Examples 3 in Symmetry about a line 7

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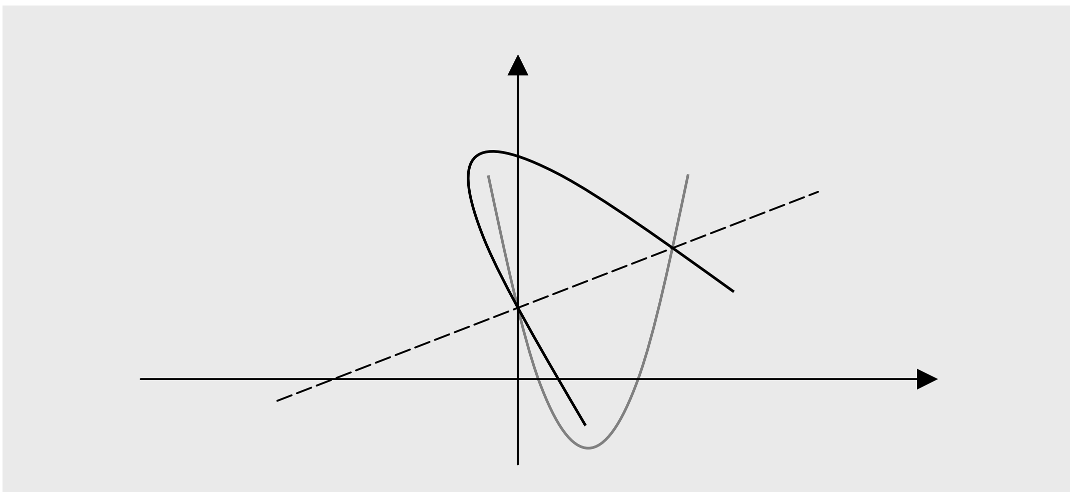


Find a curve which is symmetric about each of the lines below to the curve of an equation $\boldsymbol{f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0}$.

0. $\boldsymbol{y = 0.5x}$ 1. $\boldsymbol{y = 2x - 3}$ 2. $\boldsymbol{y = -2x + 6}$

Suggestions or Solutions
To the Problem in the Example 0

Find the equation symmetric about a line $y = 0.5x$ to the equation below.
$f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0$.

Assuming g is the equation we want, we get $g(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0$.

If not sure of the idea behind the solution above, follow the steps below.

To begin with, assuming L is a transformation symmetric about a line $y = ax + b$, we can put L the way as follows. $L: (x, y) \longrightarrow \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$.

And applying L to $f(x, y) = 0$, we get a new equation, which is symmetric about the line to $f(x, y) = 0$. So we want to apply such a transformation, and thus, want to get the transformation first. So let's get it now.

Suppose first, (x, y) is the original arbitrary point, and (s, t) is the new arbitrary point. Then, (x, y) becomes (s, t), and the two are symmetric about the line $y = 0.5x$, so both are the same distance away from the axis of symmetry, that is, the line $y = 0.5x$.

So assuming (c, d) is the midpoint between (x, y) and (s, t), we get $c = \frac{x+s}{2}$, and $d = \frac{y+t}{2}$.

Next, since the midpoint (c, d) is in the line $y = 0.5x$, we get $d = 0.5c$.

That is, we get $(y + t)/2 = (x + s)/4$. So we get $2(y + t) = x + s$.

Next the line segment connecting (x, y) and (s, t) is perpendicular to the line $y = 0.5x$, so the product of the slopes of the line segment and the line is -1. So the slope of the line segment is -2 since 0.5 is the slope of the axis of symmetry $y = 0.5x$.

And thus, we get $(y - t)/(x - s) = -2 \Rightarrow y - t = 2(s - x)$.

So we get a system of two equations where $2t - s = x - 2y$, and $t + 2s = 2x + y$.

And thus, solving the system for t first, we get

$4t - 2s + (t + 2s) = 2x - 4y + (2x + y) \Rightarrow 5t = 4x - 3y \Rightarrow t = (4x - 3y)/5$.

