Examples 3 in Polynomial Arithmetic

Click this to start.

Examples 3 in Polynomial Arithmetic

0. Assuming that $\boldsymbol{x+\frac{1}{x}=4}$, find the values of $\boldsymbol{P}$ and $\boldsymbol{Q}$ as follows.

0.0. $\boldsymbol{P=x-\frac{1}{x}}$

0.1. $\boldsymbol{Q=x^3-\frac{1}{x^3}}$

1. Assuming that $\boldsymbol{x^3-\frac{1}{x^3}=8\sqrt{5}}$ where $\boldsymbol{x>0}$, find the values of $\boldsymbol{P}$ and $\boldsymbol{Q}$ as follows.

1.0. $\boldsymbol{P=x^3+\frac{1}{x^3}}$

1.1. $\boldsymbol{Q=x+\frac{1}{x}}$

Suggestions or Solutions
To the Problem in the Example 0

Assuming that $x+\frac{1}{x}=4$, find the values of P and Q as follows.

0.0. $P=x-\frac{1}{x}$

0.1. $Q=x^3-\frac{1}{x^3}$

To begin with, letting $a = x$, and $b=\frac{1}{x}$, we get $a + b = 4$, and $ab=x\cdot\frac{1}{x}=1$.

So next, we can get $a + b = 4 \Rightarrow (a + b)^2 = a^2 + 2ab + b^2 = a^2 + 2\cdot1 + b^2 = 4^2 = 16$

$\Rightarrow a^2 + b^2 = 16 - 2 = 14 \Rightarrow a^2 + b^2 = 14$.

0.0. $P = a - b \Rightarrow P^2 = (a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2\cdot1 = 14 - 2 = 12 \Rightarrow P^2 = 12$.

Thus, we get $P=\pm\sqrt{12}=\pm2\sqrt{3}$.

0.1. Let's begin with setting again, $a = x$, and $b=\frac{1}{x}$.

Then first, in the problem **0.0**, we have $P=\pm2\sqrt{3}$, and $P = a - b$.

So we can set $a-b=\pm2\sqrt{3}$. Thus, we can get

$(a - b)^3 = (a - b)(a - b)(a - b) = (a^2 - ab - ab + b^2)(a - b) = (a^2 - 2ab + b^2)(a - b)$

$= a^3 - a^2b - 2a^2b + 2ab^2 + ab^2 - b^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= a^3 - 3ab(a - b) - b^3 = (\pm2\sqrt{3})^3 = \pm2^3 3\sqrt{3} = \pm8\cdot3\sqrt{3} = \pm24\sqrt{3}$.

And we can put it this way, too: $(\pm 2\sqrt{3})^3 = (\pm\sqrt{12})^3 = \pm 12\sqrt{12} = \pm 12 \cdot 2\sqrt{3} = \pm 24\sqrt{3}$.

So we now have $a^3 - 3ab(a - b) - b^3 = \pm 24\sqrt{3}$.

Also, we can have $ab = 1$, too, because $a = x$, and $b = \dfrac{1}{x}$. So?

So we get $a^3 - 3ab(a-b) - b^3 = a^3 - 3 \cdot 1 \cdot (\pm 2\sqrt{3}) - b^3 = a^3 - (\pm 6\sqrt{3}) - b^3 = \pm 24\sqrt{3}$.

And we know $Q = a^3 - b^3$.

Thus, we get

$$a^3 - (\pm 6\sqrt{3}) - b^3 = a^3 - b^3 - (\pm 6\sqrt{3}) = Q - (\pm 6\sqrt{3}) = (\pm 24\sqrt{3})$$

$$\Rightarrow Q = (\pm 6\sqrt{3}) + (\pm 24\sqrt{3}) = \pm 30\sqrt{3}.$$

And if knowing polynomial factorizations, we can do it the way below, too.

$Q = a^3 - b^3 = (a - b)(a^2 + ab + b^2) = P(a^2 + 2ab + b^2 - ab)$, since $P = a - b$.

