Examples in Parametric Transformations

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Examples in Parametric Transformations

Assume that each curve stated below is in the ***x-y*** plane.

0. Find the curve produced by the parametric transformation below.

$\boldsymbol{P}$: $\boldsymbol{t} \longrightarrow \mathbf{(3}\boldsymbol{t}\mathbf{ + 1, 2}\boldsymbol{t}^{\mathbf{2}}\mathbf{ + }\boldsymbol{t}\mathbf{ + 1)}$.

1. Find the curve produced by the parametric transformation below.

$\boldsymbol{P}$: $\boldsymbol{t} \longrightarrow \mathbf{(3^{-\boldsymbol{t}}, 9^{\boldsymbol{t}-1})}$.

2. Find the curve produced by the parametric transformation below.

$\boldsymbol{P}$: $\boldsymbol{t} \longrightarrow \{f(t), g(t)\}$, where $x = f(t) = t^2 + 1$, and $y = g(t) = \frac{t^4+t^2+2}{9-t^2}$.

3. Find the curve produced by the parametric transformation below.

$\boldsymbol{P}$: $\boldsymbol{t} \longrightarrow \{f(t), g(t)\}$, where $x = f(t) = \frac{1}{t^2+1}$, and $y = g(t) = \frac{t^2-2}{t^2+1}$.

4. Find the curve produced by the parametric transformation below.

$\boldsymbol{P}$: $\boldsymbol{t} \longrightarrow (\cos t, \sin t)$.

Suggestions or Solutions
To the Problem in the Example 0

Find the curve produced by the parametric transformation below.
$\boldsymbol{P}$**:** $\boldsymbol{t} \longrightarrow (3t + 1, 2t^2 + t + 1)$**.**

To begin with, $(3t + 1, 2t^2 + t + 1)$ is the new point in $\boldsymbol{P}$, and thus, is no other than the arbitrary point in the curve we want to get.

And the curve is in the x-y plane. So the arbitrary point in the curve is (x, y). And thus, putting the transformation $\boldsymbol{P}$ more specifically, we can put it the way as follows.

$\boldsymbol{P}: t \longrightarrow (x, y) = (3t + 1, 2t^2 + t + 1)$.

Thus, we get $x = 3t + 1$, and $y = 2t^2 + t + 1$.

So we can say that parameterizing x and y using t as the parameter, we get

$x = 3t + 1$, and $y = 2t^2 + t + 1$, each of which has its curve.

And thus, the x-value changes as the t-value changes, and so does the y-value.
And doing the transformation $\boldsymbol{P}$, we can get an equation where the y-value changes as the x-value changes, that is, we can get the connective equation between x and y.

And doing a parametric transformation, we can say that we mix one curve into the other.
We mix the curve of $x = 3t + 1$ into that of $y = 2t^2 + t + 1$.

And we can get the mixture by means of the parameter t, that is, using t as the medium.

And getting the mixture, we get the two variables x and y connected.

That is, we get the expression connecting x and y.

So getting the connective expression, we can get first, $x = 3t + 1 \Rightarrow t = (x - 1)/3$.
Thus next, we can get

$$y = 2t^2 + t + 1 = 2(x - 1)^2/3^2 + (x - 1)/3 + 1 = 2(x^2 - 2x + 1)/3^2 + (x - 1)/3 + 1$$

$$= 2x^2/9 - 4x/9 + 2/9 + x/3 - 1/3 + 1 = 2x^2/9 - x/9 + 8/9.$$

So we get $\boldsymbol{y = 2x^2/9 - x/9 + 8/9}$, which is a parabola, and is the curve we wanted to get.

And in the calculation process above, we have $\boldsymbol{y = 2(x - 1)^2/3^2 + (x - 1)/3 + 1}$.

So we can notice that moving 1 to the left hand side, we can get
$\boldsymbol{y - 1 = 2(x - 1)^2/9 + (x - 1)/3}$, which is the curve we can get moving the curve below in the amount of 1 along the $\boldsymbol{x}$-axis, and also, in the amount of 1 along the $\boldsymbol{y}$-axis.
$\boldsymbol{y = 2x^2/9 + x/3}$.

In short:

The curve is in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane. So we can set $\boldsymbol{(x, y) = (3t + 1, 2t^2 + t + 1)}$.

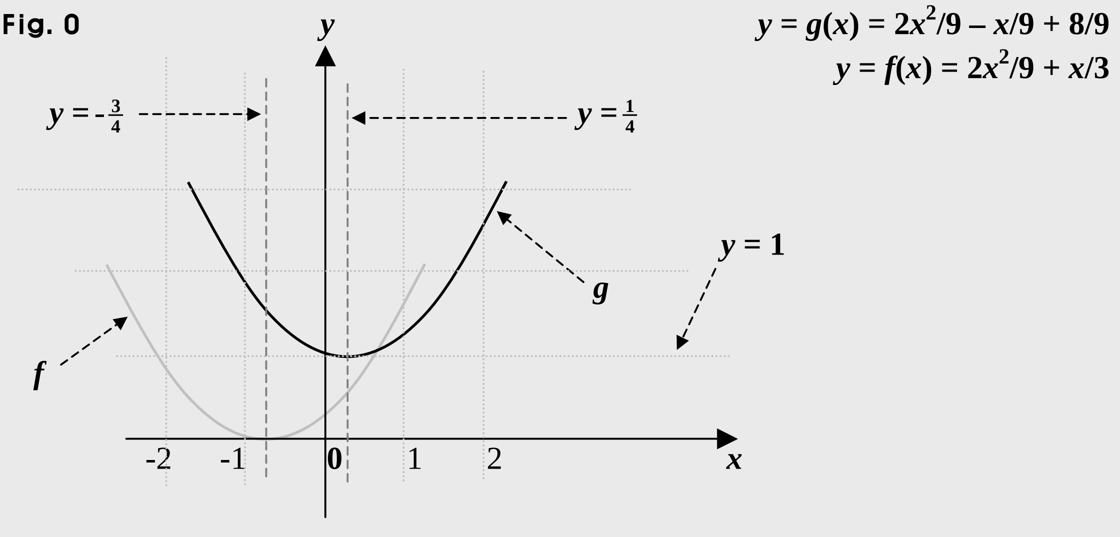
Thus, we get $\boldsymbol{x = 3t + 1}$, and $\boldsymbol{y = 2t^2 + t + 1}$. So next, connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, we can get first, $\boldsymbol{x = 3t + 1 \Rightarrow t = (x - 1)/3}$. Thus next, we can get

