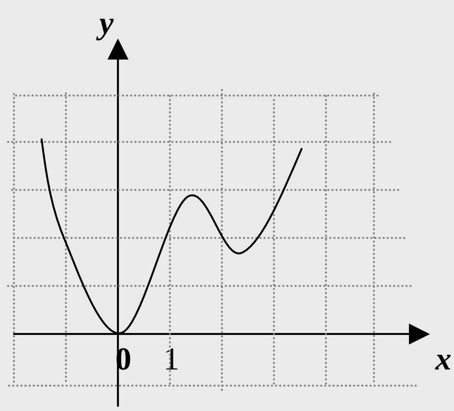
Examples in General Transformations

Click this to start.

Examples in General Transformations

0. Assuming a function $y = f(x)$ has the curve below, construct the graph of each function g listed below.

Fig. 0



0.0. $y = g(x) = f(x) + 1$.

0.1. $y = g(x) = f(x - 2) + 1$.

0.2. $y = g(x) = 2f(x - 2) + 2$.

0.3. $y = g(x) = 2f(2x) + 2$.

0.4. $y = g(x) = -2f(\frac{x}{2}) + 2$.

0.5. $y = g(x) = -2f(\frac{x}{2} - 2) + 2$.

1. Apply the transformation $\boldsymbol{T}$ below to each of **(0, 0)**, **(1, -2)**, **(-3, -1)**, and **(3, 8)**.

$\boldsymbol{T}$: $(x, y) \longrightarrow (s, t)$ where $s = 2xy - 1$, and $t = x + y + 1$.

2. Apply to a line $x + 2y + 1 = 0$ the transformation $\boldsymbol{T}$ below.

$\boldsymbol{T}$: $(x, y) \longrightarrow (s, t)$ where $s = x + y + 1$, and $t = x - y + 1$.

3. Apply to a line $2x - y + 1 = 0$ the transformation $\boldsymbol{T}$ below.

$\boldsymbol{T}$: $(x, y) \longrightarrow (s, t)$ where $3s + t = x + y$, and $st = \frac{xy}{3}$.

4. Assuming a and b are constant, $g = T \bullet f$, where $y = f(x) = x^2 + 8x + 3$, $y = g(x) = -x^2$, and $\boldsymbol{T}$: $(x, y) \longrightarrow (x + a, b - y)$, find a and b.

5. Find the curve of the function $t = f(s)$ produced by the transformation $\boldsymbol{T}$ below.

$\boldsymbol{T}$: $x \longrightarrow (3x, 9x^2 - 6x + 2)$ for $x \geq -1$.

6. Find the curve of the function $w = f(u, v)$ produced by the transformation $\boldsymbol{T}$ below.

$\boldsymbol{T}$: $(x, y) \longrightarrow (u, v, w)$ where $u = 2x + y - 1$, $v = x - y + 3$, and $w = 2xy$.

And then, find $f(0, 0)$, $f(1, -2)$, $f(-3, -1)$, and $f(3, 8)$.

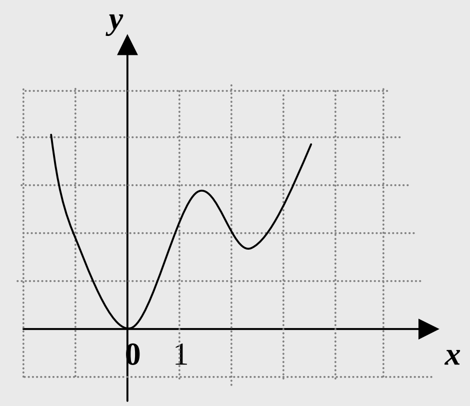
7. Assuming $w = g(u, v)$, and $g = Z \bullet f$, where $t = f(s) = s + 1$, and

$\boldsymbol{Z}$: $(s, t) \longrightarrow (2s, 2t - 1, s^2 + t)$, find the function g.

Suggestions or Solutions
To the Problem 0 in the Example 0

Assuming a function $y = f(x)$ has a curve below, construct the graph of a function $y = g(x) = f(x) + 1$.

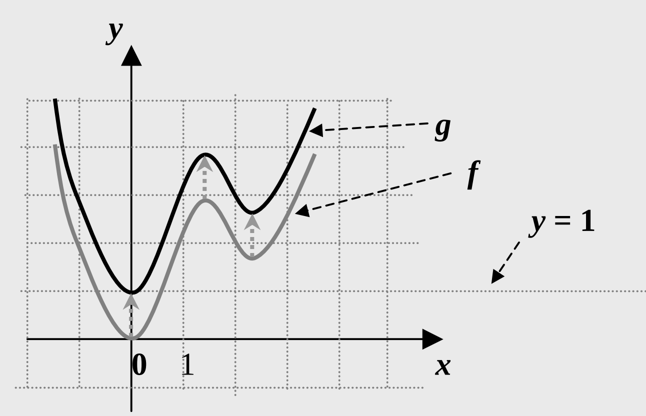
Fig. 0



Assuming first, $\boldsymbol{T}$ is a parallel transformation, we can put it the way below.
$\boldsymbol{T}$**:** $(x, y) \longrightarrow (x + a, y + b)$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.
So assuming (s, t) is the new point, we get $(s, t) = (x + a, y + b)$.
Thus, we get $s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.
So assuming f is the original, we get $y = f(x) \Rightarrow t - b = f(s - a)$.
Then, the new is $y - b = f(x - a)$, which is $y = f(x - a) + b$.
So assuming $g = T \bullet f$, we get $y = g(x) = f(x - a) + b$.
And the equation of the curve of g is $y - b = f(x - a)$, which is $y = f(x - a) + b$.

Now, we have $y = g(x) = f(x) + 1 \Rightarrow y - 1 = f(x)$. So we get $a = 0$, and $b = 1$. And thus, translating the curve of f in the amount of 1 along the y-axis, we get the curve of g.

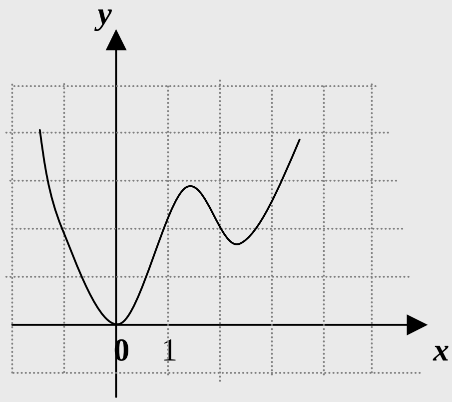
Fig. 1



Suggestions or Solutions
To the Problem 1 in the Example 0

Assuming a function $y = f(x)$ has a curve below, construct the graph of a function $y = g(x) = f(x-2) + 1$.

Fig. 0



Assuming first, $\boldsymbol{T}$ is a parallel transformation, we can put it the way below.
$\boldsymbol{T}$: $(x, y) \longrightarrow (x + a, y + b)$ where a and b are constant.

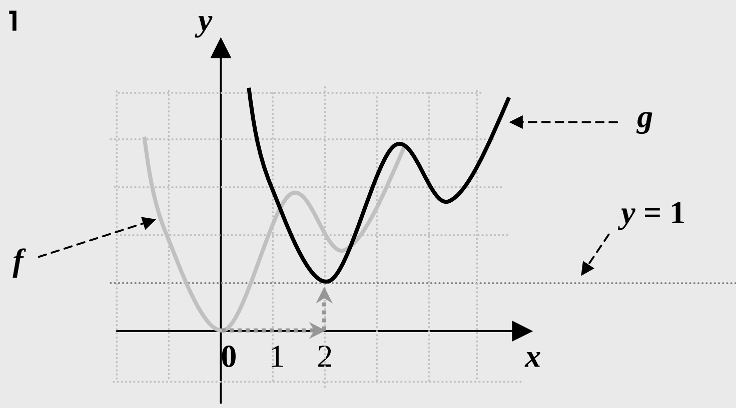
So assuming (s, t) is the new point, and f is the original curve, we get
$y = f(x) \Rightarrow t - b = f(s - a)$.