And next, putting the t above into $2t - s = x - 2y$, we get $2(4x - 3y)/5 - s = x - 2y$

$\Rightarrow s = 2(4x - 3y)/5 - 5(x - 2y)/5 = (3x + 4y)/5 \Rightarrow s = (3x + 4y)/5$.

And thus, assuming this transformation is T, we can put T the way below.

$T: (x, y) \longrightarrow (s, t) = \{(3x + 4y)/5, (4x - 3y)/5\}$.

So the transformation we apply here is T, symmetric about the line $y = 0.5x$. And the original curve is a circle, where the center is $(2, 1)$, and the radius is $\sqrt{2}$.

So the new curve we get is symmetric to the original curve about the line $y = 0.5x$, and thus, is a circle, too, where the radius is the same as that of the original circle, that is, the circle f.
And thus, assuming g is the new circle, we can say that the two circles f and g are symmetric about the line $y = 0.5x$. What then, about the center of the new circle g?

The center gets transformed exactly the way the circle gets transformed, too. So the center of g is symmetric to the center of f about the line $y = 0.5x$.

And knowing the center and the radius of a circle, we can readily get the circle. Using the standard form for the equation for circles, we can easily get the equation of the circle.

The equation of the circle f, that is, the original equation is given in fact, in the standard form. So let's first, get the new circle finding the center of the new circle.

We can get the new center applying the transformation T to the original center, that is, the center of the circle f. So let's now apply T to the original center $(2, 1)$.

To begin with, (x, y) changes to (s, t) through this: $\{(3x + 4y)/5, (4x - 3y)/5\}$.

So the new center is $\{(3\cdot 2 + 4\cdot 1)/5, (4\cdot 2 - 3\cdot 1)/5\} = (2, 1)$, which is the same as the center of the original circle f. And we know both circles have the same radius, too.
So the equations of both circles are the same, also.

So the circle g is simply $(x - 2)^2 + (y - 1)^2 - 2 = 0$. And the new equation g can be put this way, too: $g(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0$.

And let's next, get the new circle using the method we used for the previous examples.

To begin with, assuming $\boldsymbol{L}$ is a transformation symmetric about a line $\boldsymbol{y = ax + b}$, we can put $\boldsymbol{L}$ the way as follows. $\boldsymbol{L: (x, y) \longrightarrow (\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2})}$.

As explained in the section 2.3, applying the transformation $\boldsymbol{L}$ to $\boldsymbol{y = f(x)}$, or $\boldsymbol{f(x, y) = 0}$, we can get the new equation just replacing $\boldsymbol{x}$ with $\frac{(1-a^2)s+2at-2ab}{1+a^2}$, and $\boldsymbol{y}$ with $\frac{2as+(a^2-1)t+2b}{1+a^2}$.

That is, the new equation is $\boldsymbol{\frac{2ax+(a^2-1)y+2b}{1+a^2} = f(\frac{(1-a^2)x+2ay-2ab}{1+a^2})}$.

Or if the original equation is $\boldsymbol{f(x, y) = 0}$, the new one is $\boldsymbol{f(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}) = 0}$.

And assuming $\boldsymbol{g(x, y) = 0}$ is the new equation, we can put it the way below.

$\boldsymbol{g(x, y) = f(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}) = 0}$.

So in this example, too, doing the replacements as above, we get

$\boldsymbol{g(x, y) = f(\frac{3x+4y}{5}, \frac{4x-3y}{5}) = 0}$, which is the new equation.

And we have $\boldsymbol{f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0}$.
So putting the new equation specifically, we can put it the way below.

$\boldsymbol{g(x, y) = f(\frac{3x+4y}{5}, \frac{4x-3y}{5}) = (\frac{3x+4y}{5} - 2)^2 + (\frac{4x-3y}{5} - 1)^2 - 2 = 0}$,which doesn't look though, quite simple. So let's now set $\boldsymbol{A = \frac{3x+4y}{5}}$, and $\boldsymbol{B = \frac{4x-3y}{5}}$.

Then, we get $\boldsymbol{g(x, y) = f(A, B) = (A - 2)^2 + (B - 1)^2 - 2 = 0}$.