So we get $Q = P\{(a+b)^2 - 1\} = P(16 - 1) = 15P$, because we have $a + b = 4$, since we have $x + \dfrac{1}{x} = 4$, and we have set $a = x$, and $b = \dfrac{1}{x}$.

Thus, we get $Q = 15P = 15(\pm 2\sqrt{3}) = \pm 30\sqrt{3}$.

Also, we can do it this way, too:

$P^3 = (a - b)^3 = a^3 - b^3 - 3ab(a - b) = Q - 3 \cdot 1 \cdot P = Q - 3P$.

So we get $Q = P^3 + 3P = P(P^2 + 3) = \pm\sqrt{12}\{(\pm\sqrt{12})^2 + 3\} = \pm 2\sqrt{3}(12 + 3) = \pm 30\sqrt{3}$.

In short:

Assuning first, $\boldsymbol{a = x}$, and $\boldsymbol{b = \frac{1}{x}}$, we get $\boldsymbol{a + b = 4}$, and $\boldsymbol{ab = x \cdot \frac{1}{x} = 1}$.

So we get $\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2 = a^2 + 2 + b^2 = 4^2 = 16 \Rightarrow a^2 + b^2 = 14}$.

0.0. $\boldsymbol{P = x - \frac{1}{x}}$

$\boldsymbol{P^2 = (a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2 = 14 - 2 = 12 \Rightarrow P = \pm\sqrt{12} = \pm 2\sqrt{3}}$.

0.1. $\boldsymbol{Q = x^3 - \frac{1}{x^3}}$

We have $\boldsymbol{a - b = \pm 2\sqrt{3}}$, which is the value of $\boldsymbol{P}$ above, and $\boldsymbol{ab = 1}$.

So we can get $\boldsymbol{(a - b)^3 = a^3 - 3ab(a - b) - b^3 = (\pm 2\sqrt{3})^3 = \pm 24\sqrt{3}}$.

Thus, we get $\boldsymbol{Q - 3ab(a - b) = Q - 3 \cdot 1 \cdot (\pm 2\sqrt{3}) = Q - (\pm 6\sqrt{3}) = \pm 24\sqrt{3}}$.

Therefore, we get $\boldsymbol{Q = (\pm 6\sqrt{3}) + (\pm 24\sqrt{3}) = \pm 30\sqrt{3}}$.

Suggestions or Solutions

To the Problem 0 in the Example 1

Assuming that $x^3 - \frac{1}{x^3} = 8\sqrt{5}$ where $x > 0$, **find the values of** $P = x^3 + \frac{1}{x^3}$.

Setting first $a = x$, and $b = \frac{1}{x}$, we can get $ab = 1$, and $a^3 - b^3 = 8\sqrt{5}$.

So next, we can get

$P = a^3 + b^3 \Rightarrow P^2 = (a^3 + b^3)^2 = (a^3)^2 + 2a^3b^3 + (b^3)^2 = (a^3)^2 - 2a^3b^3 + (b^3)^2 + 4a^3b^3$

$= (a^3 - b^3)^2 + 4a^3b^3 = (8\sqrt{5})^2 + 4 = 8^2{\cdot}5 + 4 = 4(2{\cdot}8{\cdot}5 + 1) = 4{\cdot}81 = 18^2$.

Thus, we get $P = 18$, because $P > 0$, since $x > 0$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first again $a = x$, and $b = \frac{1}{x}$, we can get $ab = 1$, and $a^3 - b^3 = 8\sqrt{5}$.

So next, we can get

$P = a^3 + b^3 \Rightarrow P^2 = (a^3 + b^3)^2 = (a^3 + b^3)(a^3 + b^3) = (a^3)^2 + 2a^3b^3 + (b^3)^2$.

Thus, we get $P^2 = (a^3 + b^3)^2 = (a^3)^2 + 2a^3b^3 + (b^3)^2$.

And we can put it this way, too:

$P^2 = (a^3)^2 + 2a^3b^3 - 2a^3b^3 + 2a^3b^3 + (b^3)^2 = (a^3)^2 - 2a^3b^3 + (b^3)^2 + 4a^3b^3$.

Thus, we get $P^2 = (a^3)^2 - 2a^3b^3 + (b^3)^2 + 4a^3b^3$.

