Examples 2 in Parallel Transformations

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0. Suppose $\boldsymbol{P}$**: $(x, y) \longrightarrow (x + 1, y - 1)$**, and $\boldsymbol{Q}$**: $(x, y) \longrightarrow (x + 2, y + 1)$**. Then,

0.0. Assuming $\boldsymbol{L(x, y) = 2x + y - 1 = 0}$, find $\boldsymbol{P \bullet L}$, $\boldsymbol{Q \bullet L}$, and $\boldsymbol{P \bullet (Q \bullet L)}$, and then, put them all in one graph.

0.1. Assuming $\boldsymbol{C(x, y) = x^2 + (y - 1)^2 - 1 = 0}$, find $\boldsymbol{P \bullet C}$, $\boldsymbol{Q \bullet C}$, and $\boldsymbol{Q \bullet (P \bullet C)}$, and then, put them all in one graph.

0.2. Assuming $\boldsymbol{D(x, y) = x^2 - y - 1 = 0}$ for $\boldsymbol{x \geq 0}$, find $\boldsymbol{P \bullet D}$, $\boldsymbol{Q \bullet D}$, and $\boldsymbol{P \bullet (Q \bullet D)}$, and then, put them all in one graph.

1. Suppose that $\boldsymbol{P}$**: $(x, y) \longrightarrow (x + 3, y - 4)$**, that $\boldsymbol{Q}$**: $(x, y) \longrightarrow (x - 2, y + 2)$**, and that applying $\boldsymbol{P}$ and $\boldsymbol{Q}$ to a curve, we get a circle $\boldsymbol{x^2 + (y - 1)^2 = 1}$. Then, find the curve.

Suggestions or Solutions
To the Problem 0 in the Example 0

Assuming P: $(x, y) \longrightarrow (x + 1, y - 1)$, and Q: $(x, y) \longrightarrow (x + 2, y + 1)$, and also, assuming $L(x, y) = 2x + y - 1 = 0$, find $P \bullet L$, $Q \bullet L$, and $P \bullet (Q \bullet L)$, and then, put them all in one graph.

$P \bullet L = 2x + y - 2 = 0$. $\quad Q \bullet L = 2x + y - 6 = 0$. $\quad P \bullet (Q \bullet L) = 2x + y - 7 = 0$.

If not sure of the idea behind the solutions above, follow the steps below.

To begin with, $P \bullet L$ is the transformation of L by P, and thus, is a new equation.

So assuming (s, t) is the new point (the arbitrary point in the new curve, which is the curve of $P \bullet L$, we can set $(s, t) = (x + 1, y - 1)$.

So we get $s = x + 1 \Rightarrow x = s - 1$, and $t = y - 1 \Rightarrow y = t + 1$.

Next, putting $x = s - 1$ and $y = t + 1$ into the original equation L, we can get the connective equation between s and t, that is, the new equation $P \bullet L$.

So doing the substitutions, we get $2x + y - 1 = 2(s - 1) + (t + 1) - 1 = 2s + t - 2 = 0$, which is $P \bullet L$ in the s-t system, which is no other than the x-y system.

Thus, to put it in the x-y system, we just replace s with x, and t with y.

So doing the replacements, we get $P \bullet L = 2x + y - 2 = 0$, which is the solution.

More specifically, we can put it this way, too: $P \bullet L(x, y) = 2x + y - 2 = 0$.

And it is no other than this equation, of course: $y = 2 - 2x$, which can be thus, taken as the solution, too.

And next, $\boldsymbol{Q \bullet L}$ is the transformation of $\boldsymbol{L}$ by $\boldsymbol{Q}$.

So assuming $\boldsymbol{(s, t)}$ is the arbitrary point in the curve of $\boldsymbol{Q \bullet L}$, we can set

$\boldsymbol{(s, t) = (x + 2, y + 1)}$. So we get $\boldsymbol{s = x + 2 \Rightarrow x = s - 2}$, and $\boldsymbol{t = y + 1 \Rightarrow y = t - 1}$.

Thus, putting $\boldsymbol{x = s - 2}$ and $\boldsymbol{y = t - 1}$ into the original equation $\boldsymbol{L}$, we get

$\boldsymbol{2x + y - 1 = 2(s - 2) + (t - 1) - 1 = 2s + t - 6 = 0}$, the equation of $\boldsymbol{Q \bullet L}$ in the $\boldsymbol{s\text{-}t}$ system.

