# Examples in Symmetry about a line 3



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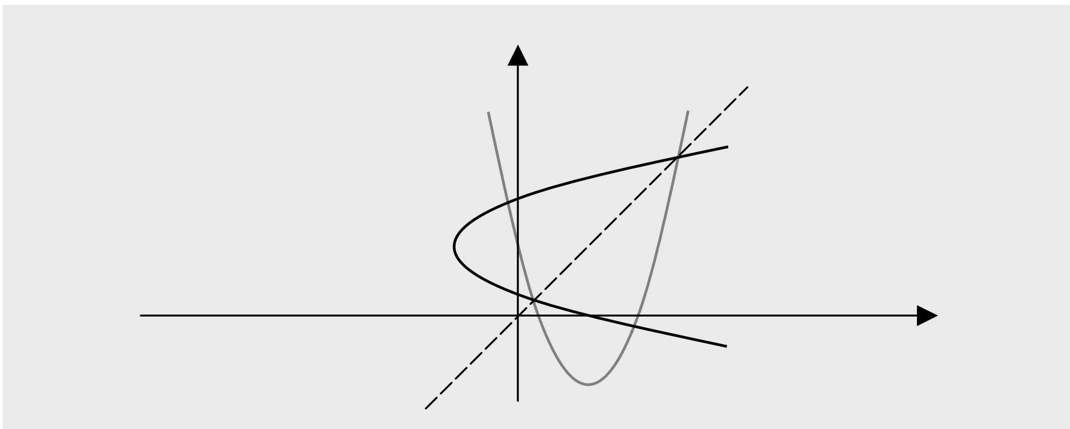


0. Find $\boldsymbol{g = R \bullet f}$, where $\boldsymbol{R: (x, y) \longrightarrow (y, x)}$, and $\boldsymbol{y = f(x) = (1 - x)^3 + 2}$.

1. Find a function, the curve of which is symmetric about the line $\boldsymbol{y = x}$ to the curve of a function $\boldsymbol{y = f(x) = \sqrt[3]{1 - x} + 2}$ for $\boldsymbol{x}$ real.

2. Assuming $\boldsymbol{R: (x, y) \longrightarrow (y, x)}$, find the transformation of each $\boldsymbol{f}$ below by $\boldsymbol{R}$.

2.0. $\boldsymbol{y = f(x) = x^2 - 2x + 1}$ for $\boldsymbol{x \geq 1}$.

2.1. $\boldsymbol{y = f(x) = (1 - x)^3 + 2}$ for $\boldsymbol{x \leq 1}$.

2.2. $\boldsymbol{y = f(x) = \sqrt[3]{1 - x} + 2}$ for $\boldsymbol{x \geq 1}$.

## Suggestions or Solutions
## To the Problem in the Example 0

**Find $g = R \bullet f$, where $R$: $(x, y) \longrightarrow (y, x)$, and $y = f(x) = (1 - x)^3 + 2$.**

Assuming first, $g$ is the new function $R \bullet f$, we can set $(x_g, y_g) = (y_f, x_f)$.

So we get $x_f = y_g$, and $y_f = x_g$. So next, we get

$$y = f(x) = (1 - x)^3 + 2 \Rightarrow y_f = f(x_f) = (1 - x_f)^3 + 2 \Rightarrow x_g = (1 - y_g)^3 + 2$$

$$\Rightarrow x_g - 2 = (1 - y_g)^3 \Rightarrow 1 - y_g = \sqrt[3]{x_g - 2} \Rightarrow y_g = 1 - \sqrt[3]{x_g - 2}$$

$$\Rightarrow y_g = g(x_g) = 1 - \sqrt{x_g - 2} \Rightarrow y = g(x) = 1 - \sqrt{x - 2}.$$

So we get $g = R \bullet f = 1 - \sqrt[3]{x - 2}$.

*If not sure of the idea behind the solution above, follow the steps below.*

To begin with, $R$ is the transformation symmetric about the line $y = x$, which is called the axis of symmetry. And the dot $\bullet$ is the transformation operator, so $g$ is the transformation of $f$ by $R$. And $f$ is a function. So is $g$ the new function?

It is. Since $f$ is one-to-one, the transformation of $f$ by $R$ is a new function, and is called the inverse. So assuming next, $g$ is the new function, $(x_g, y_g)$ is the new point, and $(x_f, y_f)$ is the original point, we can set $(x_g, y_g) = (y_f, x_f)$. Then, we get $x_f = y_g$, and $y_f = x_g$.

