# Examples 1 in Polynomial Factorizations



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# Examples 1

Factorize (that is, factor) the polynomials below.

0. $\mathbf{2}\boldsymbol{a} + \mathbf{4}\boldsymbol{b} + \mathbf{16}\boldsymbol{c}$

1. $\mathbf{2}\boldsymbol{ab} + \mathbf{6}\boldsymbol{bc}$

2. $\mathbf{2}\boldsymbol{ac} + \mathbf{6}\boldsymbol{bc} + \boldsymbol{ad} + \mathbf{3}\boldsymbol{bd}$

3. $\mathbf{3}\boldsymbol{k}^{\mathbf{2}}(\boldsymbol{b} + \mathbf{2}\boldsymbol{b}^{\mathbf{2}}\boldsymbol{c}) + \mathbf{9}\boldsymbol{bk} + \mathbf{27}\boldsymbol{b}^{\mathbf{2}}\boldsymbol{k}^{\mathbf{4}}$

## Suggestions or Solutions
### To the **Problem** in the Example **0**

This example is just a warming-up, and we have $\mathbf{2a + 4b + 16c}$, which is a polynomial.

And doing this example, we are just going over the basics in polynomial factorization processes.

So to begin with, what are we looking for when factorizing a polynomial?

We are looking for factors, which divide the polynomial.
So we want to find a divisor. Finding it though, we don't just find a divisor that divides the whole polynomial at once.    What else then do we do?

Normally, we begin with a divisor that can divide every term the polynomial has.
And such a divisor is said to be common to all the terms in the polynomial, and thus, is called a common divisor.

So a common divisor can divide every term in the polynomial, and thus, divides the polynomial, and is a factor of the polynomial.    So what does a factor do?

A factor is a divisor, and a factor of a polynomial divides the polynomial.
A divisor common to all the terms in a polynomial is a factor of the polynomial.

Suppose now, $\boldsymbol{P = 2a + 4b + 16c}$.

Then, examining all terms in the polynomial $\boldsymbol{P}$, we can see 2 is a divisor common to all the terms, and thus, 2 is a factor of $\boldsymbol{P}$.

We have $\mathbf{2a = 2 \cdot a}$, $\mathbf{4b = 2 \cdot 2b}$, and $\mathbf{16c = 2 \cdot 8c}$, so $\mathbf{2}$ is a common divisor.    So what?

So factorizing the polynomial ***P***, we divide by 2 every term in ***P***, put every quotient in a pair of parentheses, and take a product of the divisor 2 and the *sum* of all the quotients.

Then, we get $\boldsymbol{P = 2a + 4b + 16c = 2(a + 2b + 8c)}$, where $\boldsymbol{a + 2b + 8c}$ is the *sum* stated above.

So we can see that 2 divides the polynomial ***P***, and so does the sum of those quotients.

Therefore, 2 and the sum are factors of ***P***.

What then about the sum, which is a polynomial, too?

If an integer or an expression as a polynomial has as divisors 1 and itself only, it is said to be prime. And if a factor is prime, the factor is called a prime factor.

Now, the sum is the polynomial $\boldsymbol{a + 2b + 8c}$, and no divisor is common to all terms in the polynomial. That is, $\boldsymbol{a + 2b + 8c}$ has no divisor other than 1 and itself.

So $\boldsymbol{a + 2b + 8c}$ is prime, is a factor of ***P***, and thus, is a prime factor of ***P***.
Besides, 2 is a prime factor of ***P***, too.

And factorizing a polynomial, we find all its prime factors, and put them in a form of a product.

Thus, the polynomial $\boldsymbol{P = 2a + 4b + 16c}$ gets (fully) factorized to $\boldsymbol{2(a + 2b + 8c)}$, which is in a form of a product.

**In short:**

$\boldsymbol{2a + 4b + 16c = 2(a + 2b + 8c)}$.