$$\boldsymbol{y = 2t^2 + t + 1 = 2(x - 1)^2/3^2 + (x - 1)/3 + 1 = 2(x^2 - 2x + 1)/3^2 + (x - 1)/3 + 1}$$

$$\boldsymbol{= 2x^2/9 - 4x/9 + 2/9 + x/3 - 1/3 + 1 = 2x^2/9 - x/9 + 8/9}.$$

So we get $\boldsymbol{y = 2x^2/9 - x/9 + 8/9}$, which is the curve.

Fig. 0



Suggestions or Solutions
To the Problem in the Example 1

Find the curve produced by the parametric transformation below.
$\boldsymbol{P}$**:** $\boldsymbol{t} \longrightarrow (3^{-t}, 9^{t-1})$.

To begin with, $(3^{-t}, 9^{t-1})$ is the new point in $\boldsymbol{P}$, and thus, is no other than the arbitrary point in the curve, that is, the new arbitrary point. And the curve is in the ***x-y*** plane. So putting the transformation $\boldsymbol{P}$ more specifically, we can put it the way below.

$\boldsymbol{P}$**:** $\boldsymbol{t} \longrightarrow (x, y) = (3^{-t}, 9^{t-1})$. And thus, we get $x = 3^{-t}$, and $y = 9^{t-1}$.

And we can connect the two variables x and y by means of the parameter t, which is however, is in the exponent. How then, can we solve $x = 3^{-t}$ for t?

Exponents are no other than logarithms, called logs, for short.

And using the definition for logs, we can get the solution.

The definition is $y = a^x \Leftrightarrow x = \log_a y$ where $y > 0$, x is real, $a > 0$, and $a \neq 1$.

So using the definition above, we can get first, $x = 3^{-t} \Rightarrow -t = \log_3 x \Rightarrow t = \log_3 \frac{1}{x}$.

And next, we get $y = 9^{t-1} \Rightarrow t - 1 = \log_9 y = \frac{1}{2}\log_3 y \Rightarrow t = \frac{1}{2}\log_3 y + 1 = \log_3 3\sqrt{y}$.

Thus, we get $\log_3 \frac{1}{x} = \log_3 3\sqrt{y} \Rightarrow \frac{1}{x} = 3\sqrt{y} \Rightarrow y = \frac{1}{9x^2}$.

So the curve we had to get is $y = \frac{1}{9x^2}$ for $x \neq 0$, isn't it?

Not really. We have $x = 3^{-t}$. So we get $x > 0$, because $3^{-t} > 0$ for t real.

And thus, the curve is $y = \frac{1}{9x^2}$ for $x > 0$. And we can get the curve the way below, too.

We have $\boldsymbol{x} = \mathbf{3}^{-t}$, and $\boldsymbol{y} = \mathbf{9}^{t-1}$. And we can notice that $\mathbf{3}^{t}$ is a factor common in $\boldsymbol{x}$ and $\boldsymbol{y}$.

We can get $\boldsymbol{x} = \mathbf{3}^{-t} = (\mathbf{3}^{t})^{-1}$. So we get $\mathbf{3}^{t} = \mathbf{1}/\boldsymbol{x}$.

And next, we can get $\boldsymbol{y} = \mathbf{9}^{t-1} = \mathbf{9}^{t}/\mathbf{9} = (\mathbf{3}^{2})^{t}/\mathbf{9} = (\mathbf{3}^{t})^{2}/\mathbf{9}$.

Thus, we get $\boldsymbol{y} = \frac{1}{9x^2}$ for $\boldsymbol{x} > \mathbf{0}$.

In short:

To begin with, we get $\boldsymbol{x} = \mathbf{3}^{-t}$, and $\boldsymbol{y} = \mathbf{9}^{t-1}$. And next, by the definition for logs, we get

$\boldsymbol{x} = \mathbf{3}^{-t} \Rightarrow -t = \log_3 x \Rightarrow t = \log_3 \frac{1}{x}$, and $\boldsymbol{y} = \mathbf{9}^{t-1} \Rightarrow t - 1 = \log_9 y \Rightarrow t = \frac{1}{2}\log_3 y + 1$
$= \log_3 3\sqrt{y} \Rightarrow \log_3 \frac{1}{x} = \log_3 3\sqrt{y} \Rightarrow \frac{1}{x} = 3\sqrt{y} \Rightarrow y = \frac{1}{9x^2}$.