So next, putting $x_f = y_g$ and $y_f = x_g$ into the original equation $y = f(x)$, we get

$$y = f(x) \Rightarrow y_f = f(x_f) \Rightarrow x_g = f(y_g) \Rightarrow x_g = (1 - y_g)^3 + 2 \Rightarrow x_g - 2 = (1 - y_g)^3$$

$$\Rightarrow 1 - y_g = \sqrt[3]{x_g - 2} \Rightarrow y_g = 1 - \sqrt[3]{x_g - 2} \Rightarrow y_g = g(x_g) = 1 - \sqrt{x_g - 2}$$

$$\Rightarrow y = g(x) = 1 - \sqrt{x - 2}.$$

So the new function $g$ is $y = g(x) = 1 - \sqrt[3]{x - 2}$.

And we can put it this way, too. $g = R \bullet f = 1 - \sqrt[3]{x - 2}$.

How do we know if $f$ is one-to-one, though?

If $a$ and $b$ are constant, $a \neq b$, and $f(a) \neq f(b)$, we can say that $f$ is one-to-one.

First, we have $f(a) = (1 - a)^3 + 2$, and $f(b) = (1 - b)^3 + 2$.

Next, assuming $s = 1 - a$, and $t = 1 - b$, we get $f(a) = s^3 + 2$ and $f(b) = t^3 + 2$, where $s \neq t$ because $a \neq b$. And next, if $f(a) - f(b) \neq 0$, we get $f(a) \neq f(b)$, so $f$ is one-to-one. Otherwise, $f$ is not one-to-one, of course. So taking the difference now, we get

$$f(a) - f(b) = s^3 + 2 - (t^3 + 2) = s^3 - t^3 = (s - t)(s^2 + st + t^2).$$

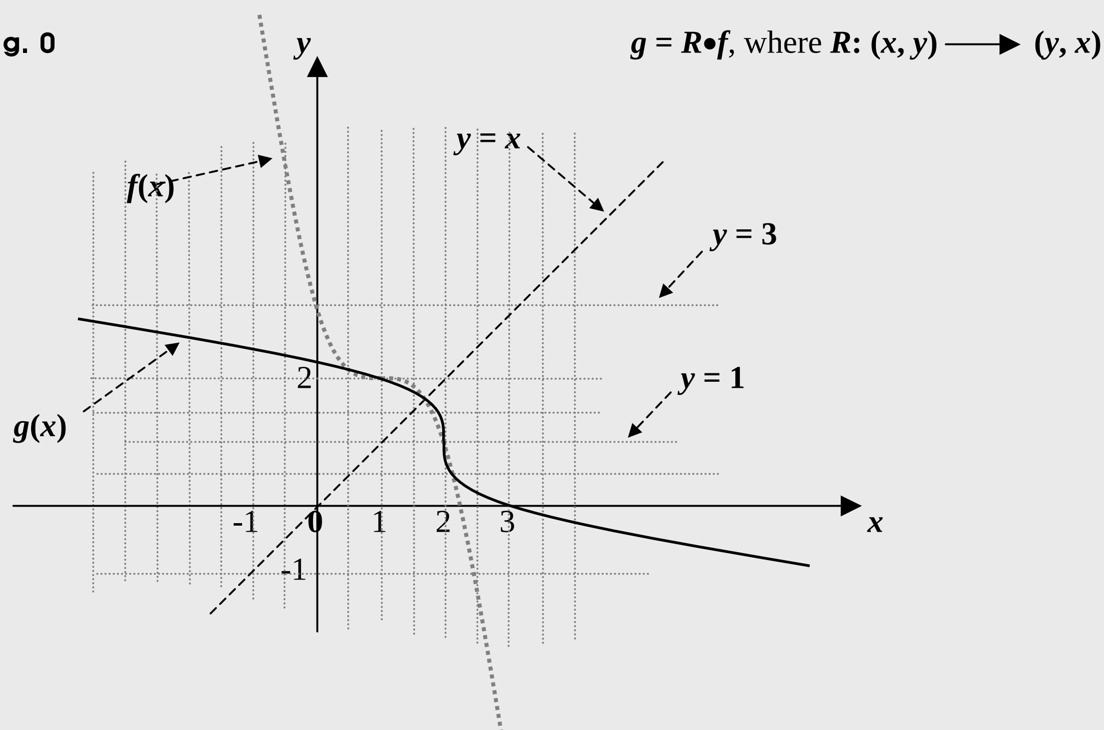
Then first, we get $s - t \neq 0$, because $s \neq t$. And next, we get

$$s^2 + st + t^2 = s^2 + st + \tfrac{t^2}{4} - \tfrac{t^2}{4} + t^2 = (s + \tfrac{t}{2})^2 + \tfrac{3t^2}{4} > 0 \Rightarrow s^2 + st + t^2 \neq 0.$$