## Suggestions or Solutions
### To the Problem in the Example 1

We have $\mathbf{2}\boldsymbol{ab} + \mathbf{6}\boldsymbol{bc}$.

Factorizing it, we get $\mathbf{2}\boldsymbol{ab} + \mathbf{6}\boldsymbol{bc} = \mathbf{2}\boldsymbol{b}(\boldsymbol{a} + \mathbf{3}\boldsymbol{c})$.

*If not quite sure of how we can get it, follow the steps below.*

To begin with, what are we looking for when doing factorizations?

We want to find divisors, which are factors.

What then can be a divisor, that is, a factor when we factorize a polynomial?

Of course, we want to find a divisor that divides the polynomial, and such a divisor is a factor of the polynomial.  What then can be such a divisor?

It can be an integer or an expression as a constant, monomial, or a polynomial.

And normally, we begin with a divisor that can divide every term the polynomial has. And such a divisor is called a common divisor.

So a common divisor can divide every term in the polynomial, and thus, divides the polynomial, and is a factor of the polynomial.  What then can be such a common divisor? That is, what divides every term in the polynomial given in this problem?

We can see 2 does, because it is a divisor common to all the terms.  Is that it though?

So does $\boldsymbol{b}$. Thus, 2 and $\boldsymbol{b}$ both can divide every term in the polynomial, so $\mathbf{2}\boldsymbol{b}$ divides the polynomial, too. That is to say that $\mathbf{2}\boldsymbol{b}$ is a divisor common to all the terms in the polynomial, and thus, is a factor of the polynomial.

So we want to divide by $\mathbf{2}\boldsymbol{b}$ every term in the polynomial, put every quotient in a pair of parentheses, and then, take a product of the divisor and the sum of all the quotients.

Suppose now, $\boldsymbol{P} = 2\boldsymbol{ab} + 6\boldsymbol{bc}$.

Then, we get $\boldsymbol{P} = 2\boldsymbol{ab} + 6\boldsymbol{bc} = 2\cdot\boldsymbol{ab} + 2\cdot 3\boldsymbol{bc} = 2\boldsymbol{b}\cdot\boldsymbol{a} + 2\boldsymbol{b}\cdot 3\boldsymbol{c} = 2\boldsymbol{b}(\boldsymbol{a} + 3\boldsymbol{c})$.

So we can see that the polynomial $\boldsymbol{a} + 3\boldsymbol{c}$ is a factor, too, of the polynomial $\boldsymbol{P}$, of course.

What factor then is $\boldsymbol{a} + 3\boldsymbol{c}$? In other words, is the factor prime?

There is no divisor common to all terms in the polynomial $\boldsymbol{a} + 3\boldsymbol{c}$.
So the polynomial $\boldsymbol{a} + 3\boldsymbol{c}$ is a prime factor of $\boldsymbol{P}$. And so are $\mathbf{2}$ and $\boldsymbol{b}$.

So all the prime factors found are an integer $\mathbf{2}$, a monomial $\boldsymbol{b}$, and a polynomial $\boldsymbol{a} + 3\boldsymbol{c}$.

And factorizing a polynomial, we put all its prime factors in a form of a product.
So the factorization of the polynomial $\boldsymbol{P} = 2\boldsymbol{ab} + 6\boldsymbol{bc}$ is $2\boldsymbol{b}(\boldsymbol{a} + 3\boldsymbol{c})$.

Then, we can say that the polynomial $\boldsymbol{P}$ gets fully factorized to $2\boldsymbol{b}(\boldsymbol{a} + 3\boldsymbol{c})$.

**In short:**

$2\boldsymbol{ab} + 6\boldsymbol{bc} = 2\boldsymbol{b}(\boldsymbol{a} + 3\boldsymbol{c})$.

By the way, we don’t normally factorize a monomial.
We can factorize it though if it has an integer factorable. And if it does, and we are asked to factorize it, we need to factorize it.
So for instance, if we really need to factorize $8\boldsymbol{b}$, and factorizing it, we get $2^3\boldsymbol{b}$.