And we can have this, too: $(a^3 - b^3)^2 = (a^3 - b^3)(a^3 - b^3) = (a^3)^2 - 2a^3b^3 + (b^3)^2$.

So we can get $P^2 = (a^3 - b^3)^2 + 4a^3b^3$.

Next, we have $\boldsymbol{ab} = \mathbf{1}$, since $\boldsymbol{a} = \boldsymbol{x}$, and $\boldsymbol{b} = \dfrac{\mathbf{1}}{\boldsymbol{x}}$. And we have this, too: $\boldsymbol{a}^3 - \boldsymbol{b}^3 = \mathbf{8}\sqrt{\mathbf{5}}$.

So we get $\boldsymbol{P}^2 = (\boldsymbol{a}^3 - \boldsymbol{b}^3)^2 + 4\boldsymbol{a}^3\boldsymbol{b}^3 = (8\sqrt{5})^2 + 4 = 8^2 \cdot 5 + 4 = 4(2 \cdot 8 \cdot 5 + 1) = 4 \cdot 81 = 18^2$.

Thus, we get $\boldsymbol{P} = \mathbf{18}$, because $\boldsymbol{P} > \mathbf{0}$, because $\boldsymbol{P} = \boldsymbol{x}^3 + \dfrac{\mathbf{1}}{\boldsymbol{x}^3}$, and $\boldsymbol{x} > \mathbf{0}$.

So in short:

Setting first $\boldsymbol{a} = \boldsymbol{x}$, and $\boldsymbol{b} = \dfrac{\mathbf{1}}{\boldsymbol{x}}$, we can get $\boldsymbol{ab} = \mathbf{1}$, and $\boldsymbol{a}^3 - \boldsymbol{b}^3 = \mathbf{8}\sqrt{\mathbf{5}}$.
So next, we can get

$$\boldsymbol{P} = \boldsymbol{a}^3 + \boldsymbol{b}^3 \Rightarrow \boldsymbol{P}^2 = (\boldsymbol{a}^3 + \boldsymbol{b}^3)^2 = (\boldsymbol{a}^3)^2 + 2\boldsymbol{a}^3\boldsymbol{b}^3 + (\boldsymbol{b}^3)^2 = (\boldsymbol{a}^3)^2 - 2\boldsymbol{a}^3\boldsymbol{b}^3 + (\boldsymbol{b}^3)^2 + 4\boldsymbol{a}^3\boldsymbol{b}^3$$

$$= (\boldsymbol{a}^3 - \boldsymbol{b}^3)^2 + 4\boldsymbol{a}^3\boldsymbol{b}^3 = (8\sqrt{5})^2 + 4 = 8^2 \cdot 5 + 4 = 4(2 \cdot 8 \cdot 5 + 1) = 4 \cdot 81 = 18^2.$$

Thus, we get $\boldsymbol{P} = \mathbf{18}$, because $\boldsymbol{P} > \mathbf{0}$, since $\boldsymbol{x} > \mathbf{0}$.

Suggestions or Solutions

To the Problem 1 in the Example 1

Assuming that $x^3 - \frac{1}{x^3} = 8\sqrt{5}$ where $x > 0$, **find the values of** $Q = x + \frac{1}{x}$.

Setting first $a = x$, and $b = \frac{1}{x}$, we can get $ab = 1$.

And next, we can have $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

So we get $Q^3 = P + 3Q$, since $P = a^3 + b^3$, and $ab = 1$. And we have $P = 18$.

Thus, we get $Q^3 - 3Q - P = Q^3 - 3Q - 18 = 0$.

And we can get $Q = 3 \Rightarrow 3^3 - 3 \cdot 3 - 18 = 27 - 9 - 18 = 0$.

So we can set $Q^3 - 3Q - 18 = (Q - 3)(uQ^2 - vQ - w)$, where u, v, and w are constant.
That is to say that $(Q - 3)$ can divide $(Q^3 - 3Q - 18)$.
So doing the synthetic division, we can get this:

$$
\begin{array}{c|cccc}
3 & 1 & 0 & -3 & -18 \\
  &   & 3 & 9 & 18 \\
\hline
  & 1 & 3 & 6 & 0
\end{array}
$$

So we get $(Q - 3)(Q^2 + 3Q + 6) = 0$.