And we have $\boldsymbol{x} > \mathbf{0}$ since $\mathbf{3}^{-t} > \mathbf{0}$.

Thus, the curve is $\boldsymbol{y} = \frac{1}{9x^2}$ for $\boldsymbol{x} > \mathbf{0}$.

Suggestions or Solutions
To the Problem in the Example 2

Find the curve produced by the parametric transformation below.
P*: *t $\longrightarrow$ **$\{f(t), g(t)\}$, where $x = f(t) = t^2 + 1$, and $y = g(t) = \frac{t^4+t^2+2}{9-t^2}$.**

Doing the transformation ***P***, we get the expression connecting ***x*** and ***y*** by means of the parameter ***t***. And the connective expression is the equation of the curve. So expressing ***t*** in terms of ***x***, and then, putting the ***t*** into ***g***, we can get the curve.

Looking at closely however, both equations ***f*** and ***g***, we can notice that both have t^2.

And also, we can notice that $9 - t^2 \neq 0 \Rightarrow t \neq \pm 3$.

So setting $s = t^2$, we get $s \geq 0$, and $s \neq 9$, and can put both equations the way below.

$x = s + 1$, and $y = \frac{s^2+s+2}{9-s}$ for $s \neq 9$.   And we want to note that $x \geq 1$, and $x \neq 10$.

Thus next, connecting ***x*** and ***y***, and beginning with $x = s + 1$, we get

$$x = s + 1 \Rightarrow s = x - 1 \Rightarrow y = \frac{s^2+s+2}{9-s} = \frac{(x-1)^2+(x-1)+2}{9-(x-1)} = \frac{x^2-2x+1+x-1+2}{9-x+1} = \frac{x^2-x+2}{10-x}.$$

So the curve we want is $y = \frac{x^2-x+2}{10-x}$ for $x \geq 1$, and $x \neq 10$.

And of course, we can put it this way, too: $y = \frac{x^2-x+2}{10-x}$ for $1 \leq x < 10$, or $x > 10$.

In short:

To begin with, setting $s = t^2$, we get $s \geq 0$, and $s \neq 9$, and can put both equations the way as follows. $x = s + 1$, and $y = \frac{s^2+s+2}{9-s}$ for $s \neq 9$. So we get $x \geq 1$, and $x \neq 10$.

And next, connecting ***x*** and ***y***, and beginning with $x = s + 1$, we get

$$x = s + 1 \Rightarrow s = x - 1 \Rightarrow y = \frac{s^2+s+2}{9-s} = \frac{(x-1)^2+(x-1)+2}{9-(x-1)} = \frac{x^2-2x+1+x-1+2}{9-x+1} = \frac{x^2-x+2}{10-x}.$$

And thus, the curve is $y = \frac{x^2-x+2}{10-x}$ for $x \geq 1$, and $x \neq 10$.

Suggestions or Solutions
To the Problem in the Example 3

Find the curve produced by the parametric transformation below.
P: $t \longrightarrow \{f(t), g(t)\}$, where $x = f(t) = \frac{1}{t^2+1}$, and $y = g(t) = \frac{t^2-2}{t^2+1}$.

As in the previous example, doing the transformation $\boldsymbol{P}$, we get the expression connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ by means of the parameter $\boldsymbol{t}$. And the connective expression is the equation of the curve. So expressing $\boldsymbol{t}$ in terms of $\boldsymbol{x}$, and then, putting the $\boldsymbol{t}$ into $\boldsymbol{g}$, we can get the curve.

This time, too, however, looking at closely both equations $\boldsymbol{f}$ and $\boldsymbol{g}$, we can notice that both have $\boldsymbol{t^2}$.

And thus, setting $\boldsymbol{s = t^2}$, we get $\boldsymbol{s \geq 0}$, and can put both equations the way below.

$x = \frac{1}{s+1}$, and $y = \frac{s-2}{s+1}$ for $s \geq 0$.

And getting the extent of x, we get $s \geq 0 \Rightarrow s + 1 \geq 1 \Rightarrow 0 < \frac{1}{s+1} \leq 1 \Rightarrow 0 < x \leq 1$.

Thus next, connecting x and y, and beginning with $x = \frac{1}{s+1}$, we get

$$x = \frac{1}{s+1} \Rightarrow s+1 = \frac{1}{x} \Rightarrow s = \frac{1}{x} - 1 \Rightarrow y = \frac{s-2}{s+1} = \frac{\frac{1}{x}-1-2}{\frac{1}{x}-1+1} = \frac{\frac{1}{x}-3}{\frac{1}{x}} = \frac{\frac{1-3x}{x}}{\frac{1}{x}} = 1 - 3x.$$

And thus, the curve we want is $\boldsymbol{y = 1 - 3x}$ for $\boldsymbol{0 < x \leq 1}$.

In short:

To begin with, setting $s = t^2$, we get $x = \frac{1}{s+1}$, and $y = \frac{s-2}{s+1}$ for $s \geq 0$.
And next, getting the extent of x, we get $s \geq 0 \Rightarrow s + 1 \geq 1 \Rightarrow 0 < \frac{1}{s+1} \leq 1 \Rightarrow 0 < x \leq 1$.
So next, connecting x and y, and beginning with $x = \frac{1}{s+1}$, we get

$$x = \frac{1}{s+1} \Rightarrow s+1 = \frac{1}{x} \Rightarrow s = \frac{1}{x} - 1 \Rightarrow y = \frac{s-2}{s+1} = \frac{\frac{1}{x}-1-2}{\frac{1}{x}-1+1} = \frac{\frac{1}{x}-3}{\frac{1}{x}} = \frac{\frac{1-3x}{x}}{\frac{1}{x}} = 1 - 3x.$$

And thus, the curve is $\boldsymbol{y = 1 - 3x}$ for $\boldsymbol{0 < x \leq 1}$.

