightarrow b^4 + b^2n^2 - a^2b^2m^2 = 0 \Rightarrow b^2 + n^2 - a^2m^2 = 0$.

Next, we know the fact that the line passes through (s, t). So we get $t = ms + n$.
Thus, we get $n = t - ms$. So we get

$b^2 + n^2 - a^2m^2 = b^2 + (t - ms)^2 - a^2m^2 = 0$
$\Rightarrow b^2 + t^2 - 2stm + s^2m^2 - a^2m^2 = m^2(s^2 - a^2) - 2stm + b^2 + t^2 = 0$.

So we get $m = \dfrac{st \pm \sqrt{s^2t^2 - (s^2 - a^2)(b^2 + t^2)}}{s^2 - a^2}$. We know however, m can have one value only. So what's inside the square root has to be 0. That is to say that the discriminant is 0.

So we get $m = \dfrac{st}{s^2 - a^2}$. And since the discriminant is 0, we get

$s^2t^2 - (s^2 - a^2)(b^2 + t^2) = a^2b^2 + a^2t^2 - s^2b^2 = 0 \Rightarrow b^2(a^2 - s^2) = -a^2t^2 \Rightarrow s^2 - a^2 = \dfrac{a^2t^2}{b^2}$.

So we get $m = \dfrac{sb^2}{ta^2}$. And we know that the line passes through (s, t).

So the tangent line is $y - t = \dfrac{sb^2}{ta^2}(x - s)$. And it can be put this way, too: $\dfrac{sx}{a^2} - \dfrac{ty}{b^2} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose first, the tangent line is $y = mx + n$, and is called T, and the hyperbola is H.

Then, since the line T is tangent to the hyperbola H, we can say that the line T meets the hyperbola H at one point.

And assuming we find the point where the line meets the hyperbola, we get to solve a system of two equations, one is $y = mx + n$, and the other is $x^2/a^2 - y^2/b^2 = 1$.

And solving the system, since $y = mx + n$, we can get first, $x^2/a^2 - (mx + n)^2/b^2 = 1$, which is $b^2x^2 - a^2(mx + n)^2 = a^2b^2$, which is a quadratic equation, which therefore, has to have a double root, because the line meets the hyperbola at one point.

So the discriminant of the quadratic equation has to be 0.
By the way, if the coefficient of x is even, we can take a quarter of the discriminant, because the result will get simplified that way.

If for instance, we have $Ax^2 + 2Bx + C = 0$, the discriminant is $B^2 - AC$.

So first, rearranging terms in the equation, we get

$$b^2x^2 - a^2(mx + n)^2 = b^2x^2 - a^2(m^2x^2 + 2mnx + n^2)$$

$$= (b^2 - a^2m^2)x^2 - 2mna^2x - a^2n^2 = a^2b^2 \Rightarrow (b^2 - a^2m^2)x^2 - 2mna^2x - a^2(n^2 + b^2) = 0.$$

So next, getting the quarter of the discriminant, we get

$$m^2n^2a^4 + (b^2 - a^2m^2)a^2(n^2 + b^2) = 0 \Rightarrow m^2n^2a^2 + (b^2 - a^2m^2)(n^2 + b^2) = 0$$
$$\Rightarrow m^2n^2a^2 + b^2n^2 + b^4 - m^2n^2a^2 - a^2b^2m^2 = b^4 + b^2n^2 - a^2b^2m^2 = 0$$
$$\Rightarrow b^2 + n^2 - a^2m^2 = 0.$$

Next, knowing the slope and a point of a line, we can get the equation of the line.

We know the tangent line T passes through (s, t), and the slope is m.
And also, s, t, a, and b are all known values.

So finding m, we can get the line T. How then, can we find m?

We now have two equations that have $\boldsymbol{m}$.

One is $\boldsymbol{b^2 + n^2 - a^2m^2 = 0}$. and the other is $\boldsymbol{y = mx + n}$.