And thus, assuming $g = T \bullet f$, we get $y = g(x) = f(x - a) + b$.
So the curve of g is $y - b = f(x - a)$, which is $y = f(x - a) + b$.

Now, we have $y = g(x) = f(x - 2) + 1 \Rightarrow y - 1 = f(x - 2)$. So we get $a = 2$, and $b = 1$.

And thus, translating the curve of f in the amount of 2 along the x-axis, and in the amount of 1 along the y-axis, we get the curve of g.

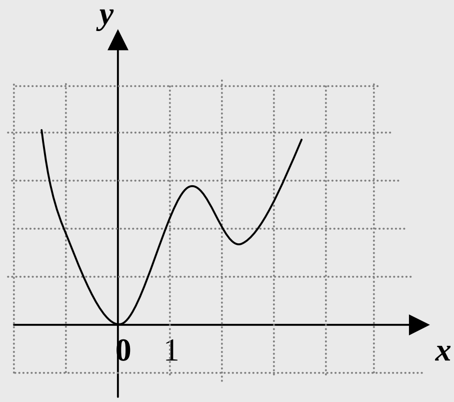
Fig. 1



Suggestions or Solutions
To the Problem 2 in the Example 0

Assuming a function $y = f(x)$ has a curve below, construct the graph of a function $y = g(x) = 2f(x - 2) + 2$.

Fig. 0



Assuming the curve of g is the new curve we get transforming f, and finding the new point in the transformation, we could get the curve of g quite easily. The new point shows the way the new curve gets made.
And we have $y = g(x) = 2f(x - 2) + 2$. So assuming $g = T \bullet f$, where T is a transformation, and tracing back from g to the original function f, we can get first, $y = 2f(x - 2) + 2 \Rightarrow y/2 = f(x - 2) + 1 \Rightarrow y/2 - 1 = f(x - 2)$. What transformation then, is T?

It is a composite transformation. And one of the transformations put together seems to be a parallel transformation. That's because moving the curve of $y = f(x)$ in the amount of 2 along the x-axis, we get $y = f(x - 2)$.
What then, about the left hand side, that is, $y/2 - 1$?

It is from $y_f = y_g/2 - 1$. And in fact, we have $x_f = x_g - 2$. Why?

Assuming (x_f, y_f) is the arbitrary point in the curve of f, and (x_g, y_g) is the arbitrary point in the curve of g, we get $y = f(x) \Rightarrow y_f = f(x_f) \Rightarrow y_g/2 - 1 = f(x_g - 2)$.
So the curve of g is $y/2 - 1 = f(x - 2)$. What then do we mean by $y_f = y_g/2 - 1$?

We can put it this way: $y_f = y_g/2 - 1 \Rightarrow y_g = 2(y_f + 1) = 2y_f + 2 \Rightarrow y_g = 2y_f + 2$.
And we can set $y_g = g(x)$, and $y_f = f(x)$. So we can put ($y_g = 2y_f + 2$) the way below.
$g(x) = 2f(x) + 2$. What then, do we mean by $2f(x)$?

For every ***x***-value, the value of $\mathbf{2}\boldsymbol{f}(\boldsymbol{x})$ is twice the value of $\boldsymbol{f}(\boldsymbol{x})$.
What then, do we mean by $\mathbf{2}\boldsymbol{f}(\boldsymbol{x}-\mathbf{2})$?

Translating the curve of $\mathbf{2}\boldsymbol{f}(\boldsymbol{x})$ in the amount of 2 along the ***x***-axis, we get the curve of $\mathbf{2}\boldsymbol{f}(\boldsymbol{x}-\mathbf{2})$. What then, do we mean by $\mathbf{2}\boldsymbol{f}(\boldsymbol{x}-\mathbf{2})+\mathbf{2}$?

Translating the curve of $\mathbf{2}\boldsymbol{f}(\boldsymbol{x})$ in the amount of 2 along the ***x***-axis, and also, in the amount of 2 along the ***y***-axis, we get the curve of $\mathbf{2}\boldsymbol{f}(\boldsymbol{x}-\mathbf{2})+\mathbf{2}$.

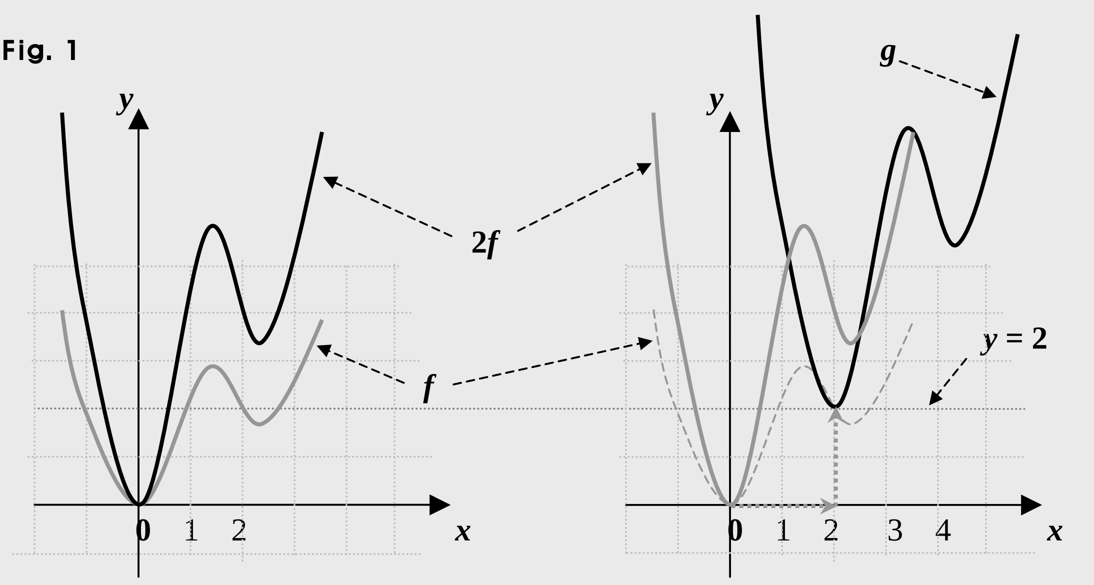
And thus, we can see two transformations get applied.
Showing the new points only, we have $(\boldsymbol{x}, \mathbf{2}\boldsymbol{y})$, and $(\boldsymbol{x}+\mathbf{2}, \boldsymbol{y}+\mathbf{2})$.

And putting them more specifically, and assuming one is ***A***, and the other is ***B***, we have
$\boldsymbol{A}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x}, \mathbf{2}\boldsymbol{y})$, and $\boldsymbol{B}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x}+\mathbf{2}, \boldsymbol{y}+\mathbf{2})$.

And combining the two together, and calling it ***C***, we get
$\boldsymbol{C}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x}+\mathbf{2}, \mathbf{2}\boldsymbol{y}+\mathbf{2})$.

And thus, expanding (stretching?) the curve of ***f*** by 2 vertically, and then, move it in the amount of 2 horizontally, and in the amount of 2 vertically, we get the curve of ***g***.

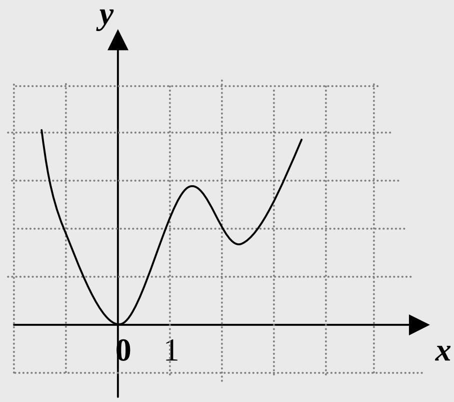
Fig. 1



Suggestions or Solutions
To the Problem 3 in the Example 0

Assuming a function $y = f(x)$ has a curve below, construct the graph of a function $y = g(x) = 2f(2x) + 2$.

Fig. 0



Assuming the curve of g is the new curve we get transforming f, let's find the new point in the transformation.