Meanwhile, $\boldsymbol{(A - 2)^2 + (B - 1)^2 - 2 = A^2 - 4A + 4 + B^2 - 2B + 1}$, where
$\boldsymbol{A^2 = \frac{1}{25}(3x + 4y)^2}$ and $\boldsymbol{B^2 = \frac{1}{25}(4x - 3y)^2}$.

Meanwhile, $\boldsymbol{(3x + 4y)^2 = 9x^2 + 24xy + 16y^2}$, and $\boldsymbol{(4x - 3y)^2 = 16x^2 - 24xy + 9y^2}$.

So we get
$\boldsymbol{g(x, y) = f(A, B) = A^2 - 4A + 4 + B^2 - 2B + 1 - 2 = A^2 - 4A + 4 + B^2 - 2B + 1 - 2}$

$= \frac{1}{25}(9x^2 + 24xy + 16y^2) - 4(3x + 4y)/5 + \frac{1}{25}(16x^2 - 24xy + 9y^2) - 2(4x - 3y)/5 + 3 = 0$

$\Rightarrow (9x^2 + 24xy + 16y^2) - 20(3x + 4y) + (16x^2 - 24xy + 9y^2) - 10(4x - 3y) + 75 = 0$

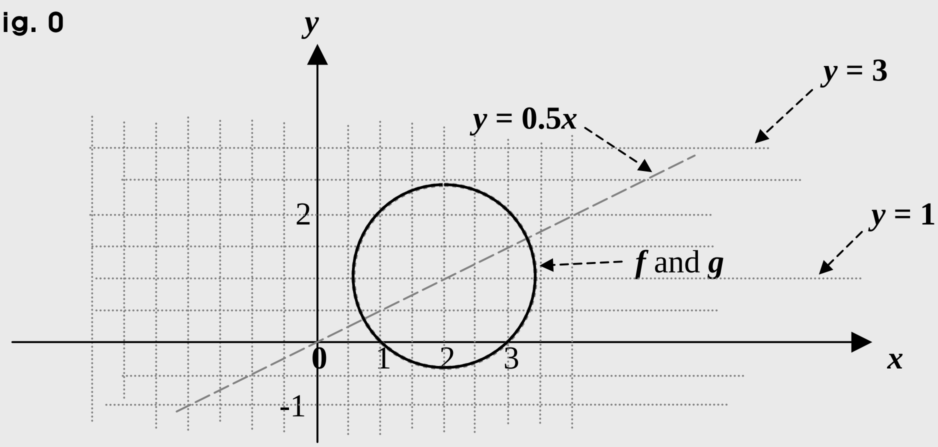
$\Rightarrow 25x^2 - 100x + 25y^2 - 50y + 75 = 0 \Rightarrow x^2 - 4x + y^2 - 2y + 3 = 0$

$\Rightarrow x^2 - 4x + 4 + y^2 - 2y + 1 - 2 = 0 \Rightarrow (x - 2)^2 + (y - 1)^2 - 2 = 0$.

So we get $g(x, y) = f(\frac{3x+4y}{5}, \frac{4x-3y}{5}) = (x - 2)^2 + (y - 1)^2 - 2 = 0$.

And thus, the curve of g is $(x - 2)^2 + (y - 1)^2 - 2 = 0$.

Fig. 0



Since the axis of symmetry includes the center of the original circle, it is natural that the new circle is the same as the original.

Suggestions or Solutions
To the Problem in the Example 1

Find the equation symmetric about a line $y = 2x - 3$ to the equation below.
$f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0$.

Assuming $\boldsymbol{g}$ is the equation we want, we get $\boldsymbol{g(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0}$.

If not sure of the idea behind the solution above, follow the steps below.

The original curve is a circle centered at (2, 1), and the line, that is, the axis of symmetry passes through the center. So the new circle is the same as the original circle.

Let's this time, too, though, for practice purposes, go through the process.

To begin with, assuming $\boldsymbol{L}$ is a transformation symmetric about a line $\boldsymbol{y = ax + b}$, we can put $\boldsymbol{L}$ the way as follows. $\boldsymbol{L}$**:** $\boldsymbol{(x, y)} \longrightarrow \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$.