And we know the fact that $\boldsymbol{T}$ passes through $\boldsymbol{(s, t)}$. So we get $\boldsymbol{t = ms + n}$.
Thus, we get $\boldsymbol{n = t - ms}$.

So we can now get $\boldsymbol{m}$ putting $\boldsymbol{n}$ into the other equation. Then, we get

$$\boldsymbol{b^2 + n^2 - a^2m^2 = b^2 + (t - ms)^2 - a^2m^2 = 0}$$
$$\Rightarrow \boldsymbol{b^2 + t^2 - 2stm + s^2m^2 - a^2m^2 = m^2(s^2 - a^2) - 2stm + b^2 + t^2 = 0}\text{, which is quadratic.}$$

So using the quadratic formula, we get $\boldsymbol{m = \dfrac{st \pm \sqrt{s^2t^2 - (s^2 - a^2)(b^2 + t^2)}}{s^2 - a^2}}$.

We know however, $\boldsymbol{m}$ can have one value only.

So what's inside the square root has to be 0. That is to say that the discriminant is 0.

So we get $\boldsymbol{m = \dfrac{st}{s^2 - a^2}}$.

And since the discriminant is 0, we get

$$\boldsymbol{s^2t^2 - (s^2 - a^2)(b^2 + t^2) = s^2t^2 - s^2b^2 - s^2t^2 + a^2b^2 + a^2t^2 = -s^2b^2 + a^2b^2 + a^2t^2 = 0}$$
$$\Rightarrow \boldsymbol{a^2b^2 - s^2b^2 + a^2t^2 = 0} \Rightarrow \boldsymbol{b^2(b^2 - s^2) = -a^2t^2} \Rightarrow \boldsymbol{b^2(s^2 - a^2) = a^2t^2} \Rightarrow \boldsymbol{s^2 - a^2 = \frac{a^2t^2}{b^2}}.$$

So we get $\boldsymbol{m = \dfrac{st}{s^2 - a^2} = st\dfrac{b^2}{a^2t^2} = \dfrac{sb^2}{ta^2}}$, which is the slope of the line $\boldsymbol{T}$.

And we know that the line $\boldsymbol{T}$ passes through $\boldsymbol{(s, t)}$.
If a line has a slope of $\boldsymbol{k}$, and has a point $\boldsymbol{(p, q)}$, the line is $\boldsymbol{y - q = k(x - p)}$.

So the line $\boldsymbol{T}$ is $\boldsymbol{y - t = \dfrac{sb^2}{ta^2}(x - s)}$, which looks a bit complicated.

So simplifying the equation above, we can get

$$y - t = \frac{sb^2}{ta^2}(x - s) \Rightarrow ta^2(y - t) = sb^2(x - s) \Rightarrow tya^2 - t^2a^2 = sxb^2 - s^2b^2$$

$$\Rightarrow sxb^2 - tya^2 = s^2b^2 - t^2a^2 \Rightarrow sx - \frac{tya^2}{b^2} = s^2 - \frac{t^2a^2}{b^2} \Rightarrow \frac{sx}{a^2} - \frac{ty}{b^2} = \frac{s^2}{a^2} - \frac{t^2}{b^2}.$$

And we know that the hyperbola $\boldsymbol{H}$ passes through the point (s, t), and $\boldsymbol{H}$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

So we get $\frac{s^2}{a^2} - \frac{t^2}{b^2} = 1$, and thus, the tangent line is $\frac{sx}{a^2} - \frac{ty}{b^2} = 1$.

So for instance, finding the line tangent to $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$ at $(4, \frac{2\sqrt{7}}{3})$, we get

$\frac{4x}{3^2} - \frac{2\sqrt{7}y}{3 \cdot 2^2} = \frac{4x}{9} - \frac{\sqrt{7}y}{6} = 1$, which is the tangent line.

And of course, we can put it this way, too: $8x - 3\sqrt{7}y - 18 = 0$.

What if we are given the slope of the tangent line and are asked to find the tangent line?

Assuming the slope given is $\boldsymbol{m}$, we can assume that the tangent line is $y = mx + n$.