To begin with, we have $y = g(x) = 2f(2x) + 2$. So assuming $g = T \bullet f$, where T is a transformation, and tracing back from g to the original function f, we can get first, $y = 2f(2x) + 2 \Rightarrow y - 2 = 2f(2x) \Rightarrow (y - 2)/2 = f(2x)$. What transformation then, is T?

It is a composite transformation. And one of the transformations put together seems to be a parallel transformation. That's because moving the curve of $y = f(x)$ in the amount of 2 in the direction of the y-axis, we get the curve of $y - 2 = f(x)$.

What then, about $2x$ in $f(2x)$? In other words, what do we mean by $f(2x)$?

Putting into $f(2x)$ an input which is half the value of the input of $f(x)$, we get the same output. In other words, assuming $h(x) = f(2x)$, we get $h(1) = f(2)$, $h(2) = f(4)$, etc.

That is to say that in the curve of f, the horizontal width gets narrowed to the half.

What then, do we mean by $2f(2x)$?

For every x-value, the value of $2f(2x)$ is twice the value of $f(2x)$.

What then, do we mean by $2f(2x) + 2$?

Translating the curve of $2f(2x)$ in the amount of 2 along the y-axis, we get the curve of $2f(2x) + 2$. And thus, we can see three transformations get applied.

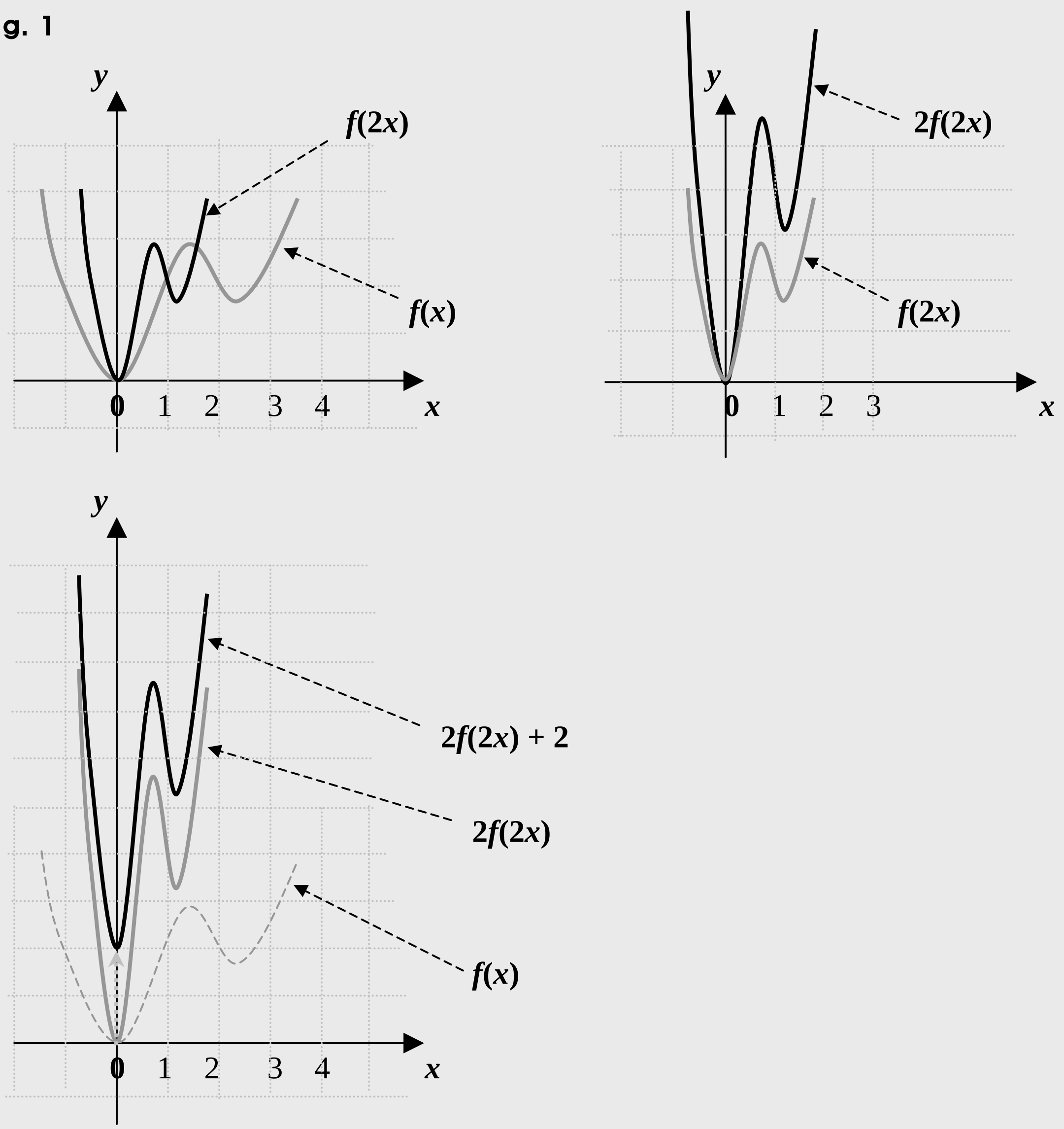
Showing the new points only, we have **(x/2, y)**, **(x, 2y)**, and **(x, y + 2)**.

And putting them more specifically, and assuming they are ***A***, ***B***, and ***C***, we have
***A*: (x, y)** $\longrightarrow$ **(x/2, y)**, ***B*: (x, y)** $\longrightarrow$ **(x, 2y)**, and ***C*: (x, y)** $\longrightarrow$ **(x, y + 2)**.

And combining the three together, that is, setting ***D* = *C*•*B*•*A***, we get
***D*: (x, y)** $\longrightarrow$ **(x/2, 2y + 2)**.

And thus, shrinking (compressing) the curve of ***f*** by half horizontally, and next, expanding (stretching) it by 2 vertically, and then, move it in the amount of 2 along the ***y***-axis, we get the curve of ***g***.

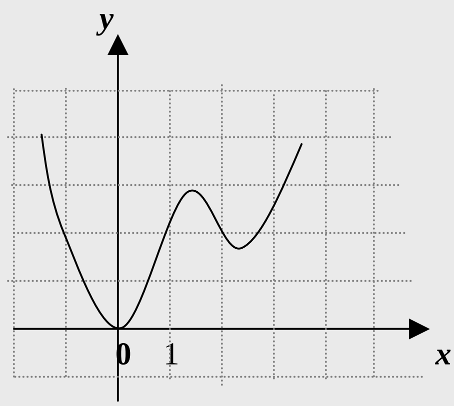
Fig. 1



Suggestions or Solutions
To the Problem 4 in the Example 0

Assuming a function $y = f(x)$ has a curve below, construct the graph of a function $y = g(x) = -2f(\frac{x}{2}) + 2$.

Fig. 0



To begin with, we have $y = g(x) = -2f(\frac{x}{2}) + 2$. And next, assuming $g = T \bullet f$, where T is a transformation, and tracing back from g to the original function f, we can get first,
$y = -2f(\frac{x}{2}) + 2 \Rightarrow y - 2 = -2f(\frac{x}{2}) \Rightarrow (2 - y)/2 = f(\frac{x}{2})$. What transformation then, is T?

It is a composite transformation. And one of the transformations put together seems to be a parallel transformation. That's because moving the curve of $y = -2f(\frac{x}{2})$ in the amount of 2 in the direction of the y-axis, we get the curve of $y - 2 = -2f(\frac{x}{2})$.
What then, about $\frac{x}{2}$ in $f(\frac{x}{2})$? In other words, what do we mean by $f(\frac{x}{2})$?

Putting into $f(\frac{x}{2})$ an input which is twice the value of the input of $f(x)$, we get the same output. In other words, assuming $h(x) = f(\frac{x}{2})$, we get $h(2) = f(1)$, $h(4) = f(2)$, etc.

That is to say that in the curve of f, the horizontal width gets doubled.
And for every x-value, the value of $-2f(\frac{x}{2})$ is negatively twice the value of $f(\frac{x}{2})$.

And translating the curve of $-2f(\frac{x}{2})$ in the amount of 2 along the y-axis, we get the curve of $-2f(\frac{x}{2}) + 2$. And thus, we can see three transformations get applied.