And putting it in the $\boldsymbol{x\text{-}y}$ system, we get $\boldsymbol{Q \bullet L = 2x + y - 6 = 0}$.

More specifically, $\boldsymbol{Q \bullet L(x, y) = 2x + y - 6 = 0}$.

And we can put it this way, too, of course: $\boldsymbol{y = 6 - 2x}$.

And next, assuming $\boldsymbol{A = Q \bullet L}$, we get $\boldsymbol{P \bullet (Q \bullet L) = P \bullet A}$, which is the transformation of $\boldsymbol{A}$ by $\boldsymbol{P}$, so $\boldsymbol{P \bullet (Q \bullet L)}$ is the transformation of $\boldsymbol{A}$ by $\boldsymbol{P}$.

And we know $\boldsymbol{A}$ is the new equation found above, that is, $\boldsymbol{Q \bullet L}$.

So we can set $\boldsymbol{A(x, y) = 2x + y - 6 = 0}$, and $\boldsymbol{A}$ is the original equation now.

Thus, assuming $\boldsymbol{(s, t)}$ is the arbitrary point in the curve of $\boldsymbol{P \bullet A}$, we can set

$\boldsymbol{(s, t) = (x + 1, y - 1)}$. So we get $\boldsymbol{s = x + 1 \Rightarrow x = s - 1}$, and $\boldsymbol{t = y - 1 \Rightarrow y = t + 1}$.

Thus, putting $\boldsymbol{x = s - 1}$ and $\boldsymbol{y = t + 1}$ into the original equation $\boldsymbol{A}$, we get

$\boldsymbol{2x + y - 6 = 2(s - 1) + (t + 1) - 6 = 2s + t - 7 = 0}$, which connects $\boldsymbol{s}$ and $\boldsymbol{t}$.

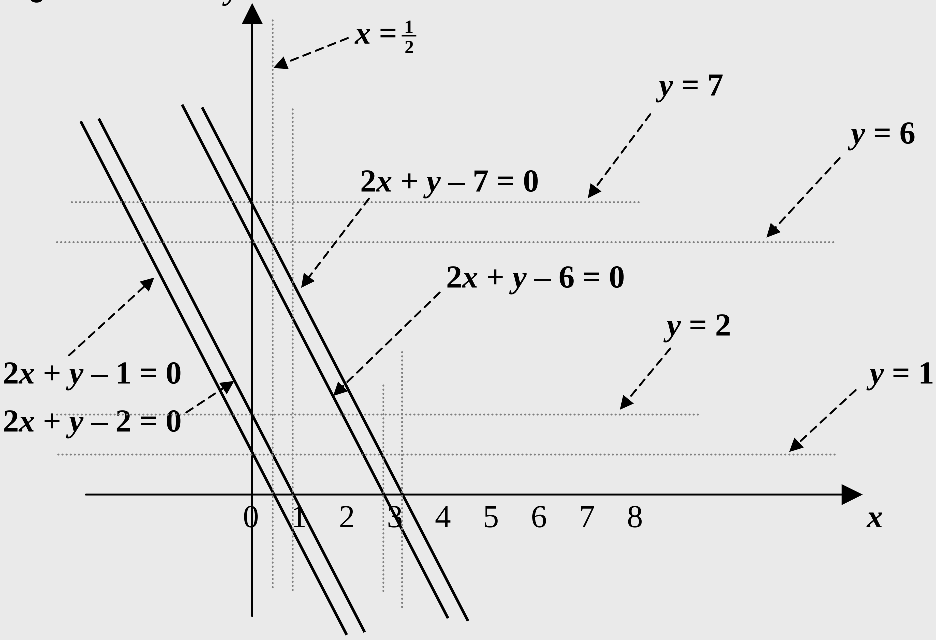
So next, putting it in the $\boldsymbol{x\text{-}y}$ plane, we get $\boldsymbol{P \bullet (Q \bullet L) = 2x + y - 7 = 0}$.

More specifically, $\boldsymbol{P \bullet (Q \bullet L(x, y)) = 2x + y - 7 = 0}$.

And we can put it this way, too, of course: $\boldsymbol{y = 7 - 2x}$.

