Examples 1 in Polynomial Arithmetic

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Examples 1 in Polynomial Arithmetic

Note:

As all the other examples do in this book, the ones here, too, cover not only practices on the topics covered, but some more ideas, and calculation tools and technics, too, which can help us do *algebra* quickly, as well as properly.

0. Assuming $\boldsymbol{4x^3 - 5x^2 + 7x + 3 = a(x-2)^3 + b(x-2)^2 + c(x-2) + d}$, find the values of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.

1. Simplify $\boldsymbol{N = \frac{3x+4}{x+1} - \frac{5x+12}{x+2} + \frac{2x+7}{x+3}}$.

2. Simplify $\boldsymbol{N = \frac{1}{x(x+2)} + \frac{1}{(x+2)(x+4)} + \frac{1}{(x+4)(x+6)}}$.

3. Find the values of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ satisfying each of the identical equations below.

3.0. $\boldsymbol{\frac{3x}{x^3+1} = \frac{3x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}}$.

3.1. $\boldsymbol{\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}}$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming $4x^3 - 5x^2 + 7x + 3 = a(x-2)^3 + b(x-2)^2 + c(x-2) + d$, find the values of *a*, *b*, *c*, and *d*.

2	4	-5	7	3	
		8	6	26	
2	4	3	13	29	$\Rightarrow$ $\boldsymbol{d = 29}$.
		8	22		
2	4	11	35	$\Rightarrow$ $\boldsymbol{c = 35}$.	
		8			
	4	19	$\Rightarrow$ $\boldsymbol{a = 4}$, and $\boldsymbol{b = 19}$.		

If not quite sure of how we can get the values, follow the steps below.

Let's begin with setting $\boldsymbol{P = 4x^3 - 5x^2 + 7x + 3}$.
Then, we can set $\boldsymbol{P = a(x-2)^3 + b(x-2)^2 + c(x-2) + d}$, too.
Then, the problem is saying that using the expression in $\boldsymbol{P = 4x^3 - 5x^2 + 7x + 3}$, we can find the values of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.   How?

By means of the polynomial $\boldsymbol{P = a(x-2)^3 + b(x-2)^2 + c(x-2) + d}$.
So examining the polynomial $\boldsymbol{P = a(x-2)^3 + b(x-2)^2 + c(x-2) + d}$, we can notice that if $\boldsymbol{x = 2}$, we get $\boldsymbol{P = d}$.   How can we get $\boldsymbol{d}$ though?

Using the expression in $\boldsymbol{P = 4x^3 - 5x^2 + 7x + 3}$, we can find it.   How?

Putting **2** into $\boldsymbol{x}$ in the polynomial $\boldsymbol{P}$ right above, we get

$\boldsymbol{P = 4x^3 - 5x^2 + 7x + 3 = 3 \cdot 2^3 - 5 \cdot 2^2 + 7 \cdot 2 + 3 = 32 - 20 + 14 + 3 = 29 \Rightarrow P = 29}$ if $\boldsymbol{x = 2}$.

Also, if $\boldsymbol{x = 2}$, we get $\boldsymbol{P = d}$, too, from $\boldsymbol{P = a(x-2)^3 + b(x-2)^2 + c(x-2) + d}$.
So we get $\boldsymbol{d = 29}$. What about the other values, though?

First, we can put $\boldsymbol{P = a(x-2)^3 + b(x-2)^2 + c(x-2) + d}$ the way below, too.

$\boldsymbol{P = a(x-2)^3 + b(x-2)^2 + c(x-2) + d = (x-2)\{a(x-2)^3 + b(x-2)^2 + c\} + d}$.

So next, setting $\boldsymbol{Q = a(x-2)^2 + b(x-2) + c}$, we get $\boldsymbol{P = (x-2)Q + d}$.

Then, we can say that if we divide $\boldsymbol{P = 4x^3 - 5x^2 + 7x + 3}$ by $\boldsymbol{x - 2}$, $\boldsymbol{Q}$ is the quotient, and $\boldsymbol{d}$ is the remainder. So let's do the division.

We can do the synthetic division the way below.
(If not sure of how we can do it the way below, refer to **Polynomial Arithmetic 5**.)

