Examples 4 in Polynomial Factorizations

Click this to start.

Examples 4

Factorize the two polynomials below.

0. $\boldsymbol{x^3 - (a + b + c)x^2 + (ab + bc + ca)\mathrm{x} - abc}$

1. $\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca)}$

Suggestions or Solutions
To the Problem in the Example 0

We have $x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc$.

Factorizing it, we can get it the way below.

$x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc$

$= x\{x^2 - (a + b + c)x + (ab + bc + ca)\} - abc$

$= x\{x^2 - (a + b)x - cx + ab + (bc + ca)\} - abc$

$= x\{x^2 - (a + b)x + ab - cx + (bc + ca)\} - abc$

$= x\{x^2 - (a + b)x + ab\} - cx^2 + (bc + ca)x - abc$.

Meanwhile,

$x\{x^2 - (a + b)x + ab\} = x(x - a)(x - b)$.

$-cx^2 + (bc + ca)x - abc = -c\{x^2 - (a + b)x + ab\} = -c(x - a)(x - b)$.

So $x\{x^2 - (a + b)x + ab\} - cx^2 + (bc + ca)x - abc = x(x - a)(x - b) - c(x - a)(x - b)$

$= (x - a)(x - b)(x - c)$.

Therefore, $x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc = (x - a)(x - b)(x - c)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $P = x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc$.

The polynomial P is no different from the one in the previous problem.

In the polynomial $x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc$, simply switching a, b, and c with $-a$, $-b$, and $-c$ respectively, we get P.

Now, we know

$x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc$ gets factorized to $(x + a)(x + b)(x + c)$.

So we can readily see that P gets factorized to $(x – a)(x – b)(x – c)$.

That is, in $x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc = (x + a)(x + b)(x + c)$, changing a, b, and c with $-a$, $-b$, and $–c$ respectively, we get

$x^3 – (a + b + c)x^2 + (ac + bc + ca)x – abc = (x – a)(x – b)(x – c)$.

Let's see though, how P can actually be factorized in a usual manner.

This time, '$x^2 – (a + b)x + ab$' is much closer to P than '$x^2 + (a + b)x + ab$', and gets factorized to $(x – a)(x – b)$, which is the solution to the problem 5 above.

So to begin with, since P is similar to $x^2 – (a + b)x + ab$, we may want to convert P to another polynomial containing a polynomial of degree 2. Then, we get

$x^3 – (a + b + c)x^2 + (ac + bc + ca)x – abc = x\{x^2 – (a + b + c)x + (ab + bc + ca)\} – abc$.

So we get $P = x\{x^2 – (a + b + c)x + (ab + bc + ca)\} – abc$, which can be put this way, too: $x\{x^2 – (a + b)x – cx + ab + (bc + ca)\} – abc$.

Then, we can get $P = x\{x^2 – (a + b)x + ab – cx + (bc + ca)\} – abc$.

And we have $x^2 – (a + b)x + ab = (x – a)(x – b)$. So we get

$P = x\{(x – a)(x – b) – cx + (bc + ca)\} – abc = x(x – a)(x – b) – cx^2 + (bc + ca)x – abc$.

Now, we can notice that $-c$ is a divisor common to every term in $-cx^2 + (bc + ca)x - abc$, because $(bc + ca)x = -c(-b - a)x$.

So we can see this:

$-cx^2 + (bc + ca)x - abc = -c\{x^2 + (-a - b)x + ab\} = -c\{x^2 - (a + b)x + ab\}$, which is equal to $-c(x - a)(x - b)$.

Thus, we get $P = x(x - a)(x - b) - c(x - a)(x - b)$, where $(x - a)(x - b)$ is a common divisor.

Therefore, we get $P = (x - a)(x - b)(x - c)$.

In short:

$x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc$

$= x\{x^2 - (a + b + c)x + (ab + bc + ca)\} - abc$

$= x\{x^2 - (a + b)x - cx + ab + (bc + ca)\} - abc$

$= x\{x^2 - (a + b)x + ab - cx + (bc + ca)\} - abc$

$= x\{x^2 - (a + b)x + ab\} - cx^2 + (bc + ca)x - abc$.

Meanwhile,

$x\{x^2 - (a + b)x + ab\} = x(x - a)(x - b)$.

$-cx^2 + (bc + ca)x - abc = -c\{x^2 - (a + b)x + ab\} = -c(x - a)(x - b)$.

So $x\{x^2 - (a + b)x + ab\} - cx^2 + (bc + ca)x - abc = x(x - a)(x - b) - c(x - a)(x - b)$
$= (x - a)(x - b)(x - c)$.

Therefore, $x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc = (x - a)(x - b)(x - c)$.

Suggestions or Solutions
To the Problem in the Example 1

We have $a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

Factorizing it, we can get it the way below.

$a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)$

$= (a + b)^2 + c^2 + 2c(a + b) = (a + b + c)^2$, because $x^2 + 2xy + y^2 = (x + y)^2$.

$a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)$

$= (a + b)^2 + c^2 + 2bc + 2ca = (a + b)^2 + c^2 + bc + bc + ca + ca$

$= (a + b)^2 + c(c + a + b) + ca + bc = (a + b)^2 + c(a + b + c) + c(a + b)$

$= (a + b)^2 + c(a + b) + c(a + b + c) = (a + b)(a + b + c) + c(a + b + c)$

$= (a + b + c)(a + b + c) = (a + b + c)^2$.