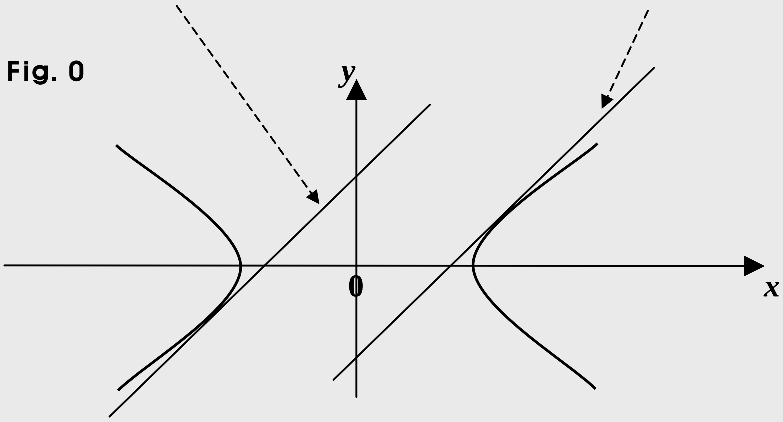
Then, finding $\boldsymbol{n}$, we get the line tangent to the hyperbola.

Finding $\boldsymbol{n}$, we just solve $b^2 + n^2 - a^2m^2 = 0$ for $\boldsymbol{n}$. And we get the equation taking the discriminant of $x^2/a^2 - (mx + n)^2/b^2 = 1$, which is $b^2x^2 - a^2(mx + n)^2 = a^2b^2$.

So solving it for $\boldsymbol{n}$, we get $n = \pm\sqrt{a^2m^2 - b^2}$. And thus, we get two tangent lines.

One is $\boldsymbol{y = mx + \sqrt{a^2m^2 - b^2}}$, and the other is $\boldsymbol{y = mx - \sqrt{a^2m^2 - b^2}}$.

Fig. 0



So for instance, finding the lines that have a slope of 2, and are tangent to $\boldsymbol{\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1}$,

we get $\boldsymbol{y = 2x \pm \sqrt{3^2 2^2 - 2^2}}$. So we get two tangent lines.

One is $\boldsymbol{y = 2x + 4\sqrt{2}}$, and the other is $\boldsymbol{y = 2x - 4\sqrt{2}}$.

What if this time, we want to find the line tangent to an hyperbola $\boldsymbol{\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1}$ at a point $\boldsymbol{(s, t)}$?

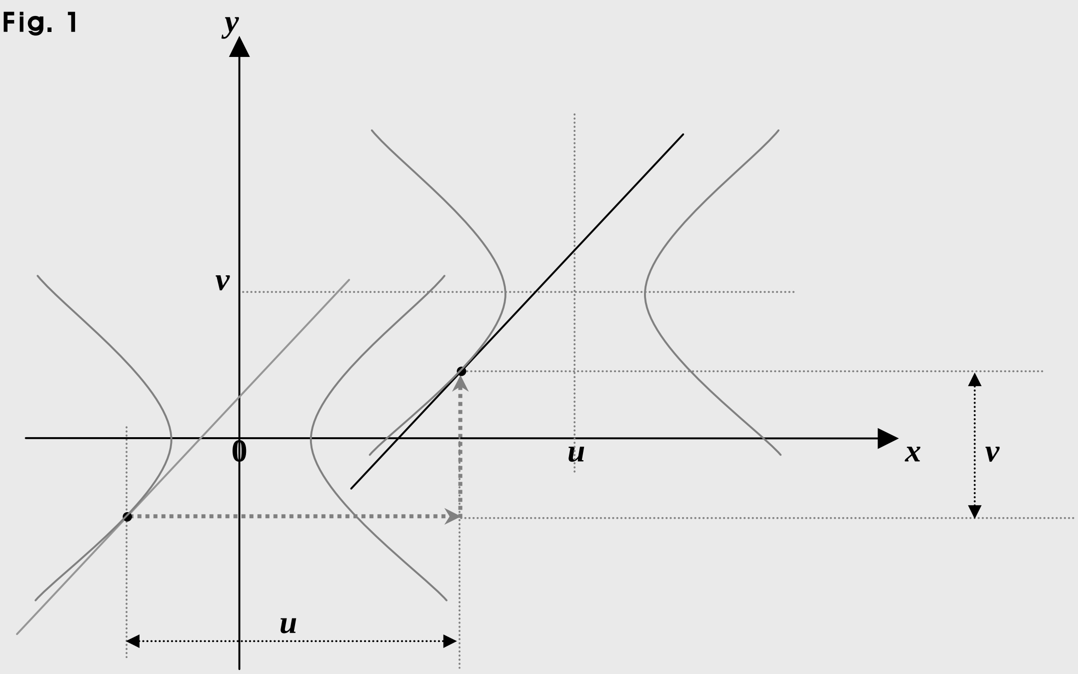
Translating the hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ in the amount of $\boldsymbol{u}$ along the $\boldsymbol{x}$-axis, and in the amount of $\boldsymbol{v}$ along the $\boldsymbol{y}$-axis, we get the hyperbola $\boldsymbol{\frac{(x-u)^2}{a^2} - \frac{(y-v)^2}{b^2} = 1}$.