Showing the new points only, we have $(2x, y)$, $(x, -2y)$, and $(x, y + 2)$.

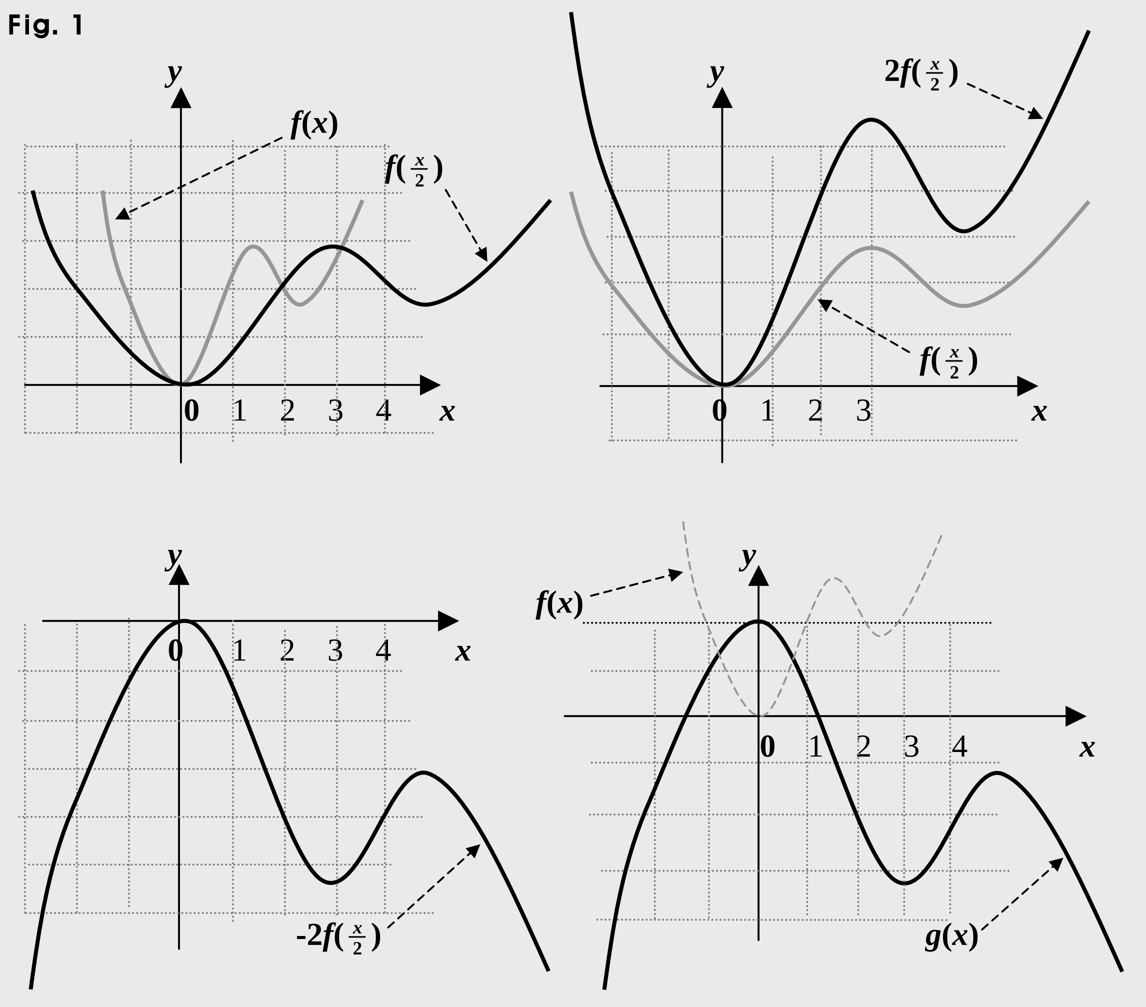
And putting them more specifically, and assuming they are $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, we have
$\boldsymbol{A}$**: $(x, y) \longrightarrow (2x, y)$**, $\boldsymbol{B}$**: $(x, y) \longrightarrow (x, -2y)$**, and $\boldsymbol{C}$**: $(x, y) \longrightarrow (x, y + 2)$**.

And some more specifically, we can take $\boldsymbol{B}$ for a composite transformation where the two transformations below get combined.
$\boldsymbol{B_1}$**: $(x, y) \longrightarrow (x, 2y)$**, and $\boldsymbol{B_2}$**: $(x, y) \longrightarrow (x, -y)$**.

And thus, combining the four together, that is, setting $\boldsymbol{D = C \bullet B_2 \bullet B_1 \bullet A}$, we get
$\boldsymbol{D}$**: $(x, y) \longrightarrow (2x, -2y + 2)$**.

And thus, expanding (stretching?) horizontally the curve of $\boldsymbol{f}$ to the double, then expanding (stretching?) it by 2 along the $\boldsymbol{y}$-axis, then making it symmetric about the $\boldsymbol{x}$-axis, (that is, folding it along the $\boldsymbol{x}$-axis) and then, moving it in the amount of 2 along the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{g}$.

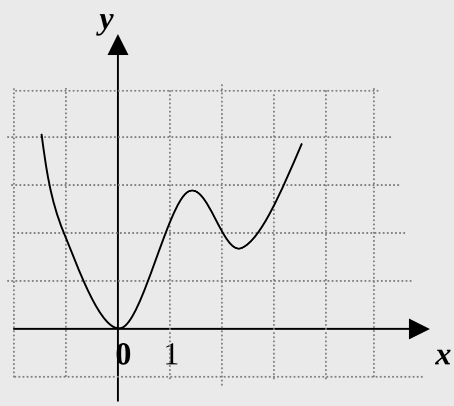
Fig. 1



Suggestions or Solutions
To the Problem 5 in the Example 0

Assuming a function $y = f(x)$ has a curve below, construct the graph of a function $y = g(x) = -2f(\frac{x}{2} - 2) + 2$.

Fig. 0



Assuming $g = T \bullet f$, where T is a transformation, and tracing back from g to the original function f, we can get first, $y = -2f(\frac{x}{2} - 2) + 2 \Rightarrow y - 2 = -2f(\frac{x}{2} - 2) \Rightarrow (2 - y)/2 = f(\frac{x}{2} - 2)$ $\Rightarrow (2 - y)/2 = f(\frac{x-4}{2})$. So T is a composite transformation. Thus next, we want to see what transformations get combined to form T.

Going back to the previous example though, we can see the only difference is that this composite transformation has one more parallel transformation, where a curve gets translated along the x-axis. That is to say that the difference is between $f(\frac{x-4}{2})$ and $f(\frac{x}{2})$. Translating the curve of $f(\frac{x}{2})$ in the amount of 4 along the x-axis, we get the curve of $f(\frac{x-4}{2})$. So we can see three transformations get applied.

Showing the new points only, we have $(2x, y)$, $(x, -2y)$, and $(x + 4, y + 2)$.