Suggestions or Solutions
To the Problem in the Example 4

Find the curve produced by the parametric transformation below.
$\boldsymbol{P}: \boldsymbol{t} \longrightarrow (\mathbf{cos}\ \boldsymbol{t}, \mathbf{sin}\ \boldsymbol{t})$.

To begin with, **(cos *t*, sin *t*)** is the new point in ***P***, and thus, is no other than the arbitrary point in the curve we want to get.

And the curve is in the ***x-y*** plane. So the arbitrary point in the curve is **(*x*, *y*)**. And thus, putting the transformation ***P*** more specifically, we can put it the way as follows.

$\boldsymbol{P}: \boldsymbol{t} \longrightarrow (x, y) = (\mathbf{cos}\ \boldsymbol{t}, \mathbf{sin}\ \boldsymbol{t})$.

And thus, we get $x = \cos t$, and $y = \sin t$. How then, can we connect ***x*** and ***y***?

We have a tool called a trig-identity $\sin^2\theta + \cos^2\theta = 1$.

So using the identity, we simply get $x^2 + y^2 = 1$, which indicates a circle, of course. Specifically, it is a unit circle centered at the origin.

And a unit circle is a circle of radius 1. So we get $-1 \le x \le 1$, and $-1 \le y \le 1$.

That is, $|x| \le 1$, and $|y| \le 1$. And of course, we have $|\cos t| \le 1$, and $|\sin t| \le 1$.

And putting $x^2 + y^2 = 1$ in a set of equations where $x = \cos t$, and $y = \sin t$, we can say that the unit circle is parameterized.

And the set of equations is called the parametric equation of the unit circle centered at the origin. Also, the transformation ***P*** can just be called the unit circle centered at the origin, too. What then, about the parametric equation for circles?

We know the standard equation for circles is $(x - a)^2 + (y - b)^2 = r^2$, where (a, b) is the center, and $|r|$ is the radius.

And we know $\sin^2\theta + \cos^2\theta = 1$. So we get $r^2(\sin^2\theta + \cos^2\theta) = r^2$.

Assuming therefore, $r\cos t = x - a$, and $r\sin t = y - b$, we get

$r^2(\sin^2 t + \cos^2 t) = (x - a)^2 + (y - b)^2 = r^2$.

And we get $r\cos t = x - a \Rightarrow x = r\cos t + a$, and $r\sin t = y - b \Rightarrow y = r\sin t + b$.

And thus, the parametric equation for circles is a set of equations where
$x = r\cos t + a$, and $y = r\sin t + b$.

And of course, we can parameterize other geometric objects, too.
Let's for instance, parameterize the basic curves.

Beginning with an ellipse, we can put it in an equation the way below.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are nonzero constants.

In equation above, if $a = b \neq 0$, we get a circle of radius of $1/a$ (or $1/b$) centered at the origin. And if in particular, $a = b = \pm 1$, we get the unit circle centered at the origin.

Now, assuming the ellipse is E, we can parameterize the ellipse E the way below.

$E: t \longrightarrow (a\cos t, b\sin t)$ where a and b are nonzero constants, and t is real.

Then first, $(a\cos t, b\sin t)$ is the new point in E, so it is no other than the arbitrary point in the ellipse E. And the ellipse is in the x-y plane. So the arbitrary point in it is (x, y).

And thus, putting the transformation E more specifically, we can put it the way below.

$E: t \longrightarrow (x, y) = (a\cos t, b\sin t)$. So we get $x = a\cos t$, and $y = b\sin t$.

How then, can we connect x and y?

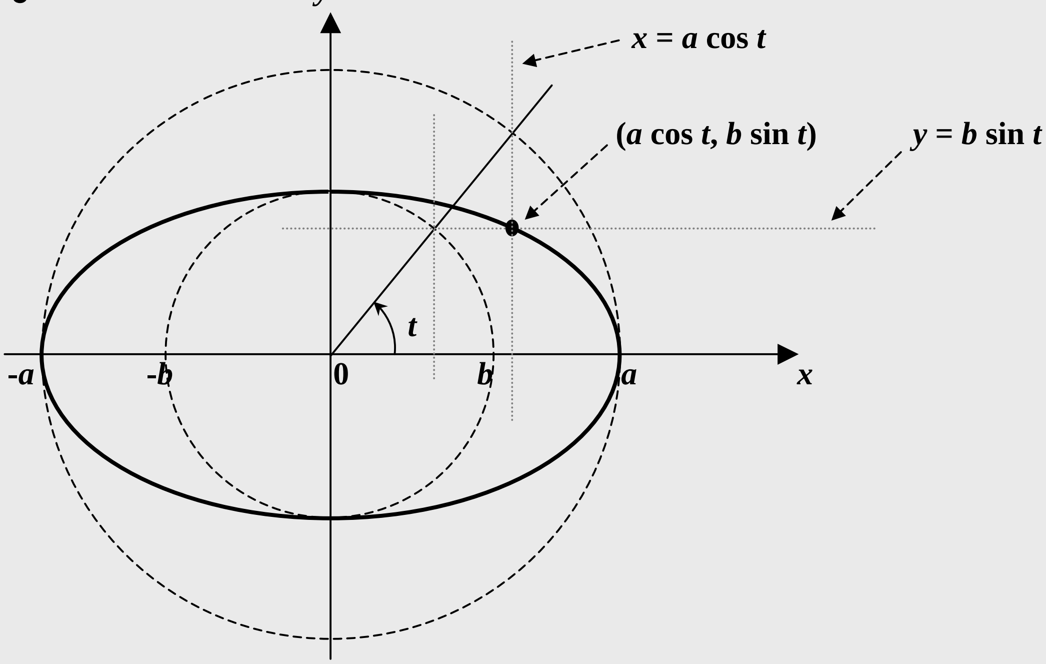
We have a tool called a trig-identity where $\sin^2\theta + \cos^2\theta = 1$.