$\therefore$ $Q = 3$, because $Q^2 + 3Q + 6 \neq 0$, because $Q > 0$, because $Q = x + \frac{1}{x}$, and $x > 0$.
And a sum of numbers positive is positive and cannot be 0.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first again $a = x$, and $b = \frac{1}{x}$, we can get $ab = 1$.

And next, we can have $\boldsymbol{(a + b)^3 = (a + b)(a + b)^2 = (a + b)(a^2 + 2ab + b^2)}$

$\boldsymbol{= a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3ab(a + b) + b^3 = a^3 + b^3 + 3ab(a + b)}$.

So we get $\boldsymbol{(a + b)^3 = a^3 + b^3 + 3ab(a + b)}$. And we have $\boldsymbol{P = a^3 + b^3}$, and $\boldsymbol{ab = 1}$.

Also, we got $\boldsymbol{P = 18}$ in the example **1.0** above. So we get $\boldsymbol{Q^3 = P + 3Q}$.

Thus, we get $\boldsymbol{Q^3 - 3Q - P = Q^3 - 3Q - 18 = 0}$.

And knowing polynomial factorizations, we can try factorizing this: $\boldsymbol{Q^3 - 3Q - 18}$ to solve the equation, $\boldsymbol{Q^3 - 3Q - 18 = 0}$. How?

Assuming $\boldsymbol{c}$ is constant, and $(\boldsymbol{Q - c})$ divides $(\boldsymbol{Q^3 - 3Q - 18})$, we can set

$\boldsymbol{Q^3 - 3Q - 18 = (Q - c)(uQ^2 + vQ + w)}$, where $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are constant.

Then, $\boldsymbol{Q^3 - 3Q - 18}$ can be said to be factorized to $\boldsymbol{(Q - c)(uQ^2 + vQ + w)}$.

Then, we can say that $\boldsymbol{Q = c}$ is a solution to the equation $\boldsymbol{Q^3 - 3Q - 18 = 0}$.

That's because $\boldsymbol{Q^3 - 3Q - 18 = 0 \Rightarrow (Q - c)(uQ^2 + vQ + w) = 0}$

$\boldsymbol{\Rightarrow Q - c = 0}$ or $\boldsymbol{uQ^2 + vQ + w = 0}$.

The value of $\boldsymbol{c}$ can be one of divisors of 18, since the equation is $\boldsymbol{Q^3 - 3Q - 18 = 0}$.
So for instance, taking 3 as the value of $\boldsymbol{c}$, we get $\boldsymbol{Q - c = 0 \Rightarrow Q - 3 = 0 \Rightarrow Q = 3}$.
Then, we get $\boldsymbol{Q = 3 \Rightarrow Q^3 - 3Q - 18 = 3^3 - 3 \cdot 3 - 18 = 27 - 9 - 18 = 0}$.

So we can set $\boldsymbol{Q^3 - 3Q - 18 = (Q - 3)(uQ^2 - vQ - w)}$, where $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are constant.
That is to say that $(\boldsymbol{Q - 3})$ can divide $(\boldsymbol{Q^3 - 3Q - 18})$.
The idea has been covered in the section **Polynomial Arithmetic 5** in the book **Polynomial Factorizations**.
And we can easily find the quotient $(\boldsymbol{uQ^2 - vQ - w})$ using the synthetic division.

So doing the synthetic division, we can get

3	1	0	-3	-18
		3	9	18
	1	3	6	0

So we get $(\boldsymbol{Q} - \mathbf{3})(\boldsymbol{Q}^2 + \mathbf{3}\boldsymbol{Q} + \mathbf{6}) = \mathbf{0}$.

Then, we get $\boldsymbol{Q} - \mathbf{3} = \mathbf{0}$ or $\boldsymbol{Q}^2 + \mathbf{3}\boldsymbol{Q} + \mathbf{6} = \mathbf{0}$.