2	4	-5	7	3
		8	6	26
	4	3	13	29

And we have $\boldsymbol{P = (x-2)Q + d}$.
So we can readily see that $\boldsymbol{Q = 4x^2 + 3x + 13}$, and the remainder is 29, which is $\boldsymbol{d}$.

Then, by the same method we did for the value of $\boldsymbol{d}$, we can find $\boldsymbol{c}$ using the polynomial $\boldsymbol{Q = 4x^2 + 3x + 13}$, together with $\boldsymbol{Q = a(x-2)^2 + b(x-2) + c}$.

That is, putting 2 into the polynomial $\boldsymbol{Q = 4x^2 + 3x + 13}$, we get $\boldsymbol{Q =}$ <u>*a particular value*</u>, and putting 2 into $\boldsymbol{x}$ in the polynomial $\boldsymbol{Q = a(x-2)^2 + b(x-2) + c}$, we get $\boldsymbol{Q = c}$, which is <u>*the particular value*</u> stated above.

And by the same token, we can find the values of $\boldsymbol{a}$ and $\boldsymbol{b}$, too, in a sequential manner.

That is, for instance, setting $\boldsymbol{R = a(x-2) + b}$, we get $\boldsymbol{Q = (x-2)R + c}$. How?

That's because $\boldsymbol{Q = a(x-2)^2 + b(x-2) + c = (x-2)\{a(x-2)^2 + b\} + c = (x-2)R + c}$.

Next, for instance, setting $\boldsymbol{S = a}$, we get $\boldsymbol{R = (x-2)S + b}$, because $\boldsymbol{R = a(x-2) + b}$.

Thus, putting threads together, we can say this:

The division of $\boldsymbol{P}$ by ($\boldsymbol{x-2}$) produces $\boldsymbol{d}$ as the remainder, the division of $\boldsymbol{Q}$ by ($\boldsymbol{x-2}$) yields $\boldsymbol{c}$ as the remainder, and dividing $\boldsymbol{R}$ by ($\boldsymbol{x-2}$), we get $\boldsymbol{a}$ as the quotient and $\boldsymbol{b}$ as the remainder. So we can sequentially do the synthetic division the way below.

$$
\begin{array}{c|cccc l}
2 & 4 & -5 & 7 & 3 & \\
 & & 8 & 6 & 26 & \\
\hline
2 & 4 & 3 & 13 & 29 & \Rightarrow \boldsymbol{Q = 4x^2 + 3x + 13}, \text{ and } \boldsymbol{d = 29}. \\
 & & 8 & 22 & & \\
\cline{2-4}
2 & 4 & 11 & 35 & & \Rightarrow \boldsymbol{R = 4x + 11}, \text{ and } \boldsymbol{c = 35}. \\
 & & 8 & & & \\
\cline{1-3}
 & 4 & 19 & & & \Rightarrow \boldsymbol{a = 4}, \text{ and } \boldsymbol{b = 19}.
\end{array}
$$

That is to say that

$$\boldsymbol{P = 4x^3 - 5x^2 + 7x + 3 = (x-2)Q + 29 = (x-2)(4x^2 + 3x + 13) + 29}.$$

$$\boldsymbol{Q = 4x^2 + 3x + 13 = (x-2)R + 35 = (x-2)(4x + 11) + 35}.$$

$$\boldsymbol{R = 4x + 11 = (x-2)a + 19 = (x-2)4 + 19}.$$

So we can see that $\boldsymbol{P = 4(x-2)^3 + 19(x-2)^2 + 35(x-2) + 29}$.

By the way, setting $\boldsymbol{P(x) = 4(x-2)^3 + 19(x-2)^2 + 35(x-2) + 29}$, we get

$$\boldsymbol{P(x+2) = 4(x+2-2)^3 + 19(x+2-2)^2 + 35(x+2-2) + 29 = 4x^3 + 19x^2 + 35x + 29}.$$

So we get $\boldsymbol{P(x+2) = 4x^3 + 19x^2 + 35x + 29}$.