And we can put the two equations $x = a\cos t$, and $y = b\sin t$ the way below.

$x = a\cos t \Rightarrow \cos t = x/a$, and $y = b\sin t \Rightarrow y/b = \sin t$.

So next, using the identity above, we simply get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is called the Cartesian equation of an ellipse, which is centered at the origin.

Fig. 4.0



And let's next, move on to the equation for parabolas.

To begin with, assuming $\boldsymbol{P}$ is a parabola where the axis of symmetry is parallel to the $\boldsymbol{y}$-axis, $\boldsymbol{Q}$ is a parabola where the axis of symmetry is parallel to the $\boldsymbol{x}$-axis, and $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant, but $\boldsymbol{a \neq 0}$, we can put the two parabolas the way below.

$\boldsymbol{P(x, y) = ax^2 + bx + c - y = 0}$, and $\boldsymbol{Q(x, y) = ay^2 + by + c - x = 0}$.

And next, parameterizing the two, we can put them the way below.

$\boldsymbol{P: t \longrightarrow (t, at^2 + bt + c)}$, and $\boldsymbol{Q: t \longrightarrow (at^2 + bt + c, t)}$, where $\boldsymbol{t}$ is real.

Then first, $\boldsymbol{(t, at^2 + bt + c)}$ is the new point in $\boldsymbol{P}$, and thus, is no other than the arbitrary point in the parabola $\boldsymbol{P}$. And $\boldsymbol{P}$ is in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane. So the arbitrary point in it is $\boldsymbol{(x, y)}$. And thus, putting the transformation $\boldsymbol{P}$ more specifically, we can put it the way below.

$\boldsymbol{P: t \longrightarrow (x, y) = (t, at^2 + bt + c)}$. And thus, we get $\boldsymbol{x = t}$, and $\boldsymbol{y = at^2 + bt + c}$.

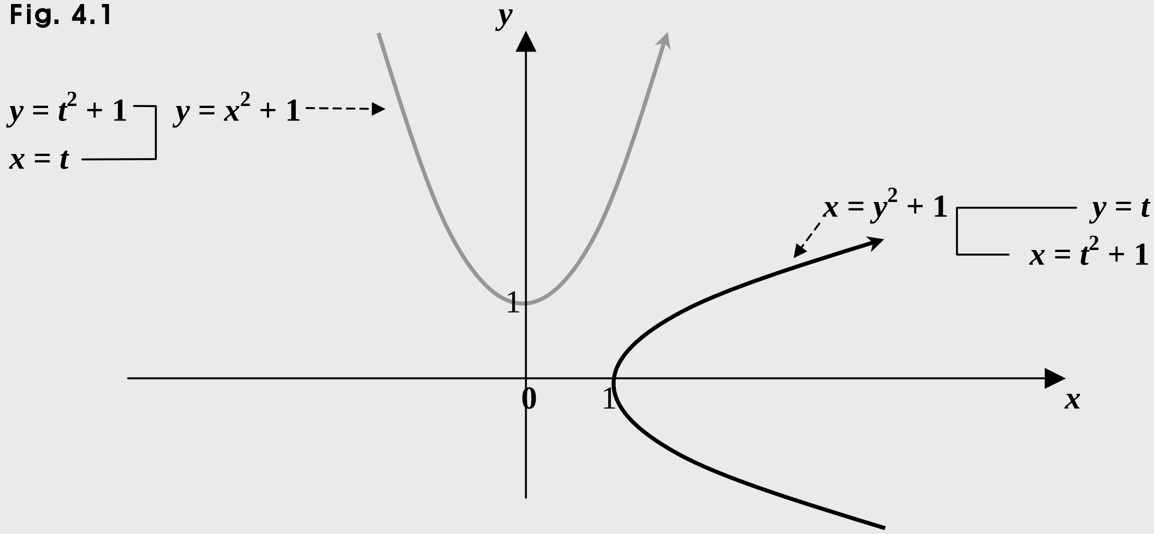
So connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, we simply get $\boldsymbol{y = ax^2 + bx + c}$.

And next, $(at^2 + bt + c, t)$ is the mech in Q, and thus, is no other than the arbitrary point in the parabola Q. And Q is in the x-y plane. So the arbitrary point in it is (x, y). And thus, putting the transformation P more specifically, we can put it the way below.

Q: $t \longrightarrow (x, y) = (at^2 + bt + c, t)$. And thus, we get $x = at^2 + bt + c$, and $y = t$.

So connecting x and y, we simply get $x = ay^2 + by + c$.

Fig. 4.1



And let's next, move on to the equation for hyperbolas.