Therefore, $(s - t)(s^2 + st + t^2) \neq 0 \Rightarrow f(a) - f(b) \neq 0 \Rightarrow f(a) \neq f(b)$.

So $f$ is one-to-one, and can have the inverse. And thus, $g$ is a new function, and is the inverse of $f$. And we say that the two functions $f$ and $g$ are symmetric about the line $y = x$.

Fig. 0



## Suggestions or Solutions
## To the Problem in the Example 1

**Find a function, the curve of which is symmetric about the line $y = x$ to the curve of a function $y = f(x) = \sqrt[3]{1-x}+2$.**

Assuming first, $g$ is the new function, we can set $(x_g, y_g) = (y_f, x_f)$.

So we get $x_f = y_g$, and $y_f = x_g$. So next, we get

$$y = f(x) \Rightarrow y_f = f(x_f) \Rightarrow x_g = f(y_g) = \sqrt[3]{1-y_g}+2 \Rightarrow x_g = \sqrt[3]{1-y_g}+2$$

$$\Rightarrow x_g - 2 = \sqrt[3]{1-y_g} \Rightarrow (x_g - 1)^3 = 1 - y_g \Rightarrow y_g = 1 - (x_g - 1)^3 = (1 - x_g)^3 + 1$$

$$\Rightarrow y_g = g(x_g) = (1 - x_g)^3 + 1 \Rightarrow y = g(x) = (1 - x)^3 + 1$$

So is the original function $f$ one-to-one?

If $a \neq b$, and $f(a) \neq f(b)$, $f$ is one-to-one. So let's see now if it is the case.

First, we can get $f(a) = \sqrt[3]{1-a}+2$, and $f(b) = \sqrt[3]{1-b}+2$.

Next, assuming $s = 1 - a$ and $t = 1 - b$, we get $f(a) = \sqrt[3]{s}+2$, $f(b) = \sqrt[3]{t}+2$, and $s \neq t$ since $a \neq b$. And if $f(a) - \mathrm{f}(b) \neq 0$, we get $f(a) \neq f(b)$, and $f$ is one-to-one. Otherwise, $f$ is not.

So taking the difference, we get $f(a) - f(b) = \sqrt[3]{s}+2-(\sqrt[3]{t}+2) = \sqrt[3]{s}-\sqrt[3]{t}$.

And we know $\sqrt[3]{s} = (\sqrt[9]{s})^3$. In other words, we have $s^{1/3} = (s^{1/9})^3$.

So we get $\sqrt[3]{s}-\sqrt[3]{t} = (\sqrt[9]{s})^3-(\sqrt[9]{t})^3 = (\sqrt[9]{s}-\sqrt[9]{t})\{(\sqrt[9]{s})^2+\sqrt[9]{s}\sqrt[9]{t}+(\sqrt[9]{t})^2\}$.

That is, we get $s^{1/3} - t^{1/3} = (s^{1/9})^3 - (t^{1/9})^3 = (s^{1/9} - t^{1/9})(s^{2/9} + s^{1/9}t^{1/9} + t^{2/9})$.