Suggestions or Solutions

To the Problem in the Example 1

Simplify $N=\frac{3x+4}{x+1}-\frac{5x+12}{x+2}+\frac{2x+7}{x+3}$.

$$N=\frac{3x+4}{x+1}-\frac{5x+12}{x+2}+\frac{2x+7}{x+3}=3+\frac{1}{x+1}-5-\frac{2}{x+2}+2+\frac{1}{x+3}=\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+3}.$$

$$=\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+2}+\frac{1}{x+3}=\frac{1}{x+1}-\frac{1}{x+2}-\left(\frac{1}{x+2}-\frac{1}{x+3}\right)$$

$$=\frac{1}{(x+1)(x+2)}-\frac{1}{(x+2)(x+3)}=\frac{1}{x+2}\left(\frac{1}{x+1}-\frac{1}{x+3}\right)$$

$$=\frac{1}{x+2}\cdot\frac{2}{(x+1)(x+3)}=\frac{2}{(x+1)(x+2)(x+3)}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, we have $\frac{ma+n}{a}=m+\frac{n}{a}$.

So we can get $\frac{3x+4}{x+1}=\frac{3x+3+1}{x+1}=\frac{3(x+1)+1}{x+1}=3+\frac{1}{x+1}$.

By the same token, we can get $\frac{5x+12}{x+2}=5+\frac{2}{x+2}$, and $\frac{2x+7}{x+3}=2+\frac{1}{x+3}$.

Therefore, $N=3+\frac{1}{x+1}-\left(5+\frac{2}{x+2}\right)+2+\frac{1}{x+3}=3+\frac{1}{x+1}-5-\frac{2}{x+2}+2+\frac{1}{x+3}$

$$=\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+3}.$$

Also, we have $\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+3}=\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+2}+\frac{1}{x+3}$, and $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$.

So we get $N=\frac{1}{x+1}-\frac{1}{x+2}-\left(\frac{1}{x+2}-\frac{1}{x+3}\right)=\frac{1}{(x+1)(x+2)}-\frac{1}{(x+2)(x+3)}$.

And we have this, too: $\frac{1}{ac}-\frac{1}{bc}=\left(\frac{1}{a}-\frac{1}{b}\right)\frac{1}{c}=\frac{1}{c}\left(\frac{1}{a}-\frac{1}{b}\right)$.

Thus, we get $N=\frac{1}{x+2}\left(\frac{1}{x+1}-\frac{1}{x+3}\right)=\frac{1}{x+2}\cdot\frac{2}{(x+1)(x+3)}=\frac{2}{(x+1)(x+2)(x+3)}$.

So we can say that $\frac{3x+4}{x+1}-\frac{5x+12}{x+2}+\frac{2x+7}{x+3}=\frac{2}{(x+1)(x+2)(x+3)}$.

In short:

$$N=\frac{3x+4}{x+1}-\frac{5x+12}{x+2}+\frac{2x+7}{x+3}=3+\frac{1}{x+1}-5-\frac{2}{x+2}+2+\frac{1}{x+3}=\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+3}.$$

$$=\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+2}+\frac{1}{x+3}=\frac{1}{x+1}-\frac{1}{x+2}-\left(\frac{1}{x+2}-\frac{1}{x+3}\right)$$

$$=\frac{1}{(x+1)(x+2)}-\frac{1}{(x+2)(x+3)}=\frac{1}{x+2}\left(\frac{1}{x+1}-\frac{1}{x+3}\right)$$

$$=\frac{1}{x+2}\cdot\frac{2}{(x+1)(x+3)}=\frac{2}{(x+1)(x+2)(x+3)}.$$

Suggestions or Solutions

To the Problem in the Example 2

Simplify $N=\dfrac{1}{x(x+2)}+\dfrac{1}{(x+2)(x+4)}+\dfrac{1}{(x+4)(x+6)}$.

To begin with, we have $\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{b-a}{ab}\Rightarrow\dfrac{1}{ab}=\dfrac{1}{b-a}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$.

So we get $\dfrac{1}{x(x+2)}=\dfrac{1}{(x+2)-x}\left(\dfrac{1}{x}-\dfrac{1}{x+2}\right)=\dfrac{1}{2}\left(\dfrac{1}{x}-\dfrac{1}{x+2}\right)$.