So finding the line tangent to $\boldsymbol{\frac{(x-u)^2}{a^2} - \frac{(y-v)^2}{b^2} = 1}$ at the point $\boldsymbol{(s, t)}$, we find first the line tangent to $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ at $\boldsymbol{(s - u, t - v)}$. Then, we get $\boldsymbol{\frac{(s-u)x}{a^2} - \frac{(t-v)y}{b^2} = 1}$.

Next, translating the line above in the amount of $\boldsymbol{u}$ along the $\boldsymbol{x}$-axis, and in the amount of $\boldsymbol{v}$ along the $\boldsymbol{y}$-axis, we get the line tangent to $\frac{(x-u)^2}{a^2}-\frac{(y-v)^2}{b^2}=1$ at the point (s, t).

Then, we get $\frac{(s-u)(x-u)}{a^2}-\frac{(t-v)(y-v)}{b^2}=1$, which is the line tangent to $\frac{(x-u)^2}{a^2}-\frac{(y-v)^2}{b^2}=1$ at the point (s, t).

Fig. 1



So for instance, finding the line tangent to $\frac{(x-1)^2}{3^2}-\frac{(y-2)^2}{2^2}=1$ at $(-1, \frac{2\sqrt{7}}{3}+2)$, we can get first, the line tangent to $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$ at $(-1-1, \frac{2\sqrt{7}}{3}+2-2)$, that is, $(-2, \frac{2\sqrt{7}}{3})$.

Then, the line is $\frac{-2x}{3^2}+\frac{\frac{2\sqrt{7}}{3}y}{2^2}=1$.

Next, translating the line above in the amount of 1 along the $\boldsymbol{x}$-axis, and in the amount of 2 along the $\boldsymbol{y}$-axis, we get the line tangent to $\frac{(x-1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$ at $(-1, \frac{2\sqrt{7}}{3}+2)$.

Then, the line is $\frac{-2(x-1)}{3^2}-\frac{\frac{2\sqrt{7}}{3}(y-2)}{2^2}=1$, which is $\frac{-2(x-1)}{9}-\frac{\sqrt{7}(y-2)}{6}=1$, which is the line tangent to $\frac{(x-1)^2}{3^2}-\frac{(y-2)^2}{2^2}=1$ at $(-1, \frac{2\sqrt{7}}{3}+2)$.