And assuming $u = \sqrt[9]{s}$, and $v = \sqrt[9]{t}$, we get

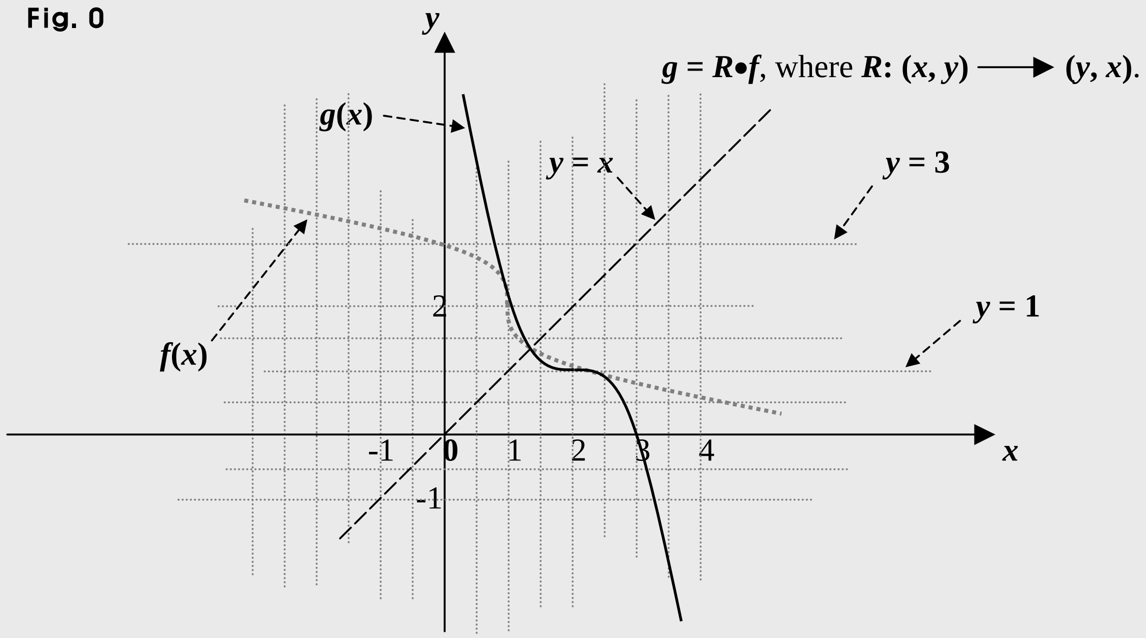
$u \neq v$ since $s \neq t$, and also, $(\sqrt[9]{s}-\sqrt[9]{t})\{(\sqrt[9]{s})^2+\sqrt[9]{s}\sqrt[9]{t}+(\sqrt[9]{t})^2\} = (u - v)(u^2 + uv + v^2)$.

And going back to the example 0, we can see this: $\boldsymbol{u^2 + uv + v^2 \neq 0}$.

So we get $\boldsymbol{f(a) - f(b) \neq 0 \Rightarrow f(a) \neq f(b)}$.

And thus, ***f*** is one-to-one, and can have its inverse.

Fig. 0



## Suggestions or Solutions
## To the Problem 0 in the Example 2

**Assuming $R$: $(x, y) \longrightarrow (y, x)$, and $y = f(x) = x^2 - 2x + 1$ for $x \geq 1$, find $R \bullet f$.**

Assuming first, $g$ is the new function, we can set $(x_g, y_g) = (y_f, x_f)$.

So we get $x_f = y_g$, and $y_f = x_g$. So next, we get

$$y = f(x) \Rightarrow y_f = f(x_f) \Rightarrow x_g = f(y_g) = (y_g^2 - 2y_g + 1) = (y_g - 1)^2 \Rightarrow x_g = (y_g - 1)^2$$

$$\Rightarrow y_g - 1 = \pm\sqrt{x_g} \Rightarrow y_g = 1 \pm \sqrt{x_g}.$$

And we have $y_g = x_f$. And we know $x_f \geq 1$. So we get $y_g \geq 1$.

Thus, we get $y_g = 1 + \sqrt{x_g}$.

And we have $x_g = y_f$, and we get $y_f = f(x_f) = x_f^2 - 2x_f + 1 = (x_f - 1)^2 \geq 0$, since $x_f \geq 1$.
So we get $y_f \geq 0$, and thus, $x_g \geq 0$, since $x_g = y_f$.

And therefore, the new function $g$ is $y = g(x) = 1 + \sqrt{x}$ for $x \geq 0$.

So is the original function $f$ one-to-one?

Putting the function $f$ this way: $y = f(x) = (x - 1)^2$ for $x \geq 1$, we can see $f$ is one-to-one, because the curve of $f$ is a right half of a parabola, and the axis of symmetry of the parabola is a line $x = 1$. And we can check to see if $f$ is one-to-one the way below, too.

If $f(a) \neq f(b)$ where $a$ and $b$ are constant, and $a \neq b$, $f$ is one-to-one.