And putting all the cases above in a graph, we get

Fig. 0



Suggestions or Solutions
To the Problem 1 in the Example 0

Assuming P: $(x, y) \longrightarrow (x + 1, y - 1)$, Q: $(x, y) \longrightarrow (x + 2, y + 1)$, and $C(x, y) = x^2 + (y - 1)^2 - 1 = 0$, find $P \bullet C$, $Q \bullet C$, and $Q \bullet (P \bullet C)$, and then, put them all in one graph.

To begin with, $P \bullet C$ is the transformation of C by P, and is a new equation.
So assuming (s, t) is the arbitrary point in the curve of $P \bullet C$, we can set
$(s, t) = (x + 1, y - 1)$. So we get $s = x + 1 \Rightarrow x = s - 1$, and $t = y - 1 \Rightarrow y = t + 1$.

Thus, putting $x = s - 1$ and $y = t + 1$ into the original equation C, we get
$x^2 + (y - 1)^2 - 1 = (s - 1)^2 + (t + 1 - 1)^2 - 1 = (s - 1)^2 + t^2 - 1 = 0$, the new equation $P \bullet C$.

And putting it in the x-y plane, we get $P \bullet C = (x - 1)^2 + y^2 - 1 = 0$.

In short:

$(s, t) = (x + 1, y - 1)$. So we get $s = x + 1 \Rightarrow x = s - 1$, and $t = y - 1 \Rightarrow y = t + 1$.

Thus, we get $x^2 + (y - 1)^2 - 1 = (s - 1)^2 + (t + 1 - 1)^2 - 1 = (s - 1)^2 + t^2 - 1 = 0$.

So we get $P \bullet C = (x - 1)^2 + y^2 - 1 = 0$.

Next, assuming (s, t) is the arbitrary point in the curve of $Q \bullet C$, we can set
$(s, t) = (x + 2, y + 1)$. So we get $s = x + 2 \Rightarrow x = s - 2$, and $t = y + 1 \Rightarrow y = t - 1$.

Thus, putting into the equation C the x and y above, we can get the connective equation between s and t, that is, the equation of the curve of $Q \bullet C$.

So we get $x^2 + (y - 1)^2 - 1 = (s - 2)^2 + (t - 1 - 1)^2 - 1 = (s - 2)^2 + (t - 2)^2 - 1 = 0$. And putting it in the x-y plane, we get $(x - 2)^2 + (y - 2)^2 - 1 = 0$, which is thus, $Q \bullet C$.

And next, assuming $\boldsymbol{A = P \bullet C}$, we get $\boldsymbol{Q \bullet (P \bullet C) = Q \bullet A}$.

And we know $\boldsymbol{A}$ is the new equation found above, that is, $\boldsymbol{P \bullet C}$.

So we can set $\boldsymbol{A(x, y) = (x - 1)^2 + y^2 - 1 = 0}$, and $\boldsymbol{A}$ is the original equation now.