And of course, we can put the line the way below, too.

$$\frac{-2(x-1)}{9}-\frac{\sqrt{7}(y-2)}{6}=1 \Rightarrow -4(x-1)-3\sqrt{7}(y-2)=18 \Rightarrow 4x+3\sqrt{7}y+14-6\sqrt{7}=0.$$

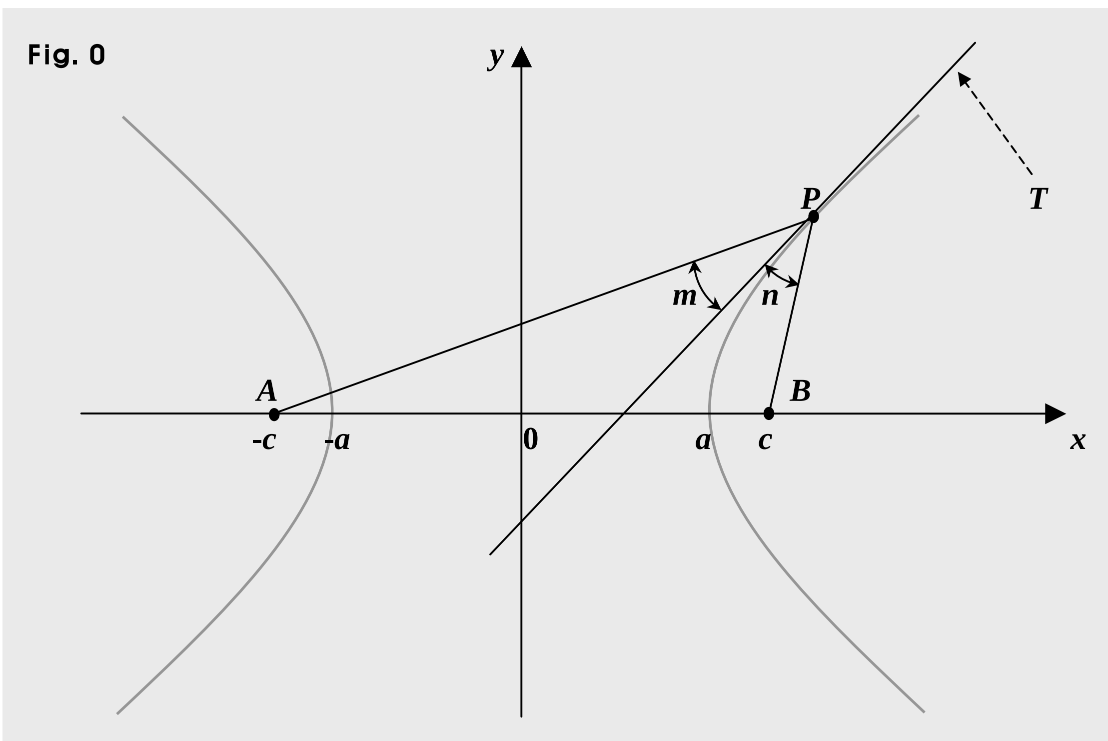
And the same is true, too, for the tangent line with the slope given.

So the lines that are tangent to $\frac{(x-u)^2}{a^2}-\frac{(y-v)^2}{b^2}=1$, and have a slope of $\boldsymbol{m}$ are as follows. $y-v=m(x-u)\pm\sqrt{a^2m^2-b^2}$.

Suggestions or Solutions
To the Problem in the Example 1

Suppose *P* is a point in an hyperbola, *A* and *B* are the foci, and *T* is a line tangent to the hyperbola at *P*. Suppose also, *m* and *n* are two acute angles, *m* is an angle between *PA* and *T*, and *n* is an angle between *PB* and *T*. Then, show that *m* = *n*.

For simplicity, let's use a horizontal hyperbola centered at the origin.
And suppose that the hyperbola is ***H***, and the equation is $x^2/a^2 - y^2/b^2 = 1$.
Then, ***a*** is half the transverse axis, ***b*** is half the conjugate axis, and we get $c^2 = a^2 + b^2$ where ***c*** is the focal distance.
And putting now, the problem description in a graph, we can put it the way below.