That is, if $f(a) - f(b) \neq 0$, $f$ is one-to-one. Otherwise, $f$ is not, and thus, is many-to-one.

So taking now the difference, we get

$$f(a) - f(b) = (a - 1)^2 - (b - 1)^2 = \{(a - 1) + (b - 1)\}\{(a - 1) - (b - 1)\} = (a + b - 2)(a - b).$$

Then first, we know $a - b \neq 0$ since $a \neq b$.

And next, we have $x \geq 1$. So we get $a \geq 1$, and $b \geq 1$. And also, we have $a \neq b$.

So we get $a + b > 2 \Rightarrow a + b - 2 > 0$, which means $a + b - 2 \neq 0$. So we get $f(a) \neq f(b)$.

And in a transformation symmetric about a line $\boldsymbol{y = x}$, the extents of the variables in the original equation can change, too. So specifying the new equation, we need to specify the new extents, that is, the new domain and range, at least the new domain if the original equation is specified with its domain.

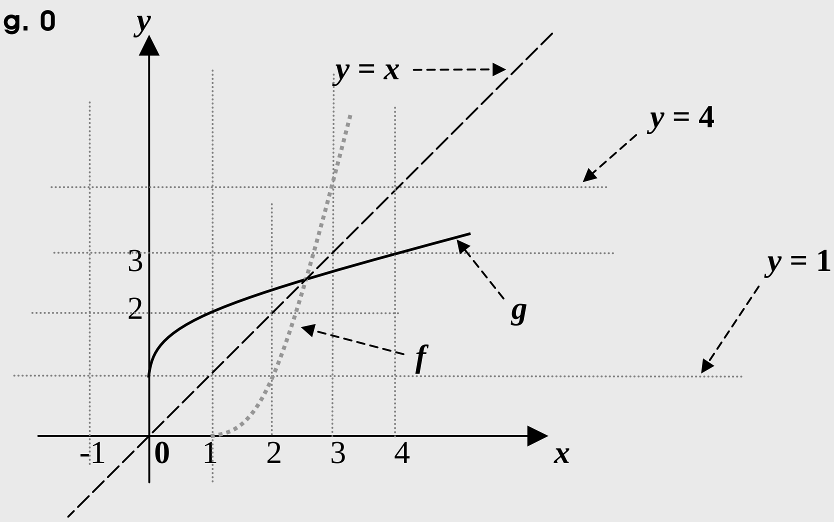
So next, moving on to the extents of the variables in the new curve, and beginning with the extent of $\boldsymbol{x_g}$, we can use $\boldsymbol{x_g = y_f}$, and $\boldsymbol{y = f(x) = x^2 - 2x + 1}$ for $\boldsymbol{x \geq 1}$.

Then, we get $\boldsymbol{y_f = f(x_f) = x_f^{\,2} - 2x_f + 1 = (x_f - 1)^2 \geq 0}$, since $\boldsymbol{x_f \geq 1}$.

So we get $\boldsymbol{x_g \geq 0}$, since $\boldsymbol{x_g = y_f}$, and $\boldsymbol{y_f \geq 0}$ as we can see in the equation above.

And next, moving on to the extent of $\boldsymbol{y_g}$, we can use $\boldsymbol{y_g = x_f}$.
We know $\boldsymbol{x_f \geq 1}$. So we get $\boldsymbol{y_g \geq 1}$.

Fig. 0




$\boldsymbol{g = R \bullet f}$, where $\boldsymbol{R: (x, y) \longrightarrow (y, x)}$, and $\boldsymbol{y = f(x) = x^2 - 2x + 1}$ for $\boldsymbol{x \geq 1}$.

## Suggestions or Solutions
## To the Problem 1 in the Example 2

**Assuming $R$: $(x, y) \longrightarrow (y, x)$, and $y = f(x) = (1-x)^3 + 2$ for $x \le 1$, find $R \bullet f$.**

Assuming first, $g$ is the new function, we can set $(x_g, y_g) = (y_f, x_f)$.

So next, we get $y = f(x) \Rightarrow y_f = f(x_f) \Rightarrow x_g = f(y_g) = (1 - y_g)^3 + 2 \Rightarrow x_g - 2 = (1 - y_g)^3$

$\Rightarrow 2 - x_g = (y_g - 1)^3 \Rightarrow \ y_g - 1 = \sqrt[3]{2 - x_g} \Rightarrow y_g = 1 + \sqrt[3]{2 - x_g}$. And we have $x_g = y_f$.