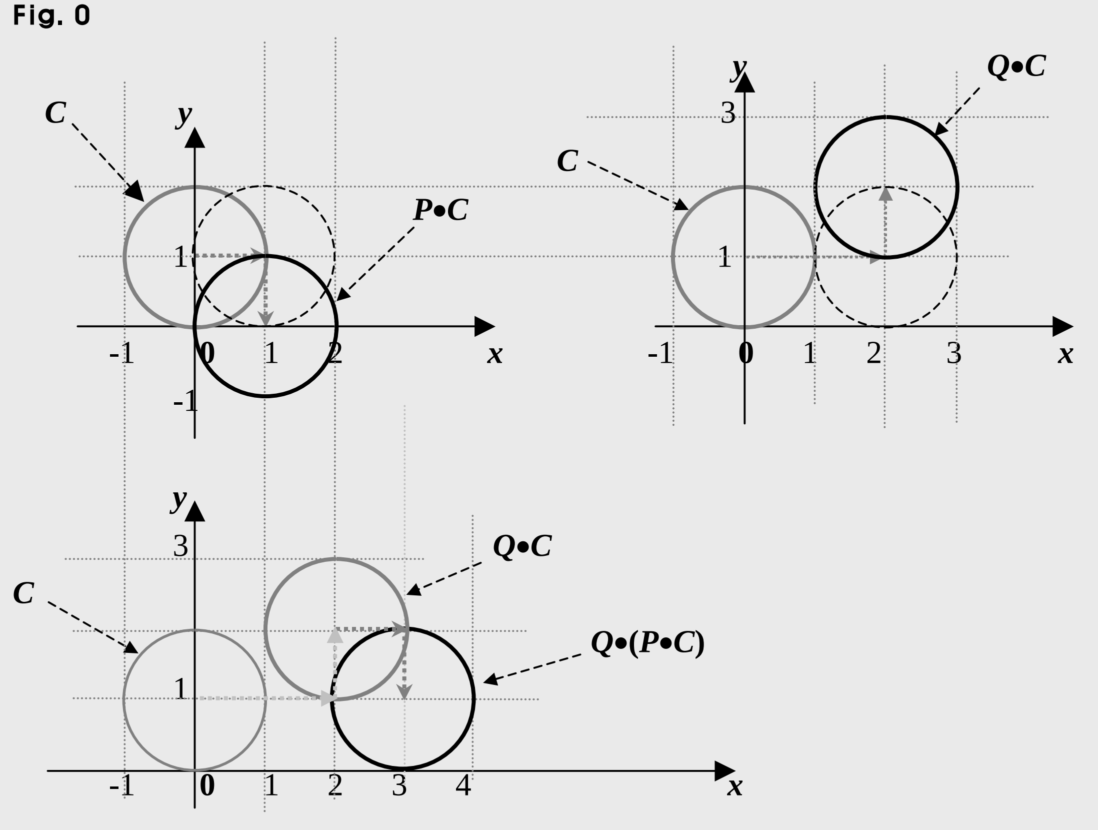
So assuming $\boldsymbol{(s, t)}$ is the arbitrary point in the curve of $\boldsymbol{P \bullet A}$, we can set
$\boldsymbol{(s, t) = (x + 2, y + 1)}$. So we get $\boldsymbol{s = x + 2 \Rightarrow x = s - 2}$, and $\boldsymbol{t = y + 1 \Rightarrow y = t - 1}$.

Thus, putting into the equation $\boldsymbol{A}$ the $\boldsymbol{x}$ and $\boldsymbol{y}$ above, we can get the connective equation between $\boldsymbol{s}$ and $\boldsymbol{t}$, that is, the equation of the curve of $\boldsymbol{P \bullet A}$. So we get

$$\boldsymbol{(x - 1)^2 + y^2 - 1 = (s - 2 - 1)^2 + (t - 1)^2 - 1 = (s - 3)^2 + (t - 1)^2 - 1 = 0}.$$

And putting it in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, we get $\boldsymbol{(x - 3)^2 + (y - 1)^2 - 1 = 0}$, which is $\boldsymbol{Q \bullet (P \bullet C)}$.

Fig. 0



Suggestions or Solutions
To the Problem 2 in the Example 0

Assuming P: $(x, y) \longrightarrow (x + 1, y - 1)$, Q: $(x, y) \longrightarrow (x + 2, y + 1)$, and $D(x, y) = x^2 - y - 1 = 0$ for $x \geq 0$, find $P \bullet D$, $Q \bullet D$, and $P \bullet (Q \bullet D)$, and then, put them all in one graph.

To begin with, assuming (s, t) is the arbitrary point in the curve of $P \bullet D$, we can set $(s, t) = (x + 1, y - 1)$. So we get $s = x + 1 \Rightarrow x = s - 1$, and $t = y - 1 \Rightarrow y = t + 1$.

Thus, putting $x = s - 1$ and $y = t + 1$ into the original equation D, we get $x^2 - y - 1 = (s - 1)^2 - (t + 1) - 1 = (s - 1)^2 - t - 2 = 0$.

Besides, $x \geq 0 \Rightarrow s = x + 1 \geq 1$. Thus, putting the new curve in the x-y plane, we get $(x - 1)^2 - y - 2 = 0$ for $x \geq 1$, which is $P \bullet D$.

In short:

$(s, t) = (x + 1, y - 1)$. So we get $s = x + 1 \Rightarrow x = s - 1$, and $t = y - 1 \Rightarrow y = t + 1$.

Thus, we get $x^2 - y - 1 = (s - 1)^2 - (t + 1) - 1 = (s - 1)^2 - t - 2 = 0$.

Besides, $x \geq 0 \Rightarrow s = x + 1 \geq 1$.

So we get $P \bullet D = (x - 1)^2 - y - 2 = 0$ for $x \geq 1$.

