Examples 2 in Polynomial Arithmetic

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0. Find the values of P and Q as follows.

0.0. $P = \dfrac{a}{ab+a+1} + \dfrac{b}{bc+b+1} + \dfrac{c}{ca+c+1}$, and $abc = 1$.

0.1. $Q = \dfrac{a+b}{(a+1)(b+1)} + \dfrac{b+c}{(b+1)(c+1)} + \dfrac{c+a}{(c+1)(a+1)}$, and $abc = -1$.

1. Assuming that $x + \dfrac{1}{x} = 1$, find the values of P, Q, R, and S as follows.

1.0. $P = x^2 + \dfrac{1}{x^2}$

1.1. $Q = x^3 + \dfrac{1}{x^3}$

1.2. $R = \dfrac{x^{10} + 1}{x^2}$

1.3. $S = x^{101} + \dfrac{1}{x^{101}}$

Suggestions or Solutions
To the Problems in the Example 0

Find the values of P and Q as follows.

0.0. $P=\dfrac{a}{ab+a+1}+\dfrac{b}{bc+b+1}+\dfrac{c}{ca+c+1}$, and $abc=1$.

0.1. $Q=\dfrac{a+b}{(a+1)(b+1)}+\dfrac{b+c}{(b+1)(c+1)}+\dfrac{c+a}{(c+1)(a+1)}$, and $abc=-1$.

0.0. Quite often, reducing the number of variables, we can make expressions simpler.

Using $abc = 1$, we can do so. So to begin with, we get $abc = 1 \Rightarrow c=\dfrac{1}{ab}$, and thus,

$$\frac{b}{bc+b+1}=\frac{b}{b\cdot\frac{1}{ab}+b+1}=\frac{b}{\frac{1}{a}+b+1}=\frac{b}{\frac{1+ab+a}{a}}=b\cdot\frac{a}{1+ab+a}=\frac{ab}{ab+a+1}.$$

$$\frac{c}{ca+c+1}=\frac{\frac{1}{ab}}{a\cdot\frac{1}{ab}+\frac{1}{ab}+1}=\frac{\frac{1}{ab}}{\frac{1}{b}+\frac{1}{ab}+1}=\frac{\frac{1}{ab}}{\frac{b+1+ab}{ab}}=\frac{1}{ab}\cdot\frac{ab}{b+1+ab}=\frac{1}{ab+b+1}.$$

Thus, we get $P=\dfrac{a}{ab+a+1}+\dfrac{ab}{ab+a+1}+\dfrac{1}{ab+a+1}=\dfrac{ab+b+1}{ab+a+1}=1$.

So what is this about?

It’s just for algebra practice, because algebra matters. And it does very much so.
Doing algebra, we manipulate math expressions.
Doing algebra very well, we can change or alter expressions keeping their values intact, so we can put the expressions the way we want. What way though?

The way the problems can be solved, of course.

And next, moving on to the problem **0.1**, we get

$$Q=\frac{a+b}{(a+1)(b+1)}+\frac{b+c}{(b+1)(c+1)}+\frac{c+a}{(c+1)(a+1)}$$

$$=\frac{(a+b)(c+1)}{(a+1)(b+1)(c+1)}+\frac{(a+1)(b+c)}{(a+1)(b+1)(c+1)}+\frac{(b+1)(c+a)}{(b+1)(c+1)(a+1)}$$

$$=\frac{ac+a+bc+b+ab+ac+b+c+bc+ab+c+a}{(a+1)(b+1)(c+1)}=\frac{2ab+2bc+2ca+2a+2b+2c}{(ab+a+b+1)(c+1)}$$

$$=\frac{2(ab+bc+ca+a+b+c)}{(abc+ab+ac+a+bc+b+c+1)}=\frac{2(ab+bc+ca+a+b+c)}{(ab+bc+ca+a+b+c)}$$, because $abc = -1$.

So we get $Q = 2$.

Suggestions or Solutions
To the Problems in the Example 1

Assuming that $x+\frac{1}{x}=1$, find the values of P, Q, R, and S as follows.

1.0. $P=x^2+\frac{1}{x^2}$

1.1. $Q=x^3+\frac{1}{x^3}$

1.2. $R=\frac{x^{10}+1}{x^2}$

1.3. $S=x^{101}+\frac{1}{x^{101}}$

Since we are given this: $x+\frac{1}{x}=1$, we should be able to take advantage of it.

1.0. $x+\frac{1}{x}=1\Rightarrow\left(x+\frac{1}{x}\right)^2=\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=x^2+1+1+\frac{1}{x^2}=1^2\Rightarrow x^2+\frac{1}{x^2}=-1.$

1.1. Suppose $a = x$, and $b=\frac{1}{x}$.

Then, since $x+\frac{1}{x}=1$, we get $a + b = 1 \Rightarrow (a + b)^3 = 1$.