So let's get now get this: $\left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$ for this example.

Suppose first, $\boldsymbol{(x, y)}$ is the original arbitrary point, and $\boldsymbol{(s, t)}$ is the new arbitrary point.
Then, since $\boldsymbol{(x, y)}$ and $\boldsymbol{(s, t)}$ are the same distance away from the line $\boldsymbol{y = 2x - 3}$.

So assuming $\boldsymbol{(c, d)}$ is the midpoint between $\boldsymbol{(x, y)}$ and $\boldsymbol{(s, t)}$, we get $c = \frac{x+s}{2}$, and $d = \frac{y+t}{2}$.

And next, the midpoint $\boldsymbol{(c, d)}$ is in the line $\boldsymbol{y = 2x - 3}$. So we get $\boldsymbol{d = 2c - 3}$.
In other words, we get $\boldsymbol{(y + t)/2 = 2(x + s)/2 - 3 \Rightarrow y + t = 2(x + s) - 6}$.

And next, we know the line segment connecting $\boldsymbol{(x, y)}$ and $\boldsymbol{(s, t)}$ is perpendicular to the line $\boldsymbol{y = 2x - 3}$, and the product of the slopes of the line segment and the line is -1.
So the slope of the line segment is -1 since 1 is the slope of the line $\boldsymbol{y = 2x - 3}$.
Thus, we get $\boldsymbol{(y - t)/(x - s) = -0.5 \Rightarrow 2(y - t) = s - x}$.

So we get a system of two equations where $\boldsymbol{t - 2s = 2x - y - 6}$, and $\boldsymbol{2t + s = x + 2y}$.

Solving thus, the system for s first, we get

$\mathbf{2t - 4s - (2t + s) = 4x - 2y - 12 - (x + 2y) \Rightarrow -5s = 3x - 4y - 12 \Rightarrow s = (-3x + 4y + 12)/5}$.

And next, putting the s above into $\mathbf{2t + s = x + 2y}$, we get $\mathbf{2t + (-3x + 4y + 12)/5 = x + 2y}$

$\mathbf{\Rightarrow 2t = 5(x + 2y)/5 - (-3x + 4y + 12)/5 = (8x + 6y - 12)/5 \Rightarrow t = (4x + 3y - 6)/5}$.

So we can now set $\mathbf{(s, t) = (\frac{-3x+4y+12}{5}, \frac{4x+3y-6}{5})}$.

And thus, assuming this transformation is Q, we can put Q the way below.

$\mathbf{Q: (x, y) \longrightarrow (s, t) = (\frac{-3x+4y+12}{5}, \frac{4x+3y-6}{5})}$.

Now, if a circle gets transformed symmetrically, its center gets transformed exactly the way the circle gets transformed, too. So the new center is symmetric to the original center of f about the line $\mathbf{y = 2x - 3}$.

And knowing the center and the radius of a circle, we can readily get the circle. Using the standard form for the equation for circles, we can easily get the equation of the circle.

The equation of the circle f, that is, the original equation is given in fact, in the standard form. So let's first, get the new circle finding the center of the new circle.

We can get the new center applying the transformation Q to the original center, that is, the center of the circle f. So let's now apply Q to the original center $\mathbf{(2, 1)}$.

To begin with, $\mathbf{(x, y)}$ changes to $\mathbf{(s, t)}$ through this: $\mathbf{\{(-3x + 4y + 12)/5, (4x + 3y - 6)/5\}}$.

So the new center is $\mathbf{\{(-3\cdot 2 + 4\cdot 1 + 12)/5, (4\cdot 2 + 3\cdot 1 - 6)/5\} = (2, 1)}$, which is the same as the center of the original circle f. And we know both circles have the same radius, too. So the equations of both circles are the same, also.

So the circle g is simply $\mathbf{(x - 2)^2 + (y - 1)^2 - 2 = 0}$. And the new equation g can be put this way, too: $\mathbf{g(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0}$.

And let's next, see if the direct method works for this example, too.