Next, assuming (s, t) is the arbitrary point in the curve of $Q \bullet D$, we can set $(s, t) = (x + 2, y + 1)$. So we get $s = x + 2 \Rightarrow x = s - 2$, and $t = y + 1 \Rightarrow y = t - 1$.

So next, putting $x = s - 2$ and $y = t - 1$ into the equation D, we get $x^2 - y - 1 = (s - 2)^2 - (t - 1) - 1 = (s - 2)^2 - t = 0$.

Besides, $x \geq 0 \Rightarrow s = x + 2 \geq 2$. Thus, putting the new curve in the x-y plane, we get $(x - 2)^2 - y = 0$ for $x \geq 2$, which is thus, $Q \bullet D$.

And next, assuming $\boldsymbol{A = Q \bullet C}$, we get $\boldsymbol{P \bullet (Q \bullet D) = P \bullet A}$.

And we know $\boldsymbol{A}$ is the new equation found above, that is, $\boldsymbol{Q \bullet D}$. And in $\boldsymbol{A}$, we have $\boldsymbol{x \geq 2}$.

So we can set $\boldsymbol{A(x, y) = (x - 2)^2 - y = 0}$ for $\boldsymbol{x \geq 2}$, and $\boldsymbol{A}$ is the original equation now.

So assuming $\boldsymbol{(s, t)}$ is the arbitrary point in the curve of $\boldsymbol{P \bullet A}$, we can set
$\boldsymbol{(s, t) = (x + 1, y - 1)}$. So we get $\boldsymbol{s = x + 1 \Rightarrow x = s - 1}$, and $\boldsymbol{t = y - 1 \Rightarrow y = t + 1}$.

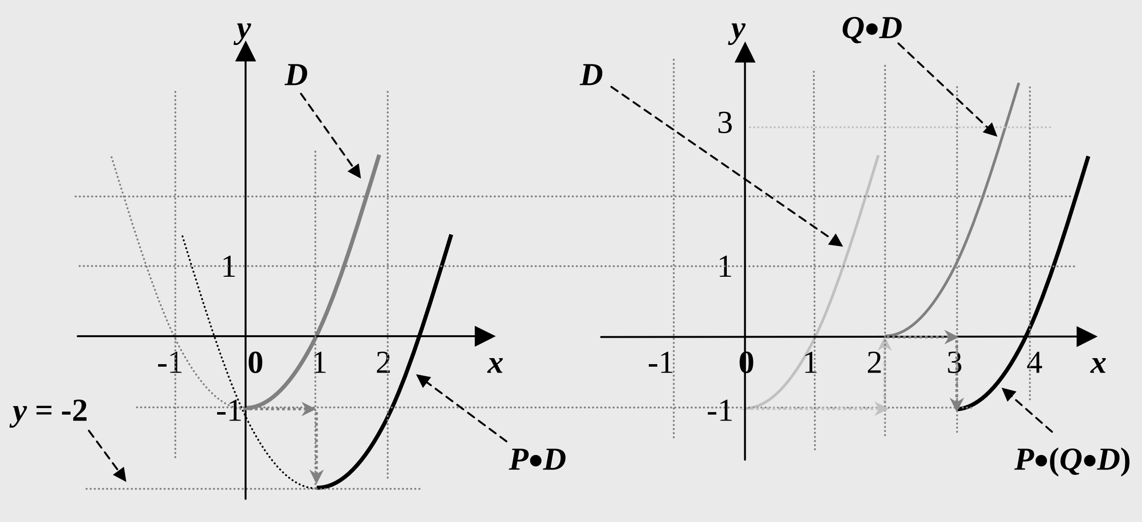
Thus, putting into the equation $\boldsymbol{A}$ the $\boldsymbol{x}$ and $\boldsymbol{y}$ above, we can get the connective equation between $\boldsymbol{s}$ and $\boldsymbol{t}$. So putting them into the equation, we get

$\boldsymbol{(x - 2)^2 - y = (s - 1 - 2)^2 - (t + 1) = (s - 3)^2 - t - 1 = 0}$.

Besides, $\boldsymbol{x \geq 2 \Rightarrow s = x + 1 \geq 3}$.

Thus, putting the curve of the new equation $\boldsymbol{P \bullet A}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, we get
$\boldsymbol{(x - 3)^2 - y - 1 = 0}$ for $\boldsymbol{x \geq 3}$, which is $\boldsymbol{P \bullet (Q \bullet D)}$.