To begin with, assuming H is a hyperbola where the directrix is parallel to the y-axis, we can put it in an equation the way below.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where a and b are nonzero constants. And a hyperbola has two branches.

So next, assuming R is the right branch of H, L is the left branch, and a and b are nonzero constants, we can parameterize the two branches the way below.

R: $t \longrightarrow (a \sec t, b \tan t)$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, and L: $t \longrightarrow (a \sec t, b \tan t)$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Then first, $(a \sec t, b \tan t)$ is the new point in either of R and L, and thus, is no other than the arbitrary point in either of the two branches.

And the hyperbola is in the ***x-y*** plane. So the arbitrary point in each is **(*x*, *y*)**. And thus, putting the transformations ***R*** and ***L*** more specifically, we can put them the way below.

$\boldsymbol{R: t \longrightarrow (x, y) = (a \sec t, b \tan t)}$ where $\boldsymbol{-\pi/2 < t < \pi/2}$.

$\boldsymbol{L: t \longrightarrow (x, y) = (a \sec t, b \tan t)}$ where $\boldsymbol{\pi/2 < t < 3\pi/2}$.

And thus, we get $\boldsymbol{x = a \sec t}$, and $\boldsymbol{y = b \tan t}$.

And for the right branch ***R***, we get $\boldsymbol{-\pi/2 < t < \pi/2 \Rightarrow \sec t \geq 1 \Rightarrow a \sec t \geq a \Rightarrow x \geq a}$.

And for the left branch ***L***, we get $\boldsymbol{\pi/2 < t < 3\pi/2 \Rightarrow \sec t \leq -1 \Rightarrow a \sec t \leq -a \Rightarrow x \leq -a}$.

How then, can we connect ***x*** and ***y***?

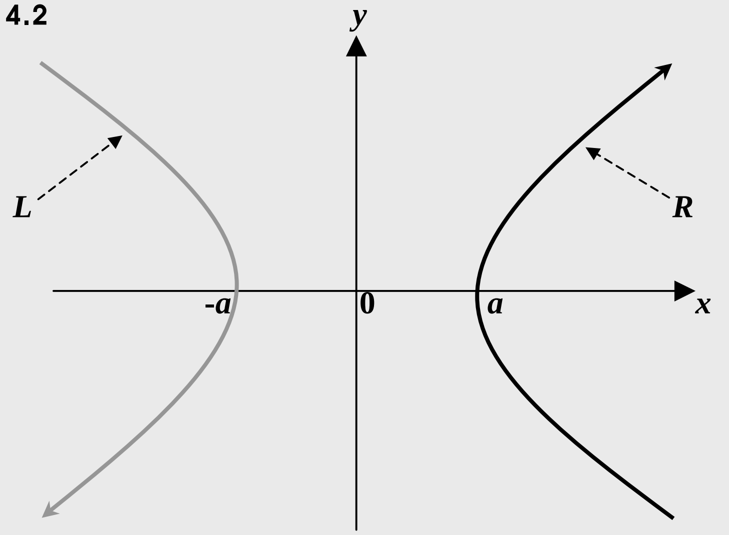
We have a tool called another trig-identity where $\boldsymbol{\sec^2\theta - \tan^2\theta = 1}$.

And we can put the two equations $\boldsymbol{x = a \sec t}$, and $\boldsymbol{y = b \tan t}$ the way below.

$\boldsymbol{x = a \sec t \Rightarrow \sec t = x/a}$, and $\boldsymbol{y = b \tan t \Rightarrow y/b = \tan t}$.

So next, using the identity above, we simply get $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ for $\boldsymbol{x \geq a}$, which is called the Cartesian equation of the right branch of the hyperbola ***H***, which is centered at the origin. And also, we get $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ for $\boldsymbol{x \leq -a}$, which is the equation of the left branch.

Fig. 4.2

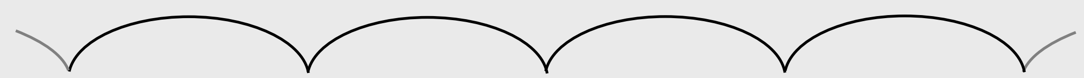


And let's for another instance, take a look at a curve called a cycloid, and another curve called a trochoid.
A cycloid is a curve traced by a point in a circle rolling along a horizontal line.
And assuming ***C*** is a cycloid, and the circle is of radius ***r***, we can put it the way below.

$\boldsymbol{C: t \longrightarrow \{r(t - \sin t), r(1 - \cos t)\}}$ where $\boldsymbol{r}$ is constant.

And a cycloid looks like the one below.