## Suggestions or Solutions
### To the Problem in the Example 2

We have $\boldsymbol{2ac + 6bc + ad + 3bd}$. And factorizing it, we get

$$\boldsymbol{2ac + 6bc + ad + 3bd = 2c(a + 3b) + d(a + 3b) = (a + 3b)(2c + d)}.$$

*If not quite sure of the idea behind the processes above, follow the steps below.*

To begin with, it's a good idea to name a polynomial.

So let's suppose first, $\boldsymbol{P = 2ac + 6bc + ad + 3bd}$.

Unlike the previous examples, we do not see this time, a divisor common to all the terms in the polynomial $\boldsymbol{P}$.
So can we say that the polynomial $\boldsymbol{P}$ itself is prime, and thus, is not factorable?

Though a polynomial doesn't seem to have a divisor common to all its terms, it can still be factorable.

A polynomial can be a sum of smaller polynomials, and each of those smaller polynomials can have a divisor common to all the terms in itself.

That is to say that there can be divisors, each of which can divide each of the smaller polynomials, the sum of which is of course, the polynomial that is to be factorized.

In addition, after all such divisions, all the quotients can be the same.
What then can the same quotient do?

The same quotient can divide the polynomial, and thus, is a factor of the polynomial.

That is to say that the polynomial is the product of the same quotient and the sum of the divisors stated above. What then do we call the sum?

The sum is called a factor of the polynomial, too, because it can divide the polynomial, too. So the polynomial can be put in a form of a product of two, one is the same quotient, and the other is the sum.

So let's now, take a closer look at the polynomial $\boldsymbol{P}$, and see if it can be portioned so that we can extract factors the way described above.

Now, examining all the terms in $\boldsymbol{P}$, we can see that $\mathbf{2}\boldsymbol{c}$ is a divisor common to two terms, $\mathbf{2}\boldsymbol{ac}$ and $\mathbf{6}\boldsymbol{bc}$, and that $\boldsymbol{d}$ is a divisor common to the other two terms, $\boldsymbol{ad}$ and $\mathbf{3}\boldsymbol{bd}$.

So $\mathbf{2}\boldsymbol{c}$ is a divisor of a polynomial $\mathbf{2}\boldsymbol{ac} + \mathbf{6}\boldsymbol{bc}$, and $\boldsymbol{d}$ divides a polynomial $\boldsymbol{ad} + \mathbf{3}\boldsymbol{bd}$.

Thus, we get

$$\boldsymbol{P} = 2ac + 6bc + ad + 3bd = (2c \cdot a + 2c \cdot 3b) + (a \cdot d + 3b \cdot d) = 2c(a + 3b) + d(a + 3b).$$

And thus, after all the divisions, we get the same quotient, which is $\boldsymbol{a} + \mathbf{3}\boldsymbol{b}$, which therefore, is a divisor common to both of $\mathbf{2}\boldsymbol{c}(\boldsymbol{a} + \mathbf{3}\boldsymbol{b})$ and $\boldsymbol{d}(\boldsymbol{a} + \mathbf{3}\boldsymbol{b})$.

So $\boldsymbol{a} + \mathbf{3}\boldsymbol{b}$ can divide $\boldsymbol{P}$, and thus, is a factor of $\boldsymbol{P}$.
Thus, dividing $\boldsymbol{P}$ by $\boldsymbol{a} + \mathbf{3}\boldsymbol{b}$, what do we get?

We get $\mathbf{2}\boldsymbol{c} + \boldsymbol{d}$, which is the sum of the two divisors $\mathbf{2}\boldsymbol{c}$ and $\boldsymbol{d}$.

In other words, $\boldsymbol{P} = \underline{2c}(a + 3b) + \underline{d}(a + 3b) = (a + 3b)(\underline{2c} + \underline{d}) \Rightarrow \boldsymbol{P} = (a + 3b)(2c + d)$.