Fig. 0



Suggestions or Solutions
To the Problem in the Example 1

Suppose that P: $(x, y) \longrightarrow (x + 3, y – 4)$, that Q: $(x, y) \longrightarrow (x – 2, y + 2)$, and that applying P and Q to a curve, we get a circle $x^2 + (y – 1)^2 = 1$. Then, find the curve.

Assuming S is a transformation we can make combining P and Q, we get
S: $(x, y) \longrightarrow (x + 3 + (-2), y + (-4) + 2) = (x + 1, y – 2)$.

And applying a parallel transformation to a circle, we get a circle.
So if A is the curve we want to find, A is a circle. And if A is centered at (a, b), and S is applied to A, the center of the new circle is $(a + 1, b + (-2))$, which is $(a + 1, b – 2)$.

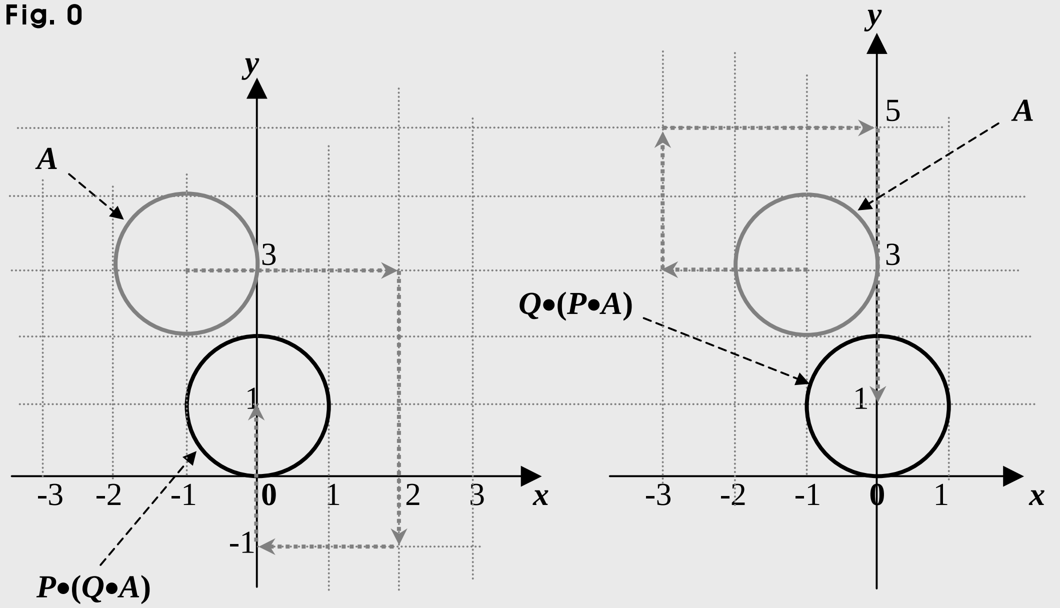
And the new circle is given, and is $x^2 + (y – 1)^2 = 1$, so the center is $(0, 1)$.

Thus, we get $a + 1 = 0 \Rightarrow a = -1$, and $b – 2 = 1 \Rightarrow b = 3$.

So the center of A is $(-1, 3)$, and the radius is 1, because the radius of the new circle is 1.

So the circle A is $(x – (-1))^2 + (y – 3)^2 = 1 \Rightarrow (x + 1)^2 + (y – 3)^2 = 1$.

Fig. 0



If not quite sure of the ideas behind the processes above, follow the steps below.

We know **P** and **Q** are parallel transformations. What do we mean by though, applying ***P*** and ***Q***? That is, which one do we need to apply first?

It doesn't matter which one we apply first. Since both are parallel transformations, we get the same result whichever of the two we apply first. How?

In a parallel transformation, both of the two curves the original and the new themselves are the same, and the only thing that can change is the location of the original curve.

That's because, applying a parallel transformation to a curve, we translate the curve, that is, we move the curve along a line without rotation and deformation. So it can get a new location only. If the curve is a line though, it can get the same location, too. How?

If ***T*** is a parallel transformation, we can define it the way below.

$\boldsymbol{T: (x, y) \longrightarrow (x + \Delta x, y + \Delta y)}$, where $\boldsymbol{\Delta x}$ is the change in the $\boldsymbol{x}$-coordinate, and $\boldsymbol{\Delta y}$ is the change in the $\boldsymbol{y}$-coordinate.