Fig. 4.3

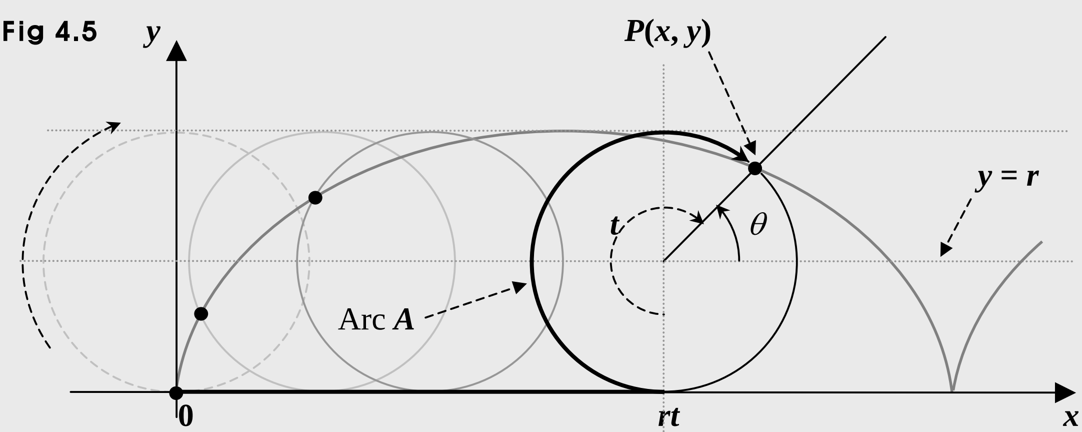


And next, a trochoid is a curve traced by a point, which is fixed on a disk rolling along a horizontal line, but is not in the circumference of the disk. And assuming ***T*** is a trochoid, the disk is of radius ***r***, and the point is ***b*** away from the center, we can put ***T*** the way as follows. $\boldsymbol{T: t \longrightarrow (rt - b \sin t, r - b \cos t)}$ where $\boldsymbol{r}$ and $\boldsymbol{b}$ are constant.
And a trochoid looks like the one below.

Fig. 4.4

And a cycloid can be produced the way below.

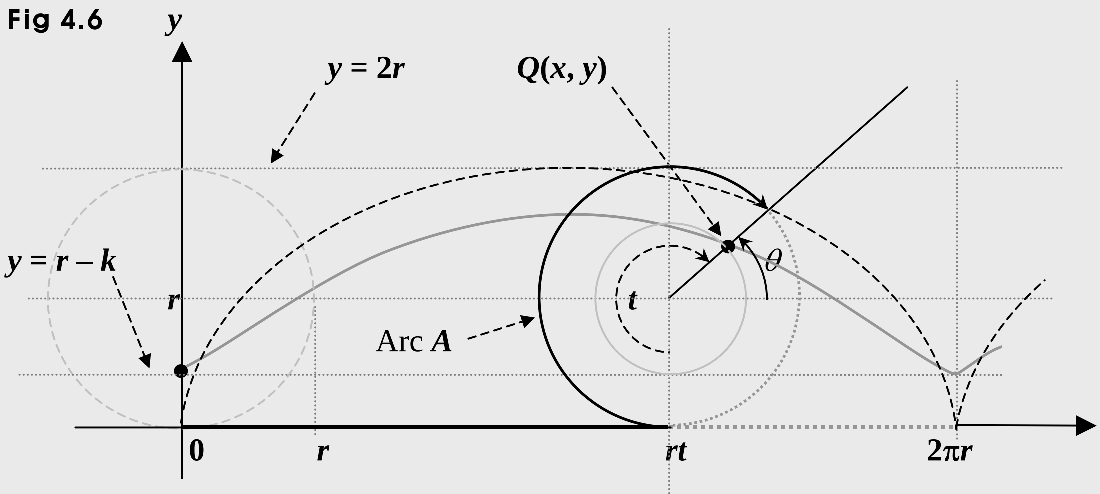
Fig 4.5



So a circular disk is rolling along the ***x***-axis, and a cycloid begins at the origin. That is, at the beginning, the center of the disk is in the ***y***-axis.
Then, the length of the circular arc ***A*** is the distance the disk moved horizontally as it rolls along the ***x***-axis. That's because the arc length is the product of the radius ***r*** and the central angle ***t***, and thus, is ***rt***.

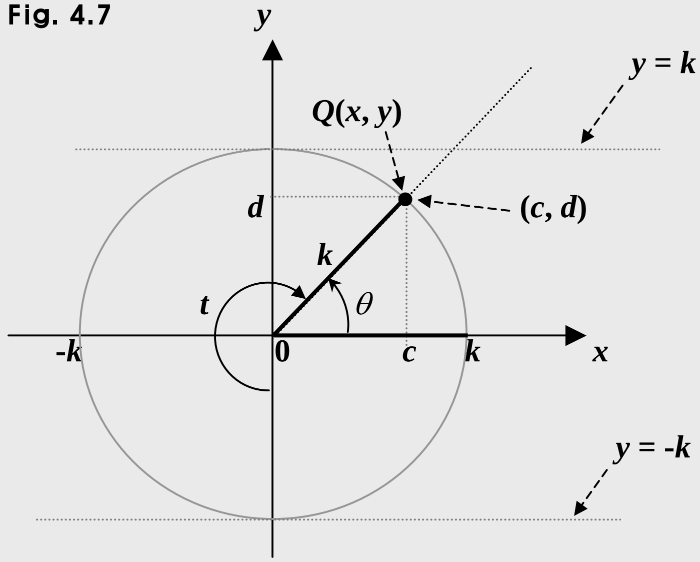
Let's first though, keep track of a point, located ***k*** units away from the center of the disk. Then, the curve traced by the point is a trochoid.
And when the center of the disk is in the ***y***-axis, the point is at $\mathbf{(0, \mathit{r} - \mathit{k})}$ where $\mathbf{0 < \mathit{k} < \mathit{r}}$.
So assuming the point is ***Q***, we can say that ***Q*** is in a circle of radius ***k***.
And of course, if $\mathbf{\mathit{k} = \mathit{r}}$, ***Q*** is in the circumference of the disk.

Fig 4.6



Let's now copy the circle of radius ***k*** in the graph above, and put the copy in a new ***x-y*** plane. And putting the center of the circle at the origin in the new plane, we get a graph as below.

