Examples 5 in Polynomial Factorizations

Click this to start.

Examples 5

Factorize the two polynomials below.

0. $\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca)}$

1. $\boldsymbol{a^2 - b^2}$

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca)}$.

Factorizing it, we can get it the way below.

$\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)}$

$\boldsymbol{= (a - b)^2 + c^2 - 2c(a - b) = (a - b - c)^2}$, because $\boldsymbol{x^2 + 2xy + y^2 = (x + y)^2}$.

$\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)}$

$\boldsymbol{= (a - b)^2 + c^2 + 2bc - 2ca = (a - b)^2 + c^2 + bc + bc - ca - ca}$

$\boldsymbol{= (a - b)^2 - c(-c - b + a) - ca + bc = (a - b)^2 - c(a - b - c) - c(a - b)}$

$\boldsymbol{= (a - b)(a - b - c) - c(a - b - c) = (a - b - c)(a - b - c) = (a - b - c)^2}$.

$\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)}$

$\boldsymbol{= (a - b)^2 + c^2 + 2bc - 2ca = (a - b)^2 + c^2 + bc + bc - ca - ca}$

$\boldsymbol{= (a - b)^2 - c(a - b) + c^2 - ca + bc = (a - b)(a - b - c) - c(-c + a - b)}$

$\boldsymbol{= (a - b - c)(a - b - c) = (a - b - c)^2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = a^2 + b^2 + c^2 - 2(ab - bc + ca)}$, we can say that the polynomial $\boldsymbol{P}$ is similar to $\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca)}$. Such a similarity makes life much easier if we can take advantage of it.

In the polynomial $\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca)}$, replacing $\boldsymbol{b}$ with $\boldsymbol{-b}$, and $\boldsymbol{c}$ with $\boldsymbol{-c}$, we get $\boldsymbol{P}$.

We have $a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2$, which is the solution to problem 9 above.
So in $(a + b + c)^2$, putting $-b$ into b, and $-c$ into c, we get $(a - b - c)^2$, to which P gets factorized.

Let's see now, how the actual factorization can proceed.

In this case, a complete square $a^2 - 2ab + b^2 = (a - b)^2$ is closer to P than $a^2 + 2ab + b^2$.

So we may want to begin with expanding the last term in P so as to extract $a^2 - 2ab + b^2$. Then, we get

$P = a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)$, where $a^2 - 2ab + b^2 = (a - b)^2$, and c is a factor of $2(-bc + ca)$.

So taking out c, we get $P = (a - b)^2 + c^2 - 2c(a - b)$.

Assuming now, $d = a - b$, we can get $P = d^2 - 2cd + c^2$, which is factorized to $(d - c)^2$.

So we now have $P = (d - c)^2 = (a - b - c)^2$, since $d = a - b$.

- And we can get the same result the way below, too.

Beginning with the step where $P = (a - b)^2 + c^2 - 2c(a - b)$, and expanding $-2c(a - b)$, we get
$P = (a - b)^2 + c^2 - 2ca + 2bc = (a - b)^2 + c^2 - ca - ca + bc + bc$.

Next, rearranging the terms, we can get $P = (a - b)^2 + c^2 - ca + bc - ca + bc$.

Then, we can say that $-c$ is a factor of $c^2 - ca + bc$, and is a factor of $-ca + bc$, too.

So we get $P = (a - b)^2 - c(-c + a - b) - c(a - b) = (a - b)^2 - c(a - b) - c(a - b - c)$.

Next, we can say that $a - b$ is a factor of $(a - b)^2 - c(a - b)$, so we can take it out.

Then, we get $P = (a - b)(a - b - c) - c(a - b - c)$, where $a - b - c$ is a common divisor.
So $a - b - c$ is a factor, and thus, we get $P = (a - b - c)(a - b - c) = (a - b - c)^2$.

• Let's now, have a look at another approach.

Beginning with the step $\boldsymbol{P = (a - b)^2 + c^2 - ca + bc - ca + bc}$, we can say that $\boldsymbol{-c}$ is a factor of $\boldsymbol{-ca + bc}$.

So we get $\boldsymbol{P = (a - b)^2 + c^2 - c(a - b) - ca + bc}$.

Next, we can say that $\boldsymbol{-c}$ is a factor of $\boldsymbol{c^2 - c(a - b)}$, and is a factor of $\boldsymbol{-ca + bc}$, too.

So $\boldsymbol{P = (a - b)^2 - c(-c + a - b) - c(a - b)}$, where $\boldsymbol{a - b}$ is common to $\boldsymbol{(a - b)^2}$ and $\boldsymbol{-c(a - b)}$.