Meanwhile,

$$(a + b)^3 = (a + b)(a + b)(a + b) = (a^2 + ab + ab + b^2)(a + b) = (a^2 + 2ab + b^2)(a + b)$$

$$= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3ab(a + b) + b^3 = 1.$$

So we now have $a^3 + 3ab(a + b) + b^3 = 1$.

And we have $Q=x^3+\frac{1}{x^3}=a^3+b^3$, $ab=x\cdot\frac{1}{x}=1$, and $a+b=x+\frac{1}{x}=1$.

So we get $Q + 3\cdot1\cdot1 = 1 \Rightarrow Q = -2$.

1.2. $R = \dfrac{x^{10}+1}{x^2}$

First, in the problem **1.1**, we found this: $Q = x^3 + \dfrac{1}{x^3} = -2$.

So setting next $a = x^3$, we can get

$$a + \frac{1}{a} = -2 \Rightarrow a\left(a + \frac{1}{a}\right) = -2a \Rightarrow a^2 + 1 = -2a \Rightarrow a^2 + 2a + 1 = 0$$

$$\Rightarrow a^2 + a + a + 1 = a(a + 1) + (a + 1)\cdot 1 = (a + 1)(a + 1) = (a + 1)^2 = 0.$$

So we get $a + 1 = 0 \Rightarrow a = -1$, and thus, $x^3 = -1$, since we've set $a = x^3$.

Now, $x^{10} = (x^3)^3 x = -x$, since $x^3 = -1$, so we get $R = \dfrac{x^{10}+1}{x^2} = \dfrac{-x+1}{x^2}$. What's next then?

We have $x + \dfrac{1}{x} = 1$. So multiplying by x both sides, we can get

$x^2 + 1 = x \Rightarrow x^2 - x + 1 = 0$. So what does it have to do with this: $R = \dfrac{-x+1}{x^2}$?

We can get $-x + 1 = -x^2$. Thus, we get $R = \dfrac{-x+1}{x^2} = \dfrac{-x^2}{x^2} = -1$.

1.3. $S = x^{101} + \dfrac{1}{x^{101}}$

In **1.2** above, we found $x^3 = -1$. And $101 = 33\cdot 3 + 2$. So we can get $x^{101} = (x^3)^{33}x^2 = -x^2$.

Thus, we get $S = x^{101} + \dfrac{1}{x^{101}} = -x^2 - \dfrac{1}{x^2} = -(x^2 + \dfrac{1}{x^2}) = -(-1)$, because $x^2 + \dfrac{1}{x^2} = -1$,

which is P in the problem **1.0**. above.

In short:

1.0. $\boldsymbol{P = x^2 + \frac{1}{x^2}}$

$$\boldsymbol{x + \frac{1}{x} = 1 \Rightarrow \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = 1 \Rightarrow P = x^2 + \frac{1}{x^2} = -1}.$$

1.1. $\boldsymbol{Q = x^3 + \frac{1}{x^3}}$

Setting $\boldsymbol{a = x}$, and $\boldsymbol{b = \frac{1}{x}}$, we get $\boldsymbol{a + b = 1}$, and $\boldsymbol{ab = 1}$.

So we get $\boldsymbol{(a + b)^3 = a^3 + 3ab(a + b) + b^3 = 1 \Rightarrow Q + 3\cdot 1\cdot 1 = 1 \Rightarrow Q = -2}$.

1.2. $\boldsymbol{R = \frac{x^{10} + 1}{x^2}}$

$$\boldsymbol{x + \frac{1}{x} = 1 \Rightarrow x^2 + 1 = x \Rightarrow x^2 - x + 1 = 0 \Rightarrow (x+1)(x^2 - x + 1) = x^3 + 1 = 0 \Rightarrow x^3 = -1}.$$

So we get $\boldsymbol{x^{10} = (x^3)^3 x = -x}$, since $\boldsymbol{x^3 = -1}$.

Thus, $\boldsymbol{R = \frac{x^{10} + 1}{x^2} = \frac{-x + 1}{x^2} = \frac{-x^2}{x^2} = -1}$, because $\boldsymbol{x^2 - x + 1 = 0 \Rightarrow -x + 1 = -x^2}$.

1.3. $\boldsymbol{S = x^{101} + \frac{1}{x^{101}}}$

We have $\boldsymbol{x^3 = -1}$, so we get $\boldsymbol{x^{101} = (x^3)^{33} x^2 = -x^2}$.

Thus, $\boldsymbol{S = x^{101} + \frac{1}{x^{101}} = -x^2 - \frac{1}{x^2} = -(x^2 + \frac{1}{x^2}) = -(-1)}$, since $\boldsymbol{x^2 + \frac{1}{x^2} = -1}$, which is from the problem **1.0**.