Examples 3 in Polynomial Factorizations

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Examples 3

Factorize the polynomials below.

0. $\boldsymbol{x^2 + (a + b)x + ab}$

1. $\boldsymbol{x^2 - (a + b)x + ab}$

2. $\boldsymbol{acx^2 + (ad + bc)\mathrm{x} + bd}$

3. $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{x^2 + (a + b)x + ab}$.

Factorizing it, we get

$\boldsymbol{x^2 + (a + b)x + ab = (x^2 + ax) + (bx + ab) = x(x + a) + b(x + a) = (x + a)(x + b)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = x^2 + (a + b)x + ab}$, we can say that the polynomial $\boldsymbol{P}$ is similar to polynomials as $\boldsymbol{x^2 + 2xy + y^2}$, which is factorized to $\boldsymbol{(x + y)^2}$.

And we can say that it is similar to $\boldsymbol{x^2 - 2xy + y^2}$, too, which is factorized to $\boldsymbol{(x - y)^2}$. How are they similar, though?

Let's put $\boldsymbol{y}$ into each of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Then, we get $\boldsymbol{x^2 + (a + b)x + ab = x^2 + (y + y)x + yy = x^2 + 2xy + y^2}$.

And we know $\boldsymbol{x^2 + 2xy + y^2}$ is factorized to $\boldsymbol{(x + y)(x + y)}$, which is $\boldsymbol{(x + y)^2}$.

Let's this time, put each of $\boldsymbol{a}$ and $\boldsymbol{b}$ into $\boldsymbol{y}$ in $\boldsymbol{(x + y)(x + y)}$.
Then, we get $\boldsymbol{(x + a)(x + b)}$.

Thus, we can expect that the factorization of $\boldsymbol{P}$ is similar to that of $\boldsymbol{x^2 + 2xy + y^2}$.   So?

So we may want to try doing the factorization of $\boldsymbol{P}$ the way we did to $\boldsymbol{x^2 + 2xy + y^2}$.

To begin with, we want to break the term $(a + b)x$ into two parts.

Then, we can get $P = x^2 + (a + b)x + ab = x^2 + ax + bx + ab = (x^2 + ax) + (bx + ab)$.

What then can we get?

Then, x is a divisor common in the first part, and b is a divisor common in the second.

So we get $P = x(x + a) + b(x + a)$, where $x + a$ is common to $x(x + a)$ and $b(x + a)$.

Thus, we get $P = (x + a)(x + b)$.

And we know $(x + a)$ is prime, and so is $(x + b)$.

So P is (fully) factorized to $(x + a)(x + b)$.

In short:

$$x^2 + (a + b)x + ab = (x^2 + ax) + (bx + ab) = x(x + a) + b(x + a) = (x + a)(x + b).$$

Suggestions or Solutions
To the Problem in the Example 1

We have $x^2 - (a + b)x + ab$.

Factorizing it, we get

$x^2 - (a + b)x + ab = (x^2 - ax) + (-bx + ab) = x(x - a) - b(x - a) = (x - a)(x - b)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $P = x^2 - (a + b)x + ab$, we can say that the polynomial P is quite similar to the one in the problem 4 above.

And in fact, it is no other than $x^2 + (a + b)x + ab$. Why?

In $x^2 + (a + b)x + ab$, replacing x with $-x$, we get $x^2 - (a + b)x + ab$.

And we know $x^2 + (a + b)x + ab = (x + a)(x + b)$.

So replacing x with $-x$, we get $(-x + a)(-x + b) = -(x - a)(-1)(x - b) = (x - a)(x - b)$.

And we can get the same the way below, too.

In $x^2 + (a + b)x + ab$, replacing a with $-a$, and b with $-b$, we get $x^2 - (a + b)x + ab$.

And also, in $(x + a)(x + b)$, replacing a with $-a$, and b with $-b$, we get $(x - a)(x - b)$.

Thus, recognizing patterns quickly, we can get to the solutions quickly, too.
How can we then recognize patterns fast?

Understanding the principles, and doing practices, we can get such a caliber in pattern recognition.

Now, the polynomial $\boldsymbol{P}$ is similar to a polynomial $\boldsymbol{x^2 + (a + b)x + ab = (x + a)(x + b)}$.