$a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)$

$= (a + b)^2 + c^2 + 2bc + 2ca = (a + b)^2 + c^2 + bc + bc + ca + ca$

$= (a + b)^2 + c(a + b + c) + c(a + b) = (a + b)(a + b + c) + c(a + b + c)$

$= (a + b + c)(a + b + c) = (a + b + c)^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $P = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, and examining the polynomial P, we can notice that the polynomial P has $a^2 + 2ab + b^2$, which is factorized to a complete square, $(a + b)^2$, and also, that the polynomial is symmetric.

Even changing each of a, b, and c with one of the others, we get the polynomial P again. So we can expect that the factorization is much likely to end up with a complete square.

Let's see now, how the factorization can proceed.

We may want to begin with expanding (simplifying) the last term in P so that we can form $a^2 + 2ab + b^2$ in the polynomial P.

Then, we get :

$P = a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)$

$= (a + b)^2 + c^2 + 2(bc + ca)$, where we can see that c is a factor of $2(bc + ca)$.

So taking out c, we get $P = (a + b)^2 + c^2 + 2c(b + a)$.

Now, suppose $d = a + b$.
Then, we get $P = d^2 + 2cd + c^2$, which is factorized to $(d + c)^2$.

So we now have $P = (d + c)^2 = (a + b + c)^2$, because $d = a + b$.

• And we can get the same result the way below, too.

Beginning with the step where $P = (a + b)^2 + c^2 + 2(bc + ca)$, and expanding $2(bc + ca)$, we get
$P = (a + b)^2 + c^2 + 2bc + 2ca = (a + b)^2 + c^2 + bc + bc + ca + ca$.

Rearranging the terms, we can get $P = (a + b)^2 + c^2 + ca + bc + ca + bc$.

Thus, we can say that c is a factor of $c^2 + ca + bc$.   So we can take it out.

Then, we get $P = (a + b)^2 + c(c + a + b) + ca + bc$.

Next, we can say that c is a factor of $ca + bc$, so we can take it out.

Then, we get $\boldsymbol{P = (a + b)^2 + c(a + b + c) + c(a + b) = (a + b)^2 + c(a + b) + c(a + b + c)}$.

Next, we can say that $\boldsymbol{a + b}$ is a factor of $\boldsymbol{(a + b)^2 + c(a + b)}$, so we can take it out.

Then, we get $\boldsymbol{P = (a + b)(a + b + c) + c(a + b + c)}$, where $\boldsymbol{a + b + c}$ is a common divisor.

So $\boldsymbol{a + b + c}$ is a factor, and thus, we get $\boldsymbol{P = (a + b + c)(a + b + c) = (a + b + c)^2}$.

- Let's have a look at another way.

Beginning with the step where $\boldsymbol{P = (a + b)^2 + c^2 + ca + bc + ca + bc}$, we can say that $\boldsymbol{c}$ is a factor of $\boldsymbol{ca + bc}$.

So we get $\boldsymbol{P = (a + b)^2 + c^2 + c(a + b) + ca + bc}$.

Then, we can say that $\boldsymbol{c}$ is a factor of $\boldsymbol{c^2 + c(a + b)}$, and is a factor of $\boldsymbol{ca + bc}$, too.

So we get $\boldsymbol{P = (a + b)^2 + c(a + b + c) + c(a + b)}$.

Next, we can see that $\boldsymbol{a + b}$ is a factor of $\boldsymbol{(a + b)^2 + c(a + b)}$.

So we get $\boldsymbol{P = (a + b)(a + b + c) + c(a + b + c)}$, where $\boldsymbol{a + b + c}$ is a common divisor.

Therefore, $\boldsymbol{a + b + c}$ is a factor, and thus, we get $\boldsymbol{P = (a + b + c)(a + b + c) = (a + b + c)^2}$.

Note:

$\boldsymbol{x^2 \pm 2xy + y^2 = (a \pm b)^2}$ means $\boldsymbol{x^2 + 2xy + y^2 = (a + b)^2}$ or $\boldsymbol{x^2 - 2xy + y^2 = (a - b)^2}$, and they are frequently used in algebraic operations.

In short:

$\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)}$

$\boldsymbol{= (a + b)^2 + c^2 + 2c(a + b) = (a + b + c)^2}$, because $\boldsymbol{x^2 + 2xy + y^2 = (x + y)^2}$.

$\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)}$

$\boldsymbol{= (a + b)^2 + c^2 + 2bc + 2ca = (a + b)^2 + c^2 + bc + bc + ca + ca}$

$\boldsymbol{= (a + b)^2 + c(c + a + b) + ca + bc = (a + b)^2 + c(a + b + c) + c(a + b)}$

$\boldsymbol{= (a + b)^2 + c(a + b) + c(a + b + c) = (a + b)(a + b + c) + c(a + b + c)}$

$\boldsymbol{= (a + b + c)(a + b + c) = (a + b + c)^2}$.

$\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + 2ab + b^2 + c^2 + 2(bc + ca)}$

$\boldsymbol{= (a + b)^2 + c^2 + 2bc + 2ca = (a + b)^2 + c^2 + bc + bc + ca + ca}$

$\boldsymbol{= (a + b)^2 + c(a + b + c) + c(a + b) = (a + b)(a + b + c) + c(a + b + c)}$

$\boldsymbol{= (a + b + c)(a + b + c) = (a + b + c)^2}$.