The new equation is $\boldsymbol{g(x, y) = f(\frac{-3x+4y+12}{5}, \frac{4x+3y-6}{5}) = 0}$.

And we have $\boldsymbol{f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0}$. So we get

$$g(x, y) = \{(-3x + 4y + 12)/5 - 2\}^2 + \{(4x + 3y - 6)/5 - 1\}^2 - 2$$

$$= \{(-3x + 4y + 2)/5\}^2 + \{(4x + 3y - 11)/5\}^2 - 2 = 0$$

$$\Rightarrow (-3x + 4y + 2)^2 + (4x + 3y - 11)^2 - 50 = 0$$

$$\Rightarrow 9x^2 + 16x^2 + 4 - 24xy + 16y - 12x + 16x^2 + 9y^2 + 121 + 24xy - 66y - 88x - 50$$

$$= 25x^2 - 100x + 25y^2 - 50y + 75 = 0$$

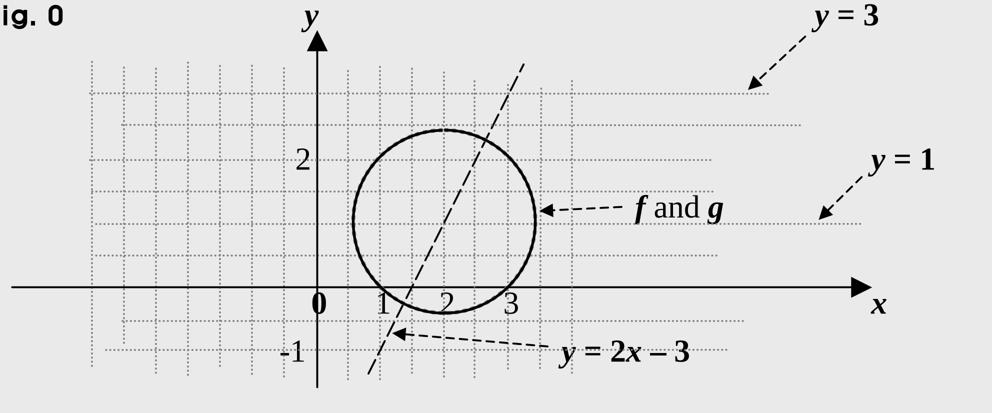
$$\Rightarrow x^2 - 4x + y^2 - 2v + 3 = 0 \Rightarrow (x^2 - 4x + 4 - 4) + (y^2 - 2y + 1 - 1) + 3 = 0$$

$$\Rightarrow (x - 2)^2 - 4 + (y - 1)^2 - 1 + 3 = 0 \Rightarrow (x - 2)^2 + (y - 1)^2 - 2 = 0.$$

So we get $\boldsymbol{g(x, y) = f\{(-3x + 4y + 12)/5, (4x + 3y - 6)/5\} = (x - 2)^2 + (y - 1)^2 - 2 = 0}$.

And thus, the curve of $\boldsymbol{g}$ is $\boldsymbol{(x - 2)^2 + (y - 1)^2 - 2 = 0}$.

Fig. 0



Since the axis of symmetry includes the center of the original circle, it is natural that the new circle is the same as the original.

Suggestions or Solutions
To the Problem in the Example 2

Find the equation symmetric about a line $y = -2x + 6$ to the equation below.
$f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0$.

Assuming g is the equation we want, we get $g(x, y) = (x - 14/5)^2 + (y - 7/5)^2 - 2 = 0$.

If not sure of the idea behind the solution above, follow the steps below.

Suppose first, (x, y) is the original arbitrary point, and (s, t) is the new arbitrary point. Then, (x, y) and (s, t) are the same distance away from the axis of symmetry $y = -2x + 6$.

So assuming next, (c, d) is the midpoint between (x, y) and (s, t), we get
$c = (x + s)/2$ and $d = (y + t)/2$.

Next, since the midpoint (c, d) is in the line $y = -2x + 6$, we get $d = -2c + 6$.
That is, we get $(y + t)/2 = -2(x + s)/2 + 6$. So we get $y + t = -2(x + s) + 12$.