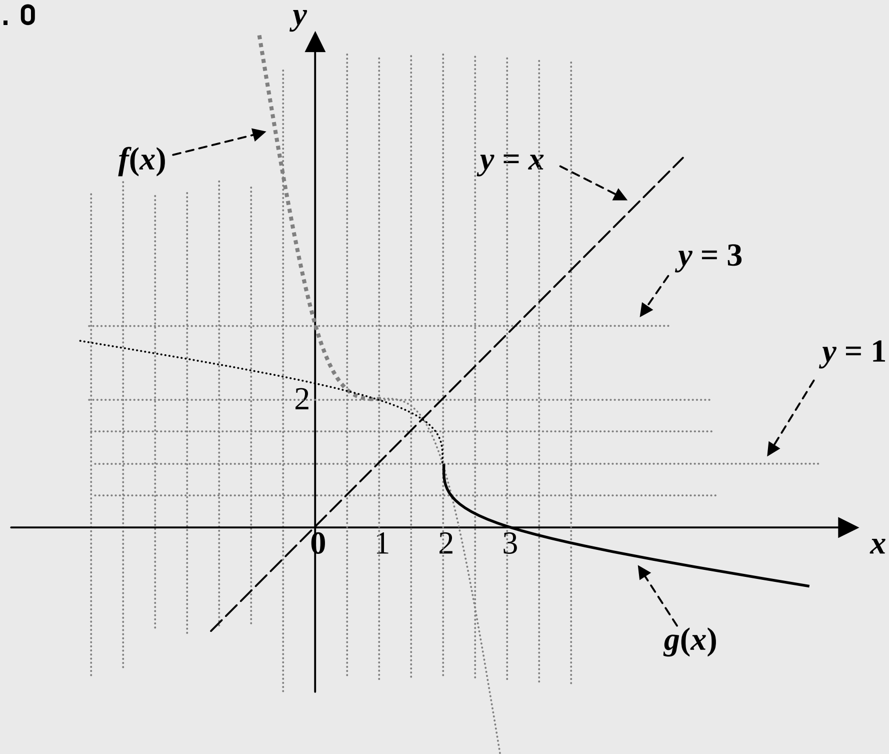
Also, we get $y_f = (1 - x_f)^3 + 2 \ge 2$, because $x_f \le 1$. So we get $y_g = 1 + \sqrt[3]{2 - x_g}$ for $x_g \ge 2$.

So the new function $g$ is $y = g(x) = 1 + \sqrt[3]{2 - x}$ for $x \ge 2$.

So is the original function $f$ one-to-one?

In the original $f$, $y$ keeps decreasing as $x$ keeps increasing, so $f$ is one-to-one.
And also, we can check to see if $f$ is one-to-one using the way we did in the example 0.

Fig. 0



## Suggestions or Solutions
## To the Problem 2 in the Example 2

**Assuming $R$: $(x, y) \longrightarrow (y, x)$, and $y = f(x) = \sqrt[3]{1-x} + 2$ for $x \geq 1$, find $R \bullet f$.**

Assuming first, $g$ is the new function, we can set $(x_g, y_g) = (y_f, x_f)$.

So next, we get $y = f(x) \Rightarrow y_f = f(x_f) \Rightarrow x_g = f(y_g) = \sqrt[3]{1-y_g} + 2 \Rightarrow x_g - 2 = \sqrt[3]{1-y_g}$

$\Rightarrow (x_g - 2)^3 = 1 - y_g \Rightarrow y_g = 1 - (x_g - 2)^3 = 1 + (2 - x_g)^3 \Rightarrow y_g = 1 + (2 - x_g)^3$

And we have $x_g = y_f$, and we get $y_f = f(x_f) = \sqrt[3]{1-x_f} + 2 \leq 2$, since $x_f \geq 1$. So $x_g \leq 2$.

And therefore, the new function $g$ is $y = g(x) = 1 + (2 - x)^3$ for $x \leq 2$.

So is the original function $f$ one-to-one?

In the original function $f$, the $y$-value keeps decreasing as the $x$-value keeps increasing, and thus, $f$ is one-to-one.
And also, we can check to see if $f$ is one-to-one using the way we did in the example 1.

Fig. 0