So let's factorize $\boldsymbol{P}$ the way we did to the polynomial $\boldsymbol{x^2 + (a + b)x + ab}$.

To begin with, breaking $\boldsymbol{P}$ into two parts, we can get

$\boldsymbol{P = x^2 - (a + b)x + ab = (x^2 - ax) + (-bx + ab)}$, where we can see $\boldsymbol{x}$ is common in the first part, and $\boldsymbol{-b}$ is common in the second part.

So we can get $\boldsymbol{P = (x^2 - ax) + (-bx + ab) = x(x - a) + \{-bx + (-a)(-b)\}}$

$\boldsymbol{= x(x - a) + (-b)\{x + (-a)\} = x(x - a) - b(x - a)}$, where $\boldsymbol{x - a}$ is common.

Thus, we get $\boldsymbol{P = x(x - a) - b(x - a) = (x - a)(x - b)}$.

In short:

$\boldsymbol{x^2 - (a + b)x + ab = (x^2 - ax) + (-bx + ab) = x(x - a) - b(x - a) = (x - a)(x - b)}$.

So for instance, we can put some polynomials the way as follows.

$\boldsymbol{x^2 - 5x + 6 = (x^2 - 2x) + (-3x + 6) = x(x - 2) - 3(x - 2) = (x - 2)(x - 3)}$.

More specifically,

$\boldsymbol{x^2 - (2 + 3)x + 2\cdot 3 = (x^2 - 2x) + (-3x + 2\cdot 3) = x(x - 2) - 3(x - 2) = (x - 2)(x - 3)}$.

So in short, $\boldsymbol{x^2 - 5x + 6 = x^2 - (2 + 3)x + 2\cdot 3 = (x - 2)(x - 3)}$.

And we have this, too:

$\boldsymbol{x^2 + (a + b)x + ab = (x^2 + ax) + (bx + ab) = x(x + a) + b(x + a) = (x + a)(x + b)}$.

So for instance, $x^2 + 5x + 6 = (x^2 + 2x) + (3x + 6) = x(x + 2) + 3(x + 2) = (x + 2)(x + 3)$.

And more specifically,

$x^2 + (2 + 3)x + 2{\cdot}3 = (x^2 + 2x) + (3x + 2{\cdot}3) = x(x + 2) + 3(x + 2) = (x + 2)(x + 3)$.

So in short, $x^2 + 5x + 6 = x^2 + (2 + 3)x + 2{\cdot}3 = (x + 2)(x + 3)$.

What then about this: $x^2 - x - 6$?

We can put it the way below.

$x^2 - x - 6 = (x^2 + 2x) + (-3x - 6) = x(x + 2) - 3(x + 2) = (x + 2)(x - 3)$.

And more specifically,

$x^2 + (2 - 3)x + 2{\cdot}(-3) = (x^2 + 2x) + \{-3x + 2{\cdot}(-3)\} = x(x + 2) + (-3)(x + 2) = (x + 2)(x - 3)$.

So in short, $x^2 - x - 6 = x^2 + (2 - 3)x + 2{\cdot}(-3) = (x + 2)(x - 3)$.

And we will get to the details on the processes above in one of the other sets of examples.

Suggestions or Solutions
To the Problem in the Example 2

We have $acx^2 + (ad + bc)x + bd$.

Factorizing it, we get

$acx^2 + (ad + bc)x + bd = (acx^2 + adx) + (bcx + bd) = ax(cx + d) + b(cx + d)$
$= (cx + d)(ax + b)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $P = acx^2 + (ad + bc)x + bd$, we can say that the polynomial P is just a bit different from the one in the problem 5 above, that is, P is quite similar to the one.

So after all, P is similar to $x^2 + 2xy + y^2$. Why?

First, we have $x^2 + 2xy + y^2 = (x + y)(x + y)$, which is $(x + y)^2$.

And in $(x + y)(x + y)$, putting b and d each into each y, and also, putting ax and cx each into each x, we can get $(ax + b)(cx + d)$.

And next, expanding the above, we get $acx^2 + (ad + bc)x + bd$.