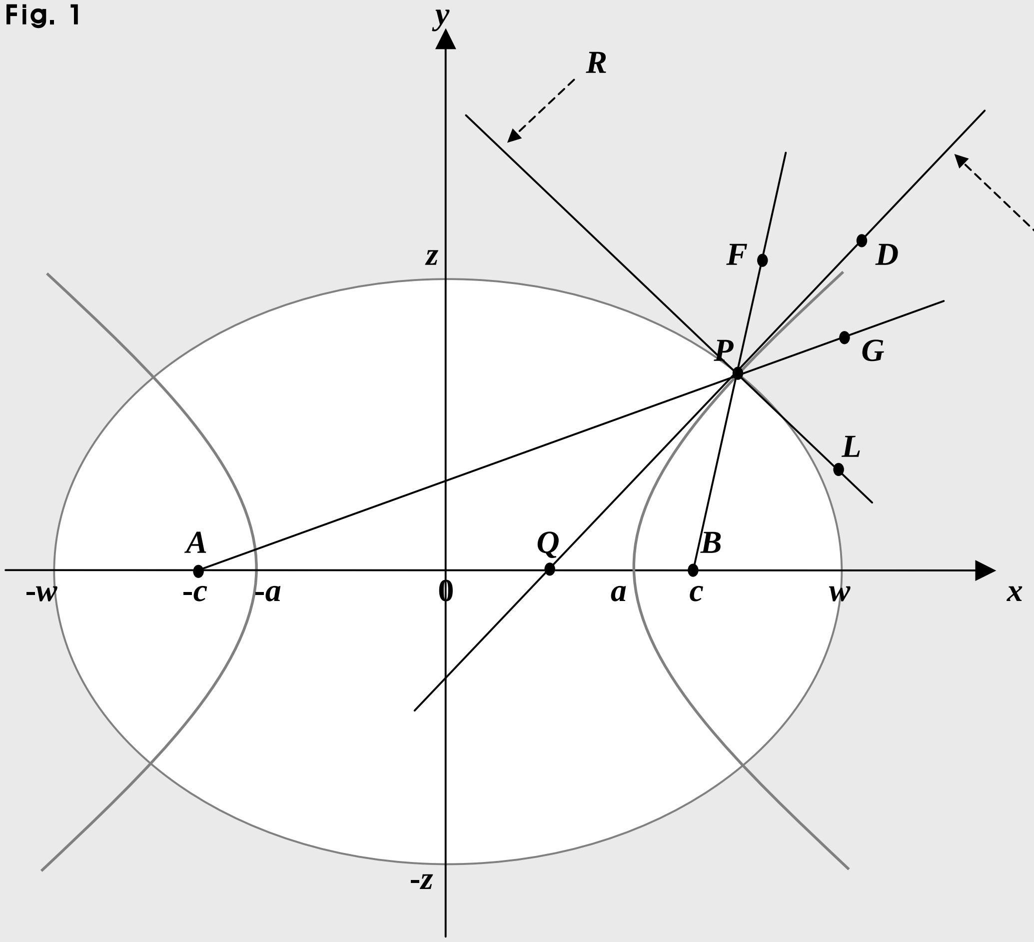


Next, we have a fact that a line tangent to an ellipse also bisects the angle made by line segments connecting the foci and a point in the ellipse. It is covered in the example 4 in **Examples 5 in Ellipses**.

So let's put in a graph, along with the hyperbola ***H***, an ellipse that has the same foci as the foci ***H*** has, and also, passes through the point ***P***, too. If an ellipse and a hyperbola have the same foci, the are said to be confocal.

Suppose the ellipse is ***E***, and the equation is $x^2/w^2 + y^2/z^2 = 1$.
Then, ***w*** is the major radius, ***z*** is the minor radius, and we get $c^2 = w^2 - z^2$ where ***c*** is the focal distance, which is the same as the focal distance of ***H***. And putting in the graph, the ellipse, together with some other line and points, we can put them the way below.

Fig. 1



Then, ***R*** is a line tangent to the ellipse ***E*** at ***P***, and the line ***R*** is perpendicular to the line ***T***. And we will get to see how it is the case shortly.

What does ***R*** though, have to do with the fact that ***T*** bisects the angle ***APB***?

First, ***R*** bisects the angle ***BPG***, so the angle ***BPL*** is the same as the angle ***GPL***.