And in this case, we get $\boldsymbol{Q} = \mathbf{3}$ only, because we get $\boldsymbol{Q}^2 + \mathbf{3}\boldsymbol{Q} + \mathbf{6} \neq \mathbf{0}$. Why?

That's because $\boldsymbol{Q} > \mathbf{0}$, because $\boldsymbol{Q} = \boldsymbol{x} + \frac{\mathbf{1}}{\boldsymbol{x}}$, and $\boldsymbol{x} > \mathbf{0}$, and thus, $\boldsymbol{Q}^2 + \mathbf{3}\boldsymbol{Q} + \mathbf{6}$ is a sum of values all positive. And a sum of numbers positive is positive and cannot be 0.

And we can show this: $\boldsymbol{Q}^2 + \mathbf{3}\boldsymbol{Q} + \mathbf{6} \neq \mathbf{0}$ the way below, too.

$$\boldsymbol{Q}^2 + 3\boldsymbol{Q} + 6 = \boldsymbol{Q}^2 + 3\boldsymbol{Q} + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + 6 = (\boldsymbol{Q} + \tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + 6 = (\boldsymbol{Q} + \tfrac{3}{2})^2 + \tfrac{15}{4} \geq \tfrac{15}{4} \text{ for all } \boldsymbol{Q}.$$

- Also, we can do this problem the way below, too.

Using $\boldsymbol{P} = \boldsymbol{x}^3 + \frac{\mathbf{1}}{\boldsymbol{x}^3} = \mathbf{18}$, and $\boldsymbol{x}^3 - \frac{\mathbf{1}}{\boldsymbol{x}^3} = \mathbf{8}\sqrt{\mathbf{5}}$, we can find the value of $\boldsymbol{x}^3$, first.

$$\boldsymbol{x}^3 + \frac{1}{\boldsymbol{x}^3} + \boldsymbol{x}^3 - \frac{1}{\boldsymbol{x}^3} = 18 + 8\sqrt{5} \Rightarrow 2\boldsymbol{x}^3 = 18 + 8\sqrt{5} \Rightarrow \boldsymbol{x}^3 = 9 + 4\sqrt{5} \Rightarrow \boldsymbol{x} = \sqrt[3]{4\sqrt{5} + 9}.$$

So we get $\boldsymbol{Q} = \boldsymbol{x} + \frac{\mathbf{1}}{\boldsymbol{x}} = \sqrt[3]{\mathbf{4}\sqrt{\mathbf{5}} + \mathbf{9}} + \frac{\mathbf{1}}{\sqrt[3]{\mathbf{4}\sqrt{\mathbf{5}} + \mathbf{9}}}$, which is equal to 3.

In addition, we can actually find the value of $\boldsymbol{x}$ directly and use the value.

That is, we can solve the equation $\boldsymbol{x^3 - \frac{1}{x^3} = 8\sqrt{5}}$ where $\boldsymbol{x > 0}$, and use the solution.

To begin with, setting $\boldsymbol{t = x^3}$ in the equation above, we get $\boldsymbol{t - \frac{1}{t} = 8\sqrt{5}}$.

Thus, we get $\boldsymbol{t - \frac{1}{t} = 8\sqrt{5} \Rightarrow t^2 - 1 = 8\sqrt{5}t \Rightarrow t^2 - 8\sqrt{5}t - 1 = 0}$.

And using the quadratic formula, we get

$$\boldsymbol{t = \frac{8\sqrt{5} \pm \sqrt{64 \cdot 5 + 4}}{2} = \frac{8\sqrt{5} \pm \sqrt{18^2}}{2} = \frac{8\sqrt{5} \pm 18}{2} = 4\sqrt{5} \pm 9}.$$

And we have $\boldsymbol{x > 0}$, and $\boldsymbol{t = x^3}$. So we get $\boldsymbol{t > 0}$. Thus, we get $\boldsymbol{t = 4\sqrt{5} + 9}$.

And next, since $\boldsymbol{t = x^3}$, we get $\boldsymbol{x = t^{\frac{1}{3}} = \sqrt[3]{t} = \sqrt[3]{4\sqrt{5} + 9}}$.