So the polynomial P is similar to the polynomial $x^2 + 2xy + y^2 = (x + y)^2$ in structure.

So let's factorize P the way we did to the polynomial $x^2 + 2xy + y^2$.

To begin with, breaking the polynomial P into two parts, we can get

$P = acx^2 + (ad + bc)x + bd = (acx^2 + adx) + (bcx + bd)$, where we can see ax is common in the first part, and b is common in the second part.

So we get $\boldsymbol{P = (acx^2 + adx) + (bcx + bd) = ax(cx + d) + b(cx + d)}$, where we can see that $\boldsymbol{cx + d}$ is common to $\boldsymbol{ax(cx + d)}$ and $\boldsymbol{b(cx + d)}$.

Thus, we get $\boldsymbol{P = ax(cx + d) + b(cx + d) = (cx + d)(ax + b)}$.

In short:

$$acx^2 + (ad + bc)x + bd = (acx^2 + adx) + (bcx + bd) = ax(cx + d) + b(cx + d)$$

$$= (cx + d)(ax + b).$$

So for instance, we can put some polynomials the way below.

$$6x^2 + 31x + 35 = (6x^2 + 10x) + (21x + 35) = 2x(3x + 5) + 7(3x + 5)$$

$$= (3x + 5)(2x + 7).$$

A bit more specifically,

$$6x^2 + (10 + 21)x + 35 = (6x^2 + 10x) + (21x + 35) = 2x(3x + 5) + 7(3x + 5)$$

$$= (3x + 5)(2x + 7).$$

And more specifically,

$$2{\cdot}3x^2 + (2{\cdot}5 + 7{\cdot}3)x + 7{\cdot}5 = (2{\cdot}3x^2 + 2{\cdot}5x) + (7{\cdot}3x + 7{\cdot}5) = 2x(3x + 5) + 7(3x + 5)$$

$$= (3x + 5)(2x + 7).$$

So in short, $\boldsymbol{6x^2 + 31x + 35 = 6x^2 + (10 + 21)x + 35 = 2{\cdot}3x^2 + (10 + 21)x + 7{\cdot}5}$

$$= (2{\cdot}3x^2 + 2{\cdot}5x) + (7{\cdot}3x + 7{\cdot}5) = 2x(3x + 5) + 7(3x + 5) = (3x + 5)(2x + 7).$$

What then about this: $\mathbf{6x^2 + 11x - 35}$?

We can put it this way:

$$6x^2 + 11x - 35 = 6x^2 + (21 - 10)x - 35 = (6x^2 - 10x) + (21x - 35)$$

$$= 2x(3x - 5) + 7(3x - 5) = (3x - 5)(2x + 7).$$

And more specifically,

$$2\cdot 3x^2 + \{7\cdot 3 + 2\cdot(-5)\}x + 7\cdot(-5) = \{2\cdot 3x^2 + 2\cdot(-5)x\} + \{7\cdot 3x + 7\cdot(-5)\}$$

$$= 2x(3x - 5) + 7(3x - 5) = (3x - 5)(2x + 7).$$

So in short, $\mathbf{6x^2 + 11x - 35 = 2\cdot 3x^2 + (21 - 10)x + 7\cdot(-5) = 2\cdot 3x^2 + (-10 + 21)x + 7\cdot(-5)}$

$$= \{2\cdot 3x^2 + 2\cdot(-5)x\} + \{7\cdot 3x + 7\cdot(-5)\} = 2x(3x - 5) + 7(3x - 5) = (3x - 5)(2x + 7).$$

And we will get to the details on the processes above in one of the other sets of examples.

Suggestions or Solutions
To the Problem in the Example 3

We have $\boldsymbol{x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc}$.

Factorizing it, we can get it the way below.

$\boldsymbol{x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc}$

$\boldsymbol{= x\{x^2 + (a + b + c)x + (ab + bc + ca)\} + abc}$

$\boldsymbol{= x\{x^2 + (a + b)x + cx + ab + (bc + ca)\} + abc}$

$\boldsymbol{= x\{x^2 + (a + b)x + ab + cx + (bc + ca)\} + abc}$

$\boldsymbol{= x\{x^2 + (a + b)x + ab\} + cx^2 + (bc + ca)x + abc}$.