Next, ***R*** is perpendicular to ***T***, the angle ***QPL*** and the angle ***DPL*** are both 90^{o}.
So the angle ***QPB*** is the same as the angle ***DPG***.

Next, the angle ***APQ*** and the angle ***DPG*** are opposite angles, so both are the same.
So the angle ***APQ*** is the same as the angle ***BPQ***.

Therefore, the tangent line ***T*** bisects the angle ***APB***.

So showing that ***R*** is perpendicular to ***T***, we show that the two angles ***m*** and ***n*** are equal.

To begin with, we have a fact that if one line is perpendicular to another, the product of the slopes of the two lines is –1.
So showing that the product of the slopes of ***R*** and ***T*** is –1, we show ***m = n***.

Suppose first, ***P*** is at **(*s, t*)**.
Then, since the hyperbola ***H*** is $x^2/a^2 - y^2/b^2 = 1$, the tangent line ***T*** is $y - t = \frac{sb^2}{ta^2}(x - s)$, found in the example 0 above. So the slope of ***T*** is $\frac{sb^2}{ta^2}$.
And of course, we can put the line ***T*** this way, too: $\frac{sx}{a^2} - \frac{ty}{b^2} = 1$.

Next, the ellipse ***E*** is $x^2/w^2 + y^2/z^2 = 1$.
And we know that the line ***R*** is tangent to ***E*** at ***P*****(*s, t*)**, too.
So the line ***R*** is $y - t = -\frac{sz^2}{tw^2}(x - s)$, found in the example 2 in **Examples 4 in Ellipses**. So the slope of ***R*** is $\frac{sz^2}{tw^2}$.
And of course, we can put the line ***R*** this way, too: $\frac{sx}{w^2} + \frac{ty}{z^2} = 1$.

Thus, the product of the slopes of ***R*** and ***T*** is $-\frac{sz^2}{tw^2} \cdot \frac{sb^2}{ta^2} = -\left(\frac{bsz}{atw}\right)^2$.

And $\boldsymbol{H}$ and $\boldsymbol{E}$ both pass through $\boldsymbol{P(s, t)}$. So we get $\boldsymbol{s^2/a^2 - t^2/b^2 = 1}$ and $\boldsymbol{s^2/w^2 + t^2/z^2 = 1}$.

And setting next, $\boldsymbol{u = s^2}$ and $\boldsymbol{v = t^2}$, we get

$\boldsymbol{s^2/a^2 - t^2/b^2 = 1 \Rightarrow u/a^2 - v/b^2 = 1 \Rightarrow ub^2 - va^2 = a^2b^2}$ …….(1)

$\boldsymbol{s^2/w^2 + t^2/z^2 = 1 \Rightarrow u/w^2 + v/z^2 = 1 \Rightarrow uz^2 + vw^2 = w^2z^2}$ …….(2)

Let' first, eliminate $\boldsymbol{v}$. Then, multiplying (1) by $\boldsymbol{w^2}$, and multiplying (2) by $\boldsymbol{a^2}$, we get

$\boldsymbol{w^2(ub^2 - va^2) = w^2(a^2b^2) \Rightarrow ub^2w^2 - va^2w^2 = a^2b^2w^2}$ ……..(3)

$\boldsymbol{a^2(uz^2 + vw^2) = a^2(w^2z^2) \Rightarrow ua^2z^2 + va^2w^2 = w^2z^2a^2}$ ………(4)

And next, taking (3) + (4), we get

$\boldsymbol{ub^2w^2 + ua^2z^2 = u(b^2w^2 + a^2z^2) = a^2b^2w^2 + w^2z^2a^2}$.

So we get $\boldsymbol{u = s^2 = \dfrac{a^2b^2w^2 + w^2z^2a^2}{b^2w^2 + a^2z^2}}$.