So we can see that $\mathbf{2}\boldsymbol{c} + \boldsymbol{d}$, too, is a divisor of $\boldsymbol{P}$, and is a factor of $\boldsymbol{P}$.

Now, ($\boldsymbol{a} + \mathbf{3}\boldsymbol{b}$) and ($\mathbf{2}\boldsymbol{c} + \boldsymbol{d}$) both are prime, and are all the factors of $\boldsymbol{P}$, so they are all the prime factors of $\boldsymbol{P}$, and therefore, the factorization of $\boldsymbol{P}$ is now complete.

**In short:**

$$2ac + 6bc + ad + 3bd = 2c(a + 3b) + d(a + 3b) = (a + 3b)(2c + d).$$

## Suggestions or Solutions
### To the Problem in the Example 3

We have $3k^2(b + 2b^2c) + 9bk + 27b^2k^4$.

Factorizing it, we get

$3k^2(b + 2b^2c) + 9bk + 27b^2k^4$
$= k\{3k(b + 2b^2c) + 9b + 27b^2k^3\}$
$= k\{3kb(1 + 2bc) + 9b(1 + 3bk^3)\}$
$= 3kb\{k(1 + 2bc) + 3(1 + 3bk^3)\}$
$= 3bk(k + 2bck + 3 + 9bk^3)$.

*If not quite sure of the idea behind the processes above, follow the steps below.*

Suppose $P = 3k^2(b + 2b^2c) + 9bk + 27b^2k^4$.

To begin with, we can see $k$ is a divisor common to all the terms, so $k$ is a factor of $P$.

Thus, we can take out $k$ from every term in $P$, since $k$ is common to all the terms in $P$.

Then, we get $P = k\{3k(b + 2b^2c) + 9b + 27b^2k^3\}$, for now.

Next, we can notice that $b$ is a divisor common to all the terms, too, so $b$ is a factor of $P$, also. How?

Suppose $Q$ is the polynomial inside the curly brackets above.

Then, cutting $Q$ into two parts, we can set $Q = 3k(b + 2b^2c) + (9b + 27b^2k^3)$.

So we get two parts, which are $3k(b + 2b^2c)$ and $9b + 27b^2k^3$.

Then, we can see that $\boldsymbol{b}$ is a divisor common to $\boldsymbol{b + 2b^2c}$, and also, is a divisor common to $\mathbf{9}\boldsymbol{b} + \mathbf{27}\boldsymbol{b^2k^3}$, too.

Thus, we get $\boldsymbol{b + 2b^2c = b(1 + 2bc)}$, and $\boldsymbol{9b + 27b^2k^3 = b(9 + 27bk^3)}$.

So we get $\boldsymbol{Q = 3k(b + 2b^2c) + (9b + 27b^2k^3) = 3kb(1 + 2bc) + b(9 + 27bk^3)}$.

Thus, we can see that $\boldsymbol{b}$ is a divisor common to $\boldsymbol{3kb(1 + 2bc)}$ and $\boldsymbol{b(9 + 27bk^3)}$.

So we get $\boldsymbol{Q = 3kb(1 + 2bc) + b(9 + 27bk^3) = b\{3k(1 + 2bc) + (9 + 27bk^3)\}}$.

Thus, we get $\boldsymbol{Q = 3k(b + 2b^2c) + 9b + 27b^2k^3 = b\{3k(1 + 2bc) + (9 + 27bk^3)\}}$.

Now, we have $\boldsymbol{P = kQ}$.

So we get $\boldsymbol{P = k\{3k(b + 2b^2c) + 9b + 27b^2k^3\} = kb\{3k(1 + 2bc) + (9 + 27bk^3)\}}$.

Therefore, $\boldsymbol{b}$ is a factor of $\boldsymbol{P}$, too.    Then, are we there now?