Meanwhile,
$\boldsymbol{x\{x^2 + (a + b)x + ab\} = x(x + a)(x + b)}$.
$\boldsymbol{cx^2 + (bc + ca)x + abc = c\{x^2 + (a + b)x + ab\} = c(x + a)(x + b)}$.

So $\boldsymbol{x\{x^2 + (a + b)x + ab\} + cx^2 + (bc + ca)x + abc = x(x + a)(x + b) + c(x + a)(x + b)}$

$\boldsymbol{= (x + a)(x + b)(x + c)}$.

Therefore, $\boldsymbol{x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc = (x + a)(x + b)(x + c)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P = x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc}$.
We can expect that the polynomial $\boldsymbol{P}$ is similar to $\boldsymbol{x^2 + (a + b)x + ab}$, which is factorized to $\boldsymbol{(x + a)(x + b)}$.

That is, we can make a logical guess that $\boldsymbol{(x + a)(x + b)(x + c)}$ will do.
Expanding (that is, simplifying) it, we will see that it's the one $\boldsymbol{P}$ gets factorized to.

Now, the polynomial $\boldsymbol{P}$ is similar to $\boldsymbol{x^2 + (a + b)x + ab}$, which is of degree 2.

So we may want to convert $\boldsymbol{P}$ to another polynomial containing a polynomial of degree 2. Then, we can get

$\boldsymbol{x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc = x\{x^2 + (a + b + c)x + (ab + bc + ca)\} + abc}$.

So we now have $\boldsymbol{P = x\{x^2 + (a + b + c)x + (ab + bc + ca)\} + abc}$.

Next, we can extract the polynomial $\boldsymbol{x^2 + (a + b)x + ab}$ from $\boldsymbol{P}$ above. How?

We can put $\boldsymbol{P}$ this way $\boldsymbol{P = x\{x^2 + (a + b)x + cx + ab + (bc + ca)\} + abc}$.

So we can get $\boldsymbol{P} = x\{\boldsymbol{x^2 + (a + b)x + ab} + cx + (bc + ca)\} + abc$.

And we have $\boldsymbol{x^2 + (a + b)x + ab = (x + a)(x + b)}$. So we get

$\boldsymbol{P = x\{(x + a)(x + b) + cx + (bc + ca)\} + abc = x(x + a)(x + b) + cx^2 + (bc + ca)x + abc}$.

What then?

We can notice that $\boldsymbol{c}$ is a divisor common to all the terms in $\boldsymbol{cx^2 + (bc + ca)x + abc}$, because $\boldsymbol{(bc + ca)x = c(b + a)x}$.

So taking care of $\boldsymbol{cx^2 + (bc + ca)x + abc}$ for now, we can see this:

$\boldsymbol{cx^2 + (bc + ca)x + abc = c\{x^2 + (a + b)x + ab\}}$, which looks quite familiar, and is equal to $\boldsymbol{c(x + a)(x + b)}$.

Thus, we get $\boldsymbol{P = x(x + a)(x + b) + c(x + a)(x + b)}$, where $\boldsymbol{(x + a)(x + b)}$ is a common divisor.

Therefore, we get $\boldsymbol{P = (x + a)(x + b)(x + c)}$.

In short:

$x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc$

$= x\{x^2 + (a + b + c)x + (ab + bc + ca)\} + abc$

$= x\{x^2 + (a + b)x + cx + ab + (bc + ca)\} + abc$

$= x\{x^2 + (a + b)x + ab + cx + (bc + ca)\} + abc$

$= x\{x^2 + (a + b)x + ab\} + cx^2 + (bc + ca)x + abc.$

Meanwhile,
$x\{x^2 + (a + b)x + ab\} = x(x + a)(x + b).$
$cx^2 + (bc + ca)x + abc = c\{x^2 + (a + b)x + ab\} = c(x + a)(x + b).$

So $x\{x^2 + (a + b)x + ab\} + cx^2 + (bc + ca)x + abc = x(x + a)(x + b) + c(x + a)(x + b)$

$= (x + a)(x + b)(x + c).$

Therefore, $x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc = (x + a)(x + b)(x + c).$