So if $\left|\frac{\Delta y}{\Delta x}\right|$, that is, the magnitude of $\frac{\Delta y}{\Delta x}$ is the same as the magnitude of the slope of the original line, the new line is no other than the original line, that is, both lines are the same, since the new equation is the same as the original.

Suppose now to a curve ***C***, we apply ***P*** first, and then, apply ***Q***.

Then, the curve ***C*** moves in the amount of 3 along the ***x***-axis, and in the amount of -4 along the ***y***-axis, and then, it moves in the amount of -2 along the ***x***-axis, and in the amount of 2 along the ***y***-axis.

That is, the net amount of movement along the ***x***-axis is $\mathbf{3 - 2 = 1}$, and the net along the ***y***-axis is $\mathbf{-4 + 2 = -2}$.

And again, to the curve ***C***, applying ***Q*** first, and then, applying ***P***, we move the curve ***C*** in the amount of -2 along the ***x***-axis, and in the amount of 2 along the ***y***-axis, and then, move it in the amount of 3 along the ***x***-axis, and in the amount of -4 along the ***y***-axis.

That is, the net amount of movement along the x-axis is $-2 + 3 = 1$, and the net along the y-axis is $2 - 4 = -2$. Thus, we get the same result whichever gets applied first.

Making a transformation combining transformations, we say that the transformation made is a composite transformation, which will be covered later in this book.

And getting a composite transformation combining parallel transformations, we can get it taking the sum of the changes in the x-coordinate, and the sum of the changes in the y-coordinate. So the composite transformation is a parallel transformation, too.

Suppose for instance, U and V are parallel transformations as follows.

U: $(x, y) \longrightarrow (x + a, y + b)$, and V: $(x, y) \longrightarrow (x + c, y + d)$.

Then, assuming W is a composite transformation made of U and V, we get

W: $(x, y) \longrightarrow (x + a + c, y + b + d)$. So $\Delta x = a + c$, and $\Delta y = b + d$.

Now, in this example, assuming S is a composite transformation made of P and Q in this example, we can combine the two the way as follows.

S: $(x, y) \longrightarrow (x + 3 + (-2), y + (-4) + 2) = (x + 1, y - 2)$.

So we get S: $(x, y) \longrightarrow (x + 1, y - 2)$.

Thus, applying S to a curve, we apply P and Q to the curve.

And this example is saying that applying S to a curve, we get a circle $x^2 + (y - 1)^2 = 1$. That is, the new curve is the circle above. What kind of curve then, is the original curve?

Since S is a parallel transformation, both curves the new and the original themselves are the same. So the original curve is a circle, too.
How then, can we find the original curve, that is, the original equation?

If a circle makes a movement, the center of the circle makes the same movement, and vice versa. So moving a circle, we can move its center. And then, using the radius, we can complete the movement with much ease.

Thus, assuming A is the original curve, and applying S to A, that is, applying S to the circle A, we move the center of A in the amount of 1 along the x-axis, and in the amount of -2 along the y-axis.

So assuming now that the center of A is (a, b), we can see that the center of the new circle is $(a + 1, b + (-2))$, which is $(a + 1, b - 2)$.

And we know the new circle is $x^2 + (y - 1)^2 = 1$. So the center of the new circle is $(0, 1)$.

Thus, we get $(a + 1, b - 2) = (0, 1)$. So we get $a + 1 = 0$, and $b - 2 = 1$.

That is, $a = -1$, and $b = 3$. Thus, the center of A is $(-1, 3)$.

How then, can we get the equation of the original circle A?

We can form a circle knowing the center and the radius. That is, knowing the center and the radius, we can get the equation of the circle. What then, is the radius of A?

We can put in an equation a circle of radius r centered at (a, b) the way below.
$(x - a)^2 + (y - b)^2 = r^2$.

And we know the new curve is a circle $x^2 + (y - 1)^2 = 1$. So the radius is 1.
Thus, so is the radius of the original circle A, where the center is $(-1, 3)$.

So putting the circle A in an equation, we get

$(x - (-1))^2 + (y - 3)^2 = 1 \Rightarrow (x + 1)^2 + (y - 3)^2 = 1$.

And putting all the ideas in a graph, we get

Fig. 0