Next, to eliminate $\boldsymbol{u}$, multiplying (1) by $\boldsymbol{z^2}$, and multiplying (2) by $\boldsymbol{b^2}$, we get

$\boldsymbol{ub^2 - va^2 = a^2b^2 \Rightarrow z^2(ub^2 - va^2) = z^2a^2b^2 \Rightarrow ub^2z^2 - va^2z^2 = z^2a^2b^2}$ ……..(5)

$\boldsymbol{uz^2 + vw^2 = w^2z^2 \Rightarrow b^2(uz^2 + vw^2) = b^2w^2z^2 \Rightarrow ub^2z^2 + vb^2w^2 = b^2w^2z^2}$ ……(6)

Subtracting (5) from (6) above, we get

$\boldsymbol{vb^2w^2 + va^2z^2 = v(b^2w^2 + a^2z^2) = b^2w^2z^2 - z^2a^2b^2}$.

So we get $\boldsymbol{v = t^2 = \dfrac{b^2w^2z^2 - z^2a^2b^2}{b^2w^2 + a^2z^2}}$.

Then, we get $\dfrac{u}{v} = \dfrac{s^2}{t^2} = s^2 \cdot \dfrac{1}{t^2} = \dfrac{a^2b^2w^2 + w^2z^2a^2}{b^2w^2 + a^2z^2} . \dfrac{b^2w^2 + a^2z^2}{b^2w^2z^2 - z^2a^2b^2} = \dfrac{a^2b^2w^2 + w^2z^2a^2}{b^2w^2z^2 - z^2a^2b^2}$.

So we get $\dfrac{s^2}{t^2} = \dfrac{a^2b^2w^2 + w^2z^2a^2}{b^2w^2z^2 - z^2a^2b^2}$.

Thus, we get $\left(\dfrac{bsz}{atw}\right)^2 = \dfrac{s^2}{t^2} \cdot \dfrac{b^2z^2}{a^2w^2} = \dfrac{a^2b^2w^2 + w^2z^2a^2}{b^2w^2z^2 - z^2a^2b^2} \cdot \dfrac{b^2z^2}{a^2w^2} = \dfrac{b^2z^2(a^2b^2w^2 + w^2z^2a^2)}{a^2w^2(b^2w^2z^2 - z^2a^2b^2)}$

$= \dfrac{b^2z^2a^2w^2(b^2 + z^2)}{a^2w^2b^2z^2(w^2 - a^2)} = \dfrac{b^2 + z^2}{w^2 - a^2}$.

So the product of the two slopes is $-\left(\dfrac{bsz}{atw}\right)^2 = -\dfrac{b^2 + z^2}{w^2 - a^2}$.

Next, we have

$c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$ in the case of the ellipse.

$c^2 = w^2 - z^2 \Rightarrow z^2 = w^2 - c^2$ in the case of the hyperbola.

So we get $b^2 + z^2 = c^2 - a^2 + w^2 - c^2 = w^2 - a^2$.

And thus, the product is $-\left(\dfrac{bsz}{atw}\right)^2 = -\dfrac{b^2 + z^2}{w^2 - a^2} = -\dfrac{w^2 - a^2}{w^2 - a^2} = -1$.

So we can now say that R is perpendicular to T, and consequently, we get $m = n$.

By the way, the two lines R and T are called normal lines.
A normal line is a line perpendicular to a surface or a curve, is defined at a point in the surface or curve, and is in particular, said to be normal to the surface or the curve at the point.
So R is said to be normal to the hyperbola H at $P(s, t)$, and T is normal to the ellipse E at $P(s, t)$.

And let’s now get back to the physical property of a hyperbola.

In the figure below, **T** is a line bisecting the angle ***APB***, so the angle ***APQ*** is the same as the angle ***FPD***.

And in fact, all the four angles, ∠***APQ***, ∠***QPB***, ∠***FPD***, and ∠***GPD*** are all the same.

Therefore, if a ray emits from the focus ***A***, and contacts ***P***, it is reflected as if it came from the focus ***B***. That is, it gets to proceed in the direction of ***PF***.

And if a ray emits from the focus ***B***, and contacts ***P***, it is reflected as if it came from the focus ***A***.   That is, it gets to proceed in the direction of ***PG***.

Fig. 2