Fig. 4.7



The circle above is in the disk where the center is at $(\mathbf{\mathit{rt}, \mathit{r}})$.

And let's see now how we can parameterize the trochoid the point $\boldsymbol{Q}$ makes in the original $\boldsymbol{x}$-$\boldsymbol{y}$ plane. That is, we want to see how we get $(\boldsymbol{rt} - \boldsymbol{k}\ \mathbf{sin}\ \boldsymbol{t}, \boldsymbol{r} - \boldsymbol{k}\ \mathbf{cos}\ \boldsymbol{t})$.

To begin with, the parameter is the angle $\boldsymbol{t}$.

And next, examining the two graphs above, since the center of the disk is at $(\boldsymbol{rt}, \boldsymbol{r})$, we can see that the coordinates of the arbitrary point $\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y})$ are $\boldsymbol{x} = \boldsymbol{rt} + \boldsymbol{c}$, and $\boldsymbol{y} = \boldsymbol{r} + \boldsymbol{d}$.

So next, we want to express $\boldsymbol{c}$ and $\boldsymbol{d}$ in terms of the parameter $\boldsymbol{t}$. How?

And examining first, the circle of radius $\boldsymbol{k}$, we get $\boldsymbol{c} = \boldsymbol{k}\ \mathbf{cos}\boldsymbol{\theta}$, and $\boldsymbol{d} = \boldsymbol{k}\ \mathbf{sin}\boldsymbol{\theta}$.

And next, we have trig-identities as follows.

$\mathbf{sin}\ (\boldsymbol{a} - \boldsymbol{b}) = (\mathbf{sin}\ \boldsymbol{a})(\mathbf{cos}\ \boldsymbol{b}) - (\mathbf{cos}\ \boldsymbol{a})(\mathbf{sin}\ \boldsymbol{b})$.

$\mathbf{cos}\ (\boldsymbol{a} - \boldsymbol{b}) = (\mathbf{cos}\ \boldsymbol{a})(\mathbf{cos}\ \boldsymbol{b}) + (\mathbf{sin}\ \boldsymbol{a})(\mathbf{sin}\ \boldsymbol{b})$.

So we get

$\mathbf{sin}\ \boldsymbol{t} = \mathbf{sin}(\frac{3\pi}{2} - \theta) = (\mathbf{sin}\frac{3\pi}{2})(\mathbf{cos}\theta) - (\mathbf{cos}\frac{3\pi}{2})(\mathbf{sin}\theta) = -\mathbf{cos}\theta \Rightarrow \mathbf{cos}\theta = -\mathbf{sin}\ \boldsymbol{t}$.

$\mathbf{cos}\ \boldsymbol{t} = \mathbf{cos}(\frac{3\pi}{2} - \theta) = (\mathbf{cos}\frac{3\pi}{2})(\mathbf{cos}\theta) + (\mathbf{sin}\frac{3\pi}{2})(\mathbf{sin}\theta) = -\mathbf{sin}\theta \Rightarrow \mathbf{sin}\theta = -\mathbf{cos}\ \boldsymbol{t}$.

Thus, we get $\boldsymbol{c} = \boldsymbol{k}\ \mathbf{cos}\theta = -\boldsymbol{k}\ \mathbf{sin}\ \boldsymbol{t}$, and $\boldsymbol{d} = \boldsymbol{k}\ \mathbf{sin}\theta = -\boldsymbol{k}\ \mathbf{cos}\ \boldsymbol{t}$.

So we get $\boldsymbol{x} = \boldsymbol{rt} + \boldsymbol{c} = \boldsymbol{rt} - \boldsymbol{k}\ \mathbf{sin}\ \boldsymbol{t}$, and $\boldsymbol{y} = \boldsymbol{r} + \boldsymbol{d} = \boldsymbol{r} - \boldsymbol{k}\ \mathbf{cos}\ \boldsymbol{t}$.

And thus, the parametric equation of the trochoid traced by a point $\boldsymbol{k}$ units away from the center of a disk of radius $\boldsymbol{r}$ is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{rt} - \boldsymbol{k}\ \mathbf{sin}\ \boldsymbol{t}, \boldsymbol{r} - \boldsymbol{k}\ \mathbf{cos}\ \boldsymbol{t})$ for $\mathbf{0} < \boldsymbol{k} < \boldsymbol{r}$, and $\boldsymbol{t}$ real.

What then, about the cycloid?

We know the cycloid is the curve traced by a point located in the circumference of the disk. And the radius of the disk is $\boldsymbol{r}$. So for the cycloid, we want to set $\boldsymbol{k} = \boldsymbol{r}$.

Then, we get $\boldsymbol{k} = \boldsymbol{r} \Rightarrow (\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{rt} - \boldsymbol{k}\ \mathbf{sin}\ \boldsymbol{t}, \boldsymbol{r} - \boldsymbol{k}\ \mathbf{cos}\ \boldsymbol{t}) = (\boldsymbol{rt} - \boldsymbol{r}\ \mathbf{sin}\ \boldsymbol{t}, \boldsymbol{r} - \boldsymbol{r}\ \mathbf{cos}\ \boldsymbol{t})$.

So we get $(\boldsymbol{x}, \boldsymbol{y}) = \{\boldsymbol{r}(\boldsymbol{t} - \mathbf{sin}\ \boldsymbol{t}), \boldsymbol{r}(\mathbf{1} - \mathbf{cos}\ \boldsymbol{t})\}$.