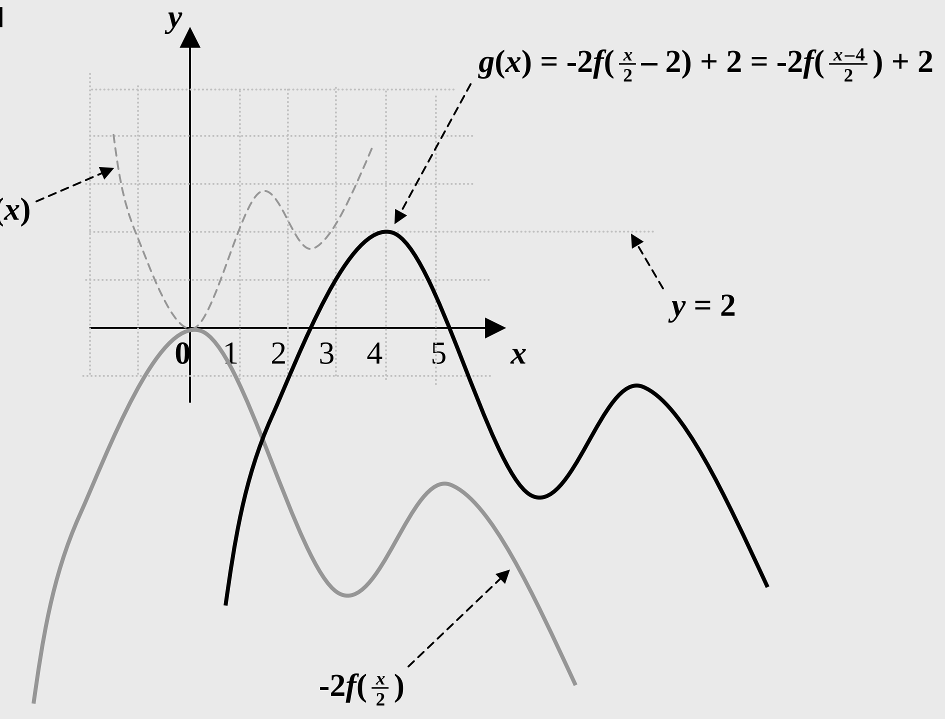
And putting them more specifically, and assuming they are A, B, and C, we have
$A: (x, y) \longrightarrow (2x, y)$, $B: (x, y) \longrightarrow (x, -2y)$, and $C: (x, y) \longrightarrow (x + 4, y + 2)$.

And some more specifically, we can take B as a composite transformation where the two transformations below get combined.
$B_1: (x, y) \longrightarrow (x, 2y)$, and $B_2: (x, y) \longrightarrow (x, -y)$.

And thus, combining the four together, that is, setting $D = C \bullet B_2 \bullet B_1 \bullet A$, we get
$D: (x, y) \longrightarrow (2x + 4, -2y + 2)$.

And thus, expanding (stretching?) horizontally the curve of ***f*** to the double, then expanding (stretching?) it by 2 along the ***y***-axis, then making it symmetric about the ***x***-axis, (that is, folding it along the ***x***-axis) and then, moving it in the amount of 4 along the ***x***-axis, and also, in the amount of 2 along the ***y***-axis, we get the curve of ***g***.

Fig. 1



Suggestions or Solutions
To the Problem in the Example 1

Apply the transformation T below to each of (0, 0), (1, -2), (-3, -1), and (3, 8).

T: $(x, y) \longrightarrow (s, t)$ **where** $s = 2xy - 1$, **and** $t = x + y + 1$.

We can put T this way, too: T: $(x, y) \longrightarrow (s, t) = (2xy - 1, x + y + 1)$, where (s, t) is the new arbitrary point.

And thus, we can simply get

$(0, 0) \Rightarrow s = 2xy - 1 = 2 \cdot 0 \cdot 0 - 1 = -1$, and $t = x + y - 1 = 0 + 0 - 1 = -1$.
And thus, $(0, 0) \longrightarrow (-1, -1)$.

$(1, -2) \Rightarrow s = 2xy - 1 = 2 \cdot 1 \cdot (-2) - 1 = -5$, and $t = x + y - 1 = 1 + (-2) - 1 = -2$.
And thus, $(1, -2) \longrightarrow (-5, -2)$.

$(-3, -1) \Rightarrow s = 2xy - 1 = 2 \cdot (-3) \cdot (-1) - 1 = 5$, and $t = x + y - 1 = -3 + (-1) - 1 = -5$.
And thus, $(-3, -1) \longrightarrow (5, -5)$.

$(3, 8) \Rightarrow s = 2xy - 1 = 2 \cdot 3 \cdot 8 - 1 = 47$, and $t = x + y - 1 = 3 + 8 - 1 = 10$.
And thus, $(3, 8) \longrightarrow (47, 10)$.

Suggestions or Solutions
To the Problem in the Example 2

Apply to a line $x + 2y + 1 = 0$ the transformation T below.
T: $(x, y) \longrightarrow (s, t)$ where $s = x + y + 1$, and $t = x - y + 1$.

$s + t = (x + y + 1) + (x - y + 1) = 2x + 2 \Rightarrow x = (s + t - 2)/2$
$\Rightarrow t = x - y + 1 = (s + t - 2)/2 - y + 1 = (s + t)/2 - y \Rightarrow y = (s + t)/2 - t = (s - t)/2$.

$(s + t - 2)/2 + 2(s - t)/2 + 1 = (s + t - 2 + 2s - 2t + 2)/2 = (3s - t)/2 = 0 \Rightarrow 3s - t = 0$.

So the new curve is a line $3x - y = 0$.

If not sure of the ideas behind the processes above, follow the steps below.

We can put T this way, too: T: $(x, y) \longrightarrow (s, t) = (x + y + 1, x - y + 1)$, where (s, t) is the new point.

So assuming $f(x, y) = x + 2y + 1 = 0$, and applying T to f, we can begin with $(s, t) = (x + y + 1, x - y + 1)$.

And next, we want to get the connective expression between s and t. And then, in the expression, replacing s with x, and t with y, we get the new curve.

And getting the expression connecting s and t, we put (x, y) in terms of s and t first. And then, putting the (x, y) into the original curve, that is, $f(x, y) = 0$, we get the expression.

Now, putting (x, y) in terms of s and t, we want to solve for x and y the system below.
$s = x + y + 1$, and $t = x - y + 1$.

So solving it, we get $s + t = (x + y + 1) + (x - y + 1) = 2x + 2 \Rightarrow x = (s + t - 2)/2$
$\Rightarrow t = x - y + 1 = (s + t - 2)/2 - y + 1 = (s + t)/2 - y \Rightarrow y = (s + t)/2 - t = (s - t)/2$.

Thus, we get $x = (s + t - 2)/2$, and $y = (s - t)/2$.

And next, putting into the original curve the (x, y) expressed in terms of s and t, we put into $f(x, y) = 0$ the x and y found above. Then, we get $f\{(s + t - 2)/2, (s - t)/2\} = 0$.

And we have $f(x, y) = x + 2y + 1 = 0$.

So we get $(s + t - 2)/2 + 2(s - t)/2 + 1 = (1/2)(s + t - 2 + 2s - 2t + 2) = (3s - t)/2 = 0$ $\Rightarrow 3s - t = 0$, which is the connective expression we want.

Assuming thus the new curve is g, we get $g(x, y) = 3x - y = 0$, which is line, of course.

In short:

Solving first for x and y the system where $s = x + y + 1$, and $t = x - y + 1$, we get

$s + t = (x + y + 1) + (x - y + 1) = 2x + 2 \Rightarrow x = (s + t - 2)/2$
$\Rightarrow t = x - y + 1 = (s + t - 2)/2 - y + 1 = (s + t)/2 - y \Rightarrow y = (s + t)/2 - t = (s - t)/2$.

And next, putting into $x + 2y + 1 = 0$ the x and y above, we get

$(s + t - 2)/2 + 2(s - t)/2 + 1 = (s + t - 2 + 2s - 2t + 2)/2 = (3s - t)/2 = 0 \Rightarrow 3s - t = 0$.

And thus, the new curve is a line $3x - y = 0$.

Suggestions or Solutions
To the Problem in the Example 3

Apply to a line $2x - y + 1 = 0$ the transformation T below.
T: $(x, y) \longrightarrow (s, t)$ where $3s + t = x + y$, and $st = \frac{xy}{3}$.

Assuming A is a transformation in **2-*D*** space, we can put it the way below.

A: $(x, y) \longrightarrow \{f(x, y), g(x, y)\}$ where $f(x, y)$ and $g(x, y)$ are expressions in terms of x and y as $x + y + 1$, $xy - 1$, $2x + 1$, etc.

Why though, is $2x + 1$ an expression in terms of x and y?

We can put it this way, too: $2x + 0 \cdot y + 1$.
In the transformation T though, what are the expressions for $f(x, y)$ and $g(x, y)$?

We are given the system where $3s + t = x + y$, and $st = \frac{xy}{3}$.
So solving for s and t the system above, we can get those expressions.
Why bother solving it, though?

We can take (s, t) as the new arbitrary point.
So finding the new curve, we want to put x and y in terms of s and t anyway.
And thus, we want to the system for x and y and not for s and t.