So we get $\boldsymbol{P = (a - b)(a - b - c) - c(a - b - c)}$, where $\boldsymbol{a - b - c}$ is a common divisor.

Therefore, $\boldsymbol{a - b - c}$ is a factor, and thus, we get $\boldsymbol{P = (a - b - c)(a - b - c) = (a - b - c)^2}$.

In short:

$\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)}$

$\boldsymbol{= (a - b)^2 + c^2 - 2c(a - b) = (a - b - c)^2}$, because $\boldsymbol{x^2 + 2xy + y^2 = (x + y)^2}$.

$\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)}$

$\boldsymbol{= (a - b)^2 + c^2 + 2bc - 2ca = (a - b)^2 + c^2 + bc + bc - ca - ca}$

$\boldsymbol{= (a - b)^2 - c(-c - b + a) - ca + bc = (a - b)^2 - c(a - b - c) - c(a - b)}$

$\boldsymbol{= (a - b)(a - b - c) - c(a - b - c) = (a - b - c)(a - b - c) = (a - b - c)^2}$.

$\boldsymbol{a^2 + b^2 + c^2 - 2(ab - bc + ca) = a^2 - 2ab + b^2 + c^2 - 2(-bc + ca)}$

$\boldsymbol{= (a - b)^2 + c^2 + 2bc - 2ca = (a - b)^2 + c^2 + bc + bc - ca - ca}$

$\boldsymbol{= (a - b)^2 - c(a - b) + c^2 - ca + bc = (a - b)(a - b - c) - c(-c + a - b)}$

$\boldsymbol{= (a - b - c)(a - b - c) = (a - b - c)^2}$.

Suggestions or Solutions
To the Problem in the Example 1

We have $a^2 - b^2$.

Let's set first, $P = a^2 - b^2$.

This time, it looks quite clear that no divisor is common to all the terms in the polynomial P, and thus, P has no divisor other than 1 and itself, so is P prime?

Of course, not, so P can be factorized.

We can notice that P is similar to a polynomial $x^2 + 2xy + y^2 = (x + y)^2$.

During the factorization process of $x^2 + 2xy + y^2$, we get

$$x^2 + 2xy + y^2 = x^2 + xy + xy + y^2 = x(x + y) + y(x + y) = \ldots$$

Thus, we can expect that a term ab can be used for the factorization of P the way we use a term xy factorizing $x^2 + 2xy + y^2$.

How can we use ab, though? The polynomial P does not have ab, does it?

Of course, it doesn't. We can still use it, though.   How?

We can add it in, and then, subtract it out.

Then, we get $P = a^2 - b^2 = a^2 - b^2 + ab - ab = a^2 + ab - b^2 - ab = (a^2 + ab) - (b^2 + ab)$.

Thus, we can see that a is a factor of $a^2 + ab$, and that b is that of $b^2 + ab$.

So we get $\boldsymbol{P = (a^2 + ab) - (b^2 + ab) = a(a + b) - b(a + b)}$, where $\boldsymbol{a + b}$ is a common divisor.

Therefore, $\boldsymbol{a + b}$ is a factor, and thus, we get $\boldsymbol{P = (a + b)(a - b)}$.

In short:

$$\boldsymbol{a^2 - b^2 = a^2 + ab - b^2 - ab = a(a + b) - b(a + b) = (a + b)(a - b)}.$$

Note:

Quite often, we take for identities all the factorizations in this set of examples.
So they are often called factorization identities.
Besides, they are frequently called factorization formulas, too.
The factorization identity $\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$ is probably one of the most often used when we do algebra.

And putting together all the other factorization identities covered until this example, we have these:

$\boldsymbol{x^2 + 2xy + y^2 = (x + y)^2}$. $\boldsymbol{x^2 - 2xy + y^2 = (x - y)^2}$.

$\boldsymbol{x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3}$. $\boldsymbol{x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3}$.

$\boldsymbol{x^2 + (a + b)x + ab = (x + a)(x + b)}$. $\boldsymbol{x^2 - (a + b)x + ab = (x - a)(x - b)}$.

$\boldsymbol{acx^2 + (ad + bc)\mathrm{x} + bd = (ax + d)(cx + b)}$.

$\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc = (x + a)(x + b)(x + c)}$.

$\boldsymbol{x^3 - (a + b + c)x^2 + (ab + bc + ca)\mathrm{x} - abc = (x - a)(x - b)(x - c)}$.

$\boldsymbol{a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2}$.