By the same token, we get

$\dfrac{1}{(x+2)(x+4)}=\dfrac{1}{2}\left(\dfrac{1}{x+2}-\dfrac{1}{x+4}\right)$, and $\dfrac{1}{(x+4)(x+6)}=\dfrac{1}{2}\left(\dfrac{1}{x+4}-\dfrac{1}{x+6}\right)$.

So we get

$$N=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)+\frac{1}{2}\left(\frac{1}{x+2}-\frac{1}{x+4}\right)+\frac{1}{2}\left(\frac{1}{x+4}-\frac{1}{x+6}\right)=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+6}\right).$$

Then again, since we have $\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{b-a}{ab}$, we get

$$N=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+6}\right)=\frac{1}{2}\cdot\frac{(x+6)-x}{x(x+6)}=\frac{1}{2}\cdot\frac{6}{x(x+6)}=\frac{3}{x(x+6)}.$$

In short:

$$N=\frac{1}{x(x+2)}+\frac{1}{(x+2)(x+4)}+\frac{1}{(x+4)(x+6)}$$

$$=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)+\frac{1}{2}\left(\frac{1}{x+2}-\frac{1}{x+4}\right)+\frac{1}{2}\left(\frac{1}{x+4}-\frac{1}{x+6}\right)=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+6}\right)=\frac{3}{x(x+6)}.$$

Suggestions or Solutions

To the Problem 0 in the Example 3

Find the values of *A*, *B* and *C* satisfying the identical equation below.

$$\frac{3x}{x^3+1}=\frac{3x}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

To begin with, $\frac{3x}{x^3+1}=\frac{3x}{x^3+1}=\frac{A(x^2-x+1)+(Bx+C)(x+1)}{x^3+1}$.

So $3x = A(x^2 - x + 1) + (x + 1)(Bx + C) = x^2(A + B) + x(C + B - A) + A + C$.

Now, $A + B = 0$, and $A + C = 0$, so $A = -C = -B$.

Thus, $C + B - A = B + B - (-B) = 3B = 3 \Rightarrow B = 1 \Rightarrow A = -1 \Rightarrow C = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

Examining first, the denominators, we can see those in the first two fractions are the same. How can we so readily and quickly notice that, though?

If much used to polynomial factorizations, we can do so.
Expanding the second one, we get $(x + 1)(x^2 - x + 1) = x^3 - x^2 + x + x^2 - x + 1 = x^3 + 1$.

Next, taking the common denominator in the fractions on the far right hand side, we get

$(x + 1)(x^2 - x + 1)$, which is $x^3 + 1$. So we get

$$\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}=\frac{A(x^2-x+1)}{(x+1)(x^2-x+1)}+\frac{(Bx+C)(x+1)}{(x+1)(x^2-x+1)}=\frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}.$$

Thus, we get $\frac{3x}{x^3+1} = \frac{3x}{x^3+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{x^3+1}$.

So we get $3x = A(x^2 - x + 1) + (x + 1)(Bx + C) = Ax^2 - Ax + A + Bx^2 + Cx + Bx + C$

$= x^2(A + B) + x(-A + C + B) + A + C \Rightarrow 3x = x^2(A + B) + x(C + B - A) + A + C$

$\Rightarrow 0 \cdot x^2 + 3x + 0 = x^2(A + B) + x(C + B - A) + A + C$.

So comparing now, the coefficients and the constant terms in both sides, we get a system of equations as follows. $A + B = 0$, $C + B - A = 3$, and $A + C = 0$.

Now, solving the system above, we can find A, B, and C, of course.

First, $A + B = 0 \Rightarrow A = -B$, and next, $A + C = 0 \Rightarrow A = -C \Rightarrow C = B$, since $A = -B$.

Thus, we get $C + B - A = B + B - (-B) = 3B = 3 \Rightarrow B = 1 \Rightarrow A = -1 \Rightarrow C = 1$.

So we can put the fraction this way, too: $\frac{3x}{x^3+1} = \frac{-1}{x+1} + \frac{x+1}{x^2-x+1}$.

In short:

To begin with, $\frac{3x}{x^3+1} = \frac{3x}{x^3+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{x^3+1}$.

So $3x = A(x^2 - x + 1) + (x + 1)(Bx + C) = x^2(A + B) + x(C + B - A) + A + C$.