And next, the line segment connecting (x, y) and (s, t) is perpendicular to the line, that is, the axis of symmetry, and the product of the slopes of the line segment and the axis of symmetry is -1. So the slope of the line segment is 0.5 since -2 is the slope of the axis of symmetry $y = -2x + 6$.

And thus, we get $(y - t)/(x - s) = 0.5 \Rightarrow 2(y - t) = x - s$.

So we get a system of two equations where $t + 2s = -2x - y + 12$, and $2t - s = 2y - x$.

Solving thus, the system for s first, we get
$2t + 4s - (2t - s) = -4x - 2y + 24 - (2y - x) \Rightarrow 5s = -3x - 4y + 24 \Rightarrow s = (-3x - 4y + 24)/5$.

And next, putting the s above into $2t - s = 2y - x$, we get $2t - (-3x - 4y + 24)/5 = 2y - x$
$\Rightarrow 2t = 5(2y - x)/5 + (-3x - 4y + 24)/5 = (-8x + 6y + 24)/5 \Rightarrow t = (-4x + 3y + 12)/5$.

And thus, assuming this transformation is R, we can put R the way below.
R: $(x, y) \longrightarrow (s, t) = \{(-3x - 4y + 24)/5, (-4x + 3y + 12)/5\}$.

And since the transformation is symmetric, and the original curve is a circle of radius $\sqrt{2}$ centered at **(2, 1)**, the new curve a circle of radius $\sqrt{2}$, also.
What then, about the center of the new circle?

The center gets transformed the same manner, too. So the center is symmetric to the center of f about the line $y = -2x + 6$.

So let's first, get the new circle finding the center of the new circle.
That is, we are now going to apply R to the original center **(2, 1)**.

To begin with, (x, y) changes to (s, t) through this: $\{(-3x - 4y + 24)/5, (-4x + 3y + 12)/5\}$.

So the new center is $\{(-3\cdot 2 - 4\cdot 1 + 24)/5, (-4\cdot 2 + 3\cdot 1 + 12)/5\} = (14/5, 7/5)$.

And the radius is $\sqrt{2}$. So the new circle is simply $(x - 14/5)^2 + (y - 7/5)^2 - 2 = 0$.

And let's next, see if the direct method works for this example, too.

The new equation is $g(x, y) = f(\frac{-3x-4y+24}{5}, \frac{-4x+3y+12}{5}) = 0$.

And we have $f(x, y) = (x - 2)^2 + (y - 1)^2 - 2 = 0$. So we get

$g(x, y) = \{(-3x - 4y + 24)/5 - 2\}^2 + \{(-4x + 3y + 12)/5 - 1\}^2 - 2$

$= \{(-3x - 4y + 14)/5\}^2 + \{(-4x + 3y + 7)/5\}^2 - 2 = 0$

$\Rightarrow (-3x - 4y + 14)^2 + (-4x + 3y + 7)^2 - 50 = 0$

$\Rightarrow 9x^2 + 16x^2 + 196 + 24xy - 112y - 84x + 16x^2 + 9y^2 + 49 - 24xy + 42y - 56x - 50$

$= 25x^2 - 140x + 25y^2 - 70y + 195 = 0$

$\Rightarrow 5x^2 - 28x + 5y^2 - 14v + 39 = 0 \Rightarrow 5(x^2 - 28x/5) + 5(y^2 - 14y/5) + 39 = 0$

$\Rightarrow 5\{x^2 - 28x/5 + (14/5)^2 - (14/5)^2\} + 5\{y^2 - 14y/5 + (7/5)^2 - (7/5)^2\} + 39 = 0$

$\Rightarrow 5(x - 14/5)^2 - 14^2/5 + 5(y - 7/5)^2 - 7^2/5 + 39 = 0$

$\Rightarrow 5(x - 14/5)^2 + 5(y - 7/5)^2 - (196 + 49 - 195)/5 = 5(x - 14/5)^2 + 5(y - 7/5)^2 - 10 = 0$

$\Rightarrow (x - 14/5)^2 + (y - 7/5)^2 - 2 = 0$.

Fig. 0