Before solving it though, we may want to put the second equation this way first: $xy = 3st$.
Then, we get $x + y = 3s + t$, and $xy = 3st$. So what?

Assuming x and y are the solution to a quadratic equation, and the unknown is u, we can put the equation the way as follows. $(u - x)(u - y) = 0$. So we get $u^2 - (x + y)u + xy = 0$.

And we have $x + y = 3s + t$, and $xy = 3st$.

So we get $u^2 - (x + y)u + xy = 0 \Rightarrow u^2 - (3s + t)u + 3st = 0 \Rightarrow (u - 3s)(u - t) = 0$
$\Rightarrow u = 3s$ or t. And we know $(u - x)(u - y) = 0 \Rightarrow u = x$ or y.

So we get two cases where $(x, y) = (3s, t)$, and $(x, y) = (t, 3s)$.

Now, the original curve is a line $\mathbf{2x - y + 1 = 0}$.
And thus, beginning with the case where $\mathbf{x = 3s}$ and $\mathbf{y = t}$, we get

$$\mathbf{2x - y + 1 = 2 \cdot 3s - t + 1 = 6s + t = 0}.$$

And next, moving on to the case where $\mathbf{x = t}$ and $\mathbf{y = 3s}$, we get

$$\mathbf{2x - y + 1 = 2 \cdot t - 3s + 1 = 2t - 3s + 1 = 0}.$$

And thus, applying the transformation $\boldsymbol{T}$ to the line $\mathbf{2x - y + 1 = 0}$, we get two lines, one is $\mathbf{6x + y = 0}$, and the other is $\mathbf{2y - 3x + 1 = 0}$.

Suggestions or Solutions
To the Problem in the Example 4

Assuming a and b are constant, $g = T \bullet f$, where $y = f(x) = x^2 + 8x + 3$, $y = g(x) = -x^2$, and T: $(x, y) \longrightarrow (x + a, b - y)$, find a and b.

This problem is rather simple. So we may want to just follow what the problem says.
It is saying that $g = T \bullet f$. And we are given the specifics of g, T, and f.
And thus, finding the transformation of f by T, and then, comparing it to g given, we should be able to get the solution.

So to begin with, assuming (s, t) is the arbitrary point in the curve of g, we can set first, $(s, t) = (x + a, b - y)$. That is, we get $s = x + a$, and $t = b - y$.

So we get $x = s - a$, and $y = b - t$. What then, is the next?

We want to find the expression connecting s and t. And finding it, we put into the original equation $y = f(x)$ the x and y above. So next, doing the substitutions, we get $y = f(x) \Rightarrow b - t = f(s - a)$. And we have $y = f(x) = x^2 + 8x + 3$. Thus, we get

$b - t = (s - a)^2 + 8(s - a) + 3$, which is the connective expression.

So the curve of g is $b - y = (x - a)^2 + 8(x - a) + 3$. And we can put it this way, too:

$y + (x - a)^2 + 8(x - a) + 3 - b = 0$. And expanding (simplifying?) it, we get

$y + x^2 - 2ax + a^2 + 8x - 8a + 3 - b = y + x^2 + (8 - 2a)x + a^2 - 8a + 3 - b = 0$.

And we have this, too: $y = -x^2$, which can be put this way, too: $y + x^2 = 0$.

So comparing the two equations above, we get a system of two equations as follows.

$8 - 2a = 0$, and $a^2 - 8a + 3 - b = 0$.

And thus, we get $a = 4 \Rightarrow a^2 - 8a + 3 - b = 16 - 32 + 3 - b = -13 - b = 0 \Rightarrow b = -13$.

So we can now see the mech in T is $(x + 4, -13 - y)$, and can notice that we can take T for a composite transformation where two transformations below get combined together.

P: $(x, y) \longrightarrow (x + 4, y + 13)$, which is a parallel transformation.

Q: $(x, y) \longrightarrow (x, -y)$, which is the transformation symmetric about the x-axis.

And thus, we can put T this way: $T = Q \bullet P$. Then, we get this, of course: $g = Q \bullet P \bullet f$.

In short:

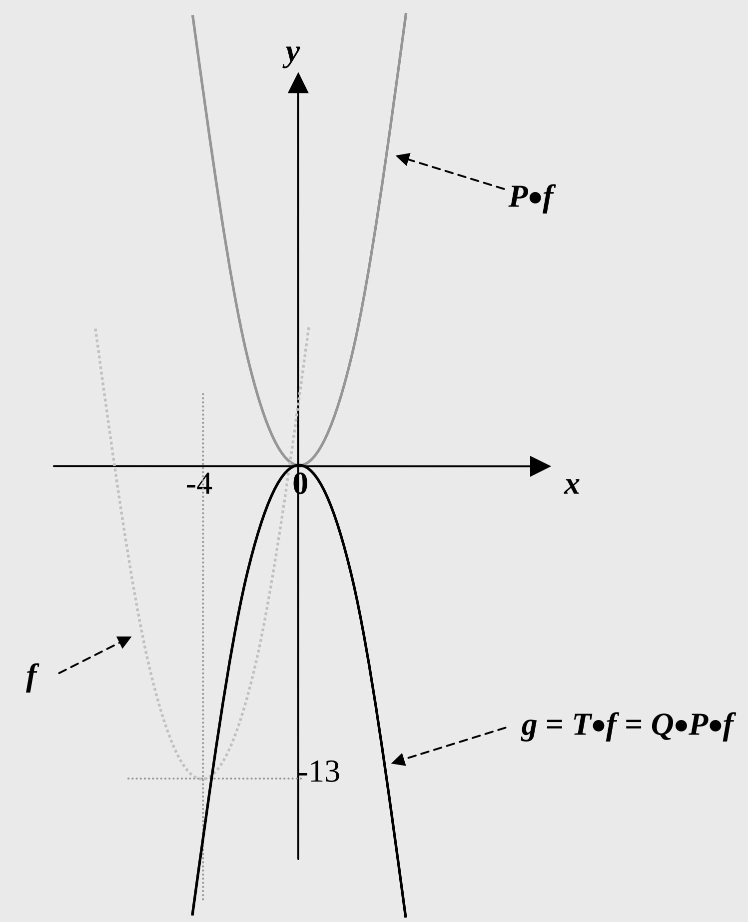
To begin with, assuming (s, t) is the arbitrary point in the curve of g, we get
$s = x + a$, and $t = b - y$. So we get $x = s - a$, and $y = b - t$.
And next, finding the expression connecting s and t, we get
$y = f(x) \Rightarrow b - t = f(s - a) \Rightarrow b - t = (s - a)^2 + 8(s - a) + 3$.

So the curve of g is $y + (x - a)^2 + 8(x - a) + 3 - b = 0$. And expanding it, we get
$y + x^2 - 2ax + a^2 + 8x - 8a + 3 - b = y + x^2 + (8 - 2a)x + a^2 - 8a + 3 - b = 0$.

And we have this, too: $y = -x^2$, which can be put this way, too: $y + x^2 = 0$.
So comparing the two equations above, we get a system of two equations as follows.
$8 - 2a = 0$, and $a^2 - 8a + 3 - b = 0$.

And thus, we get $a = 4 \Rightarrow a^2 - 8a + 3 - b = 16 - 32 + 3 - b = -13 - b = 0 \Rightarrow b = -13$.

Fig. 4.0



Suggestions or Solutions
To the Problem in the Example 5

Find the curve of the function $t = f(s)$ produced by the transformation T below.
T: $x \longrightarrow (3x, 9x^2 - 6x + 2)$ for $x \geq -1$.

$(s, t) = (3x, 9x^2 - 6x + 2)$ for $x \geq -1$.
$s = 3x \Rightarrow x = s/3 \Rightarrow t = 9(s/3)^2 - 6(s/3) + 2 = s^2 - 2s + 2$.
$x \geq -1 \Rightarrow 3x \geq -3 \Rightarrow s \geq -3$.
So $t = f(s) = s^2 - 2s + 2$ for $s \geq -3$.

If not sure of the ideas behind the processes above, follow the steps below.