Now, $A + B = 0$, and $A + C = 0$, so $A = -C = -B$.

Thus, $C + B - A = B + B - (-B) = 3B = 3 \Rightarrow B = 1 \Rightarrow A = -1 \Rightarrow C = 1$.

Suggestions or Solutions

To the Problem 1 in the Example 3

Find the values of *A*, *B* and *C* satisfying the identical equation below.

$$\frac{1}{x^3+x^2}=\frac{1}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}.$$

To begin with, $\frac{1}{x^3+x^2}=\frac{1}{x^2(x+1)}=\frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$.

So $1 = Ax(x+1) + B(x+1) + Cx^2 = (A+C)x^2 + (A+B)x + B$.

Thus, $A + C = 0$, $A + B = 0$, and $B = 1$.

So $B = 1 \Rightarrow A + B = A + 1 = 0 \Rightarrow A = -1$, and thus, $A + C = -1 + C = 0 \Rightarrow C = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

Examining the denominators, we can see those in the first two fractions are the same.

Next, taking the common denominator in the fractions on the far right hand side, that is, taking the LCM (the least common multiple) of x, x^2, and $x + 1$, we get

$x^2(x+1)$, which is x^3+x^2. So we get

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}=\frac{Ax(x+1)}{x^2(x+1)}+\frac{B(x+1)}{x^2(x+1)}+\frac{Cx^2}{x^2(x+1)}=\frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}.$$

Thus, we get $\frac{1}{x^2(x+1)}=\frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$

So we get

$\mathbf{1 = \mathit{Ax}(\mathit{x} + 1) + \mathit{B}(\mathit{x} + 1) + \mathit{Cx}^2 = \mathit{Ax}^2 + \mathit{Ax} + \mathit{Bx} + \mathit{B} + \mathit{Cx}^2 = (\mathit{A} + \mathit{C})\mathit{x}^2 + (\mathit{A} + \mathit{B})\mathit{x} + \mathit{B}}$.

That is, we get $\mathbf{0 \cdot \mathit{x}^2 + 0 \cdot \mathit{x} + 1 = (\mathit{A} + \mathit{C})\mathit{x}^2 + (\mathit{A} + \mathit{B})\mathit{x} + \mathit{B}}$.

So comparing the coefficients and the constant terms in both sides of the equality above, we get a system of equations as follows. $\mathbf{\mathit{A} + \mathit{C} = 0}$, $\mathbf{\mathit{A} + \mathit{B} = 0}$, and $\mathbf{\mathit{B} = 1}$.

Thus, we get $\mathbf{\mathit{B} = 1 \Rightarrow \mathit{A} + \mathit{B} = \mathit{A} + 1 = 0 \Rightarrow \mathit{A} = -1}$, so $\mathbf{\mathit{A} + \mathit{C} = -1 + \mathit{C} = 0 \Rightarrow \mathit{C} = 1}$.

So we can put the fraction this way, too: $\dfrac{1}{x^3 + x^2} = \dfrac{1}{x^2(x+1)} = \dfrac{-1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x+1}$.

And we can notice that we can break apart a fraction the way above.

In short:

To begin with, $\dfrac{1}{x^3 + x^2} = \dfrac{1}{x^2(x+1)} = \dfrac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$.

So $\mathbf{1 = \mathit{Ax}(\mathit{x} + 1) + \mathit{B}(\mathit{x} + 1) + \mathit{Cx}^2 = (\mathit{A} + \mathit{C})\mathit{x}^2 + (\mathit{A} + \mathit{B})\mathit{x} + \mathit{B}}$.

Thus, $\mathbf{\mathit{A} + \mathit{C} = 0}$, $\mathbf{\mathit{A} + \mathit{B} = 0}$, and $\mathbf{\mathit{B} = 1}$.

So we get $\mathbf{\mathit{B} = 1 \Rightarrow \mathit{A} + \mathit{B} = \mathit{A} + 1 = 0 \Rightarrow \mathit{A} = -1}$, and thus, $\mathbf{\mathit{A} + \mathit{C} = -1 + \mathit{C} = 0 \Rightarrow \mathit{C} = 1}$.