Not quite. We have another factor of $\boldsymbol{P}$, which is 3.    How?

We can see that $\boldsymbol{(9 + 27bk^3)}$ has a divisor, which is $\mathbf{9}$.

So we get $\boldsymbol{P = kb\{3k(1 + 2bc) + (9 + 27bk^3)\} = kb\{3k(1 + 2bc) + 9(1 + 3bk^3)\}}$.

And in turn, we can see 3 is a common divisor in what's inside the curly brackets above.

What's inside the brackets can be taken as a polynomial with two terms, and 3 is common to the two.    What then are the two?

One is $\boldsymbol{3k(1 + 2bc)}$, and the other is $\boldsymbol{9(1 + 3bk^3)}$.

So we can take it out of each of the two terms, and then, put it outside the brackets.

Then, we get

$\boldsymbol{P = kb\{3k(1 + 2bc) + 9(1 + 3bk^3)\} = kb[3\{k(1 + 2bc) + 3(1 + 3bk^3)\}]}$

$\boldsymbol{= 3kb\{k(1 + 2bc) + 3(1 + 3bk^3)\}}$. Are we there then?

Almost.
We want to expand the two terms inside the brackets, and then, check to see if we can proceed further with the factorization. So expanding the two first, we get

$\boldsymbol{P = 3kb\{k(1 + 2bc) + 3(1 + 3bk^3)\} = 3bk(k + 2bck + 3 + 9bk^3)}$.

And next, we want to check to see if what's inside the parentheses can be factorized. What's inside the parentheses is a polynomial, and we can put it the way below.

$\boldsymbol{k + 2bck + 3 + 9bk^3 = k(1 + 2bc) + 3(1 + 3bk^3)}$, which has however, no divisor common to $\boldsymbol{k(1 + 2bc)}$ and $\boldsymbol{3(1 + 3bk^3)}$.

We can put it the way below, too.

$\boldsymbol{k + 2bck + 3 + 9bk^3 = k + 2bck + 9bk^3 + 3 = k(1 + 2bc + 9bk^2) + 3}$, which however, has no divisor common to $\boldsymbol{k(1 + 2bc + 9bk^2)}$ and $\boldsymbol{3}$.

We can put it this way, too: $\boldsymbol{k(1 + 2bc + 9bk^2) + 3 = k\{1 + b(2c + 9k^2)\} + 3}$, which however, does not give us any divisor.

We can put it the way below, too.

$\boldsymbol{k + 2bck + 3 + 9bk^3 = k + 3 + 2bck + 9bk^3 = k + 3 + bk(2c + 9k^2)}$, which however, has no divisor common to $\boldsymbol{k}$, $\boldsymbol{3}$, and $\boldsymbol{bk(2c + 9k^2)}$.

So we can conclude that the polynomial $\boldsymbol{k + 2bck + 3 + 9bk^3}$ has no divisor other than 1 and itself, and therefore, is prime.

Thus, the factorization can now be completed as follows.

$\boldsymbol{P = 3bk(k + 2bck + 3 + 9bk^3)}$.

And all the prime factors of $\boldsymbol{P}$ are 3, $\boldsymbol{b}$, $\boldsymbol{k}$, and $\boldsymbol{k + 2bck + 3 + 9bk^3}$.

**In short:**

$\boldsymbol{3k^2(b + 2b^2c) + 9bk + 27b^2k^4}$

$\boldsymbol{= k\{3k(b + 2b^2c) + 9b + 27b^2k^3\}}$

$\boldsymbol{= k\{3kb(1 + 2bc) + 9b(1 + 3bk^3)\}}$

$\boldsymbol{= 3kb\{k(1 + 2bc) + 3(1 + 3bk^3)\}}$

$\boldsymbol{= 3bk(k + 2bck + 3 + 9bk^3)}$.

And of course, there can be other ways we can get the factorization, too.