The transformation T is a parametric transformation. What then, is the parameter?

It is x. In this parametric transformation, what gets transformed is a number ray, that is, a part of a number line. And what we get is a part of a parabola. So in T, a straight ray gets transformed to a parabolic ray. How then, can we get the parabolic ray?

A ray is a curve, too. So finding the curve, we use the new point the transformation makes, and the new point is in this case, $(3x, 9x^2 - 6x + 2)$. So assuming (s, t) is the new point, we can set $(s, t) = (3x, 9x^2 - 6x + 2)$. What then, is the next?

We need to connect s and t, that is, we want to get the equation connecting s and t. Then, the connective equation is the equation of the curve. And we can get the equation through the medium of x. How?

We want s to be the input variable in the function f. So solving for x the equation $s = 3x$, and putting the x into the other equation $t = 9x^2 - 6x + 2$, we get the function f.

So we get $s = 3x \Rightarrow x = s/3 \Rightarrow t = 9(s/3)^2 - 6(s/3) + 2 = s^2 - 2s + 2$.

And thus, the function f is $t = f(s) = s^2 - 2s + 2$. Is it though?

If not specified, the domain of a function is assumed to be a set of all real numbers.
In this case however, the domain is not a set of all real numbers.

That's because the original curve, that is, the input to the transformation is not the entire number line but a number ray. So we want to find the domain of the function f.

Thus, finding it, we get $x \geq -1 \Rightarrow 3x \geq -3 \Rightarrow s \geq -3$.
So the function f is $t = f(s) = s^2 - 2s + 2$ for $s \geq -3$.

In short:

We have $(s, t) = (3x, 9x^2 - 6x + 2)$ for $x \geq -1$.
So we get $s = 3x \Rightarrow x = s/3 \Rightarrow t = 9(s/3)^2 - 6(s/3) + 2 = s^2 - 2s + 2$.
And next, finding the domain, we get $x \geq -1 \Rightarrow 3x \geq -3 \Rightarrow s \geq -3$.
So the function f is $t = f(s) = s^2 - 2s + 2$ for $s \geq -3$.

Suggestions or Solutions
To the Problem in the Example 6

Find the curve of the function $w = f(u, v)$ produced by the transformation T below.
T: $(x, y) \longrightarrow (u, v, w)$ where $u = 2x + y - 1$, $v = x - y + 3$, and $w = 2xy$.

What kind of transformation is T?

It is a parametric transformation. What then, is the parameter?

There are more than one parameter. And they are x and y. In this parametric transformation, what gets transformed is a plane. More specifically, it's a coordinate plane. And what we get is a curved surface. So in T, a straight surface (a plane) gets transformed to a curved surface. How then, can we get the curved surface?

A surface is a curve, too.
So finding the curve, we use the new point the transformation makes, and the new point is in this case, $(2x + y - 1, x - y + 3, 2xy)$. So assuming (u, v, w) is the new point, we can set $(u, v, w) = (2x + y - 1, x - y + 3, 2xy)$. What then, is the next?

We need to connect u, v, and w, that is, we want to get the expression connecting u, v, and w. Then, the connective expression is the equation of the curve, that is, the curved surface. And we can get the expression through the medium of x and y. How?

We want w to be the output variable in the function f. So solving for x and y the system of two equations $u = 2x + y - 1$ and $v = x - y + 3$, and putting the x and y into the other equation $w = 2xy$, we get the function f. And thus, solving the system, we get

$$u + v = (2x + y - 1) + (x - y + 3) = 3x + 2 \Rightarrow x = (u + v - 2)/3$$
$$\Rightarrow v = x - y + 3 = (u + v - 2)/3 - y + 3 \Rightarrow y = (u + v - 2)/3 - v + 3 = (u - 2v + 7)/3.$$

So we get $w = 2xy = (2/9)(u + v - 2)(u - 2v + 7)$.

And thus, the function f is $w = f(u, v) = \frac{2}{9}(u + v - 2)(u - 2v + 7)$.

Note that the function f is 3-D.

Suggestions or Solutions
To the Problem in the Example 7

Assuming $w = g(u, v)$, and $g = Z \bullet f$, where $t = f(s) = s + 1$, and Z: $(s, t) \longrightarrow (2s, 2t - 1, s^2 + t)$, find the function g.

$(u, v, w) = (2s, 2t - 1, s^2 + t)$.

$u = 2s \Rightarrow s = u/2$, $v = 2t - 1 \Rightarrow t = (v + 1)/2$, and $w = s^2 + t$.

$w = s^2 + t = (u/2)^2 + (v + 1)/2 = (u^2 + 2v + 2)/4$.

$t = s + 1 \Rightarrow (v + 1)/2 = u/2 + 1 \Rightarrow v + 1 = u + 2 \Rightarrow v = u + 1$.

$w = g(u, v) = (u^2 + 2v + 2)/4$ where $v = u + 1$.

If not sure of the ideas behind the processes above, follow the steps below.

We know Z is a transformation. So finding g, we find $Z \bullet f$. So finding g, we just take the transformation of f by Z.   What transformation then, is Z?

It is a transformation where a curve in 2-D gets transformed to a curve in 3-D.

And the original curve in this case is the curve of f, which is a line, which is a part of a plane, so the curve of f is a 2-D object.   Isn't a line a 1-D object though?

A line can be in 1-D space. And if a line is in 1-D space, the line is a 1-D object.

In this case however, the curve of f, that is, the line is not in 1-D but in 2-D.
So the line that is the curve of f is a 2-D object.

And there is another reason that the line is a 2-D object.
A line is a set of points. The same is true though, for any other object in geometry, too.

If a line is in 2-D space, that is, it is in a plane, every point in the line has a pair of coordinates. In other words, each of all the points in the line has two coordinates.

And thus, each of all the points in the line is a 2-D object, and so is the line.

Suppose however, a line is in 1-D space. A number line is a 1-D object, for instance. A number line is a set of points, each of which has a number, which is in fact, a coordinate. So every point in a number line has one coordinate, and thus, is a 1-D object.

What then, do we mean by a line in 3-D?

A line can be a 3-D object if every point it has three coordinates.

So for instance, a line in a cube is a set of points, each of which has three coordinates, and thus, the line is a 3-D object. And the same is true, too, for a curved line in any-D.

Let's now take the transformation of $\boldsymbol{f}$ by $\boldsymbol{Z}$. What then, do we want to begin with?

We know the curve of $\boldsymbol{g}$ is a 3-D object. So we can take $\boldsymbol{(u, v, w)}$ as the arbitrary point the curve of $\boldsymbol{g}$. And then, we can begin with setting $\boldsymbol{(u, v, w) = (2s, 2t - 1, s^2 + t)}$. That is, we get $\boldsymbol{u = 2s}$, $\boldsymbol{v = 2t - 1}$, and $\boldsymbol{w = s^2 + t}$. What then, is the next?

We need to connect $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, that is, we want to get the expression connecting $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. Then, the connective equation is the equation of the curve, that is, the curve of $\boldsymbol{g}$. And we can get the equation through the medium of $\boldsymbol{s}$ and $\boldsymbol{t}$. In other words, $\boldsymbol{Z}$ is a parametric transformation where the parameters are $\boldsymbol{s}$ and $\boldsymbol{t}$.

So we want $\boldsymbol{w}$ to be the output variable in the function $\boldsymbol{g}$.

And thus, solving for $\boldsymbol{s}$ and $\boldsymbol{t}$ the system of two equations $\boldsymbol{u = 2s}$ and $\boldsymbol{v = 2t - 1}$, and then, putting the $\boldsymbol{s}$ and $\boldsymbol{t}$ into the other equation $\boldsymbol{w = s^2 + t}$, we get the curve of $\boldsymbol{g}$.

And the system is rather simple, and solving it, we get $\boldsymbol{u = 2s \Rightarrow s = u/2}$, and $\boldsymbol{v = 2t - 1}$ $\boldsymbol{\Rightarrow t = (v + 1)/2}$.

Thus, we get $\boldsymbol{w = s^2 + t = (u/2)^2 + (v + 1)/2 = (u^2 + 2v + 2)/4}$.

So is $\boldsymbol{w = g(u, v) = (u^2 + 2v + 2)/4}$ the function $\boldsymbol{g}$?

If the function $\boldsymbol{g}$ is $\boldsymbol{w = g(u, v) = (u^2 + 2v + 2)/4}$, the domain is the entire $\boldsymbol{u}$-$\boldsymbol{v}$ plane. In this case however, it is not the case. What then, is the domain?

We know the original curve is the curve of $\boldsymbol{f}$, and thus, is $\boldsymbol{t = s + 1}$.

And $\boldsymbol{u}$ and $\boldsymbol{v}$ are the input variables used in the function $\boldsymbol{g}$.

So since the original curve is $\boldsymbol{t = s + 1}$, the input variables $\boldsymbol{u}$ and $\boldsymbol{v}$ are subject to $\boldsymbol{t = s + 1}$. And we have $\boldsymbol{s = u/2}$, and $\boldsymbol{t = (v + 1)/2}$. So we get

$$\boldsymbol{t = s + 1 \Rightarrow (v + 1)/2 = u/2 + 1 \Rightarrow v + 1 = u + 2 \Rightarrow v = u + 1}.$$

And thus, the domain of $\boldsymbol{g}$ is a line $\boldsymbol{v = u + 1}$ in the $\boldsymbol{u}$-$\boldsymbol{v}$-$\boldsymbol{w}$ space.

So the function $\boldsymbol{g}$ is $\boldsymbol{w = g(u, v) = (u^2 + 2v + 2)/4}$ where $\boldsymbol{v = u + 1}$.

And thus, only the values of $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfying $\boldsymbol{v = u + 1}$ are eligible for the input to the function $\boldsymbol{g}$. And of course, the line $\boldsymbol{v = u + 1}$ is in the $\boldsymbol{u}$-$\boldsymbol{v}$ plane in the $\boldsymbol{u}$-$\boldsymbol{v}$-$\boldsymbol{w}$ space.

So in other words, the set of all the points in the line $\boldsymbol{v = u + 1}$ in the $\boldsymbol{u}$-$\boldsymbol{v}$ plane is the domain of the function $\boldsymbol{w = g(u, v)}$, and any other point is not allowed for $\boldsymbol{w = g(u, v)}$.

So for instance, we can use (1, 2) in the $\boldsymbol{u}$-$\boldsymbol{v}$ plane since we can have $\boldsymbol{g(1, 2)}$, but we cannot use (1, 3) in the plane because we cannot have $\boldsymbol{g(1, 3)}$.

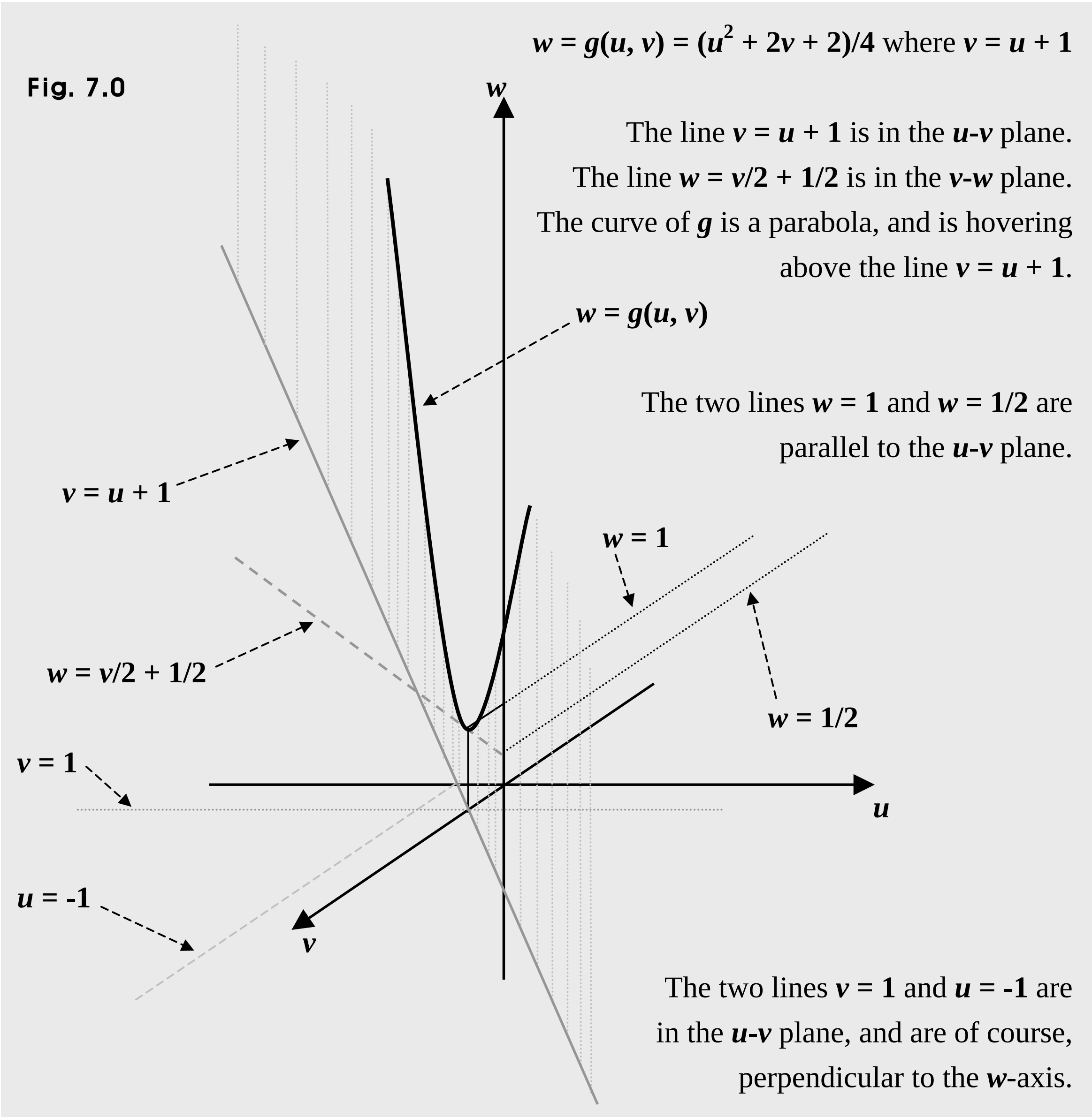
What then, exactly the curve of $\boldsymbol{g}$?

If the domain of $\boldsymbol{g}$ is the entire $\boldsymbol{u}$-$\boldsymbol{v}$ plane, the curve of $\boldsymbol{g}$ is a curved surface, which is a parabolic trough. We know however, the domain of $\boldsymbol{g}$ is the line $\boldsymbol{v = u + 1}$.

So the curve of $\boldsymbol{g}$ is in a plane vertical to the $\boldsymbol{u}$-$\boldsymbol{v}$ plane, cutting through the trough, and including the line $\boldsymbol{v = u + 1}$.

In other words, the cross section is the curve of $\boldsymbol{g}$, and is a parabola. And thus, the curve of $\boldsymbol{g}$ is a parabola hovering above the line in the $\boldsymbol{u}$-$\boldsymbol{v}$-$\boldsymbol{w}$ space, and is shown below.

Fig. 7.0



In short:

To begin with, we can set $(u, v, w) = (2s, 2t - 1, s^2 + t)$.
So we get $u = 2s$, $v = 2t - 1$, and $w = s^2 + t$. That is, we get $s = u/2$, and $t = (v + 1)/2$.

And next, connecting u, v, and w, we get $w = s^2 + t = (u/2)^2 + (v + 1)/2 = (u^2 + 2v + 2)/4$.

And next, getting the domain of g, since the original curve is $t = s + 1$, we get

$t = s + 1 \Rightarrow (v + 1)/2 = u/2 + 1 \Rightarrow v + 1 = u + 2 \Rightarrow v = u + 1$.

And thus, the function g is $w = g(u, v) = (u^2 + 2v + 2)/4$ where $v = u + 1$.