Examples 6 in Polynomial Factorizations

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Examples 6

Factorize the two polynomials below.

0. $\boldsymbol{u^2 + 15u + 56}$

1. $\boldsymbol{x^2 - 3x - 54}$

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{u^2 + 15u + 56}$.

In this example, we have several things to discuss besides the solution.
And the things are about growing your algebra, which is the purpose of this book.
That is to say that they are for the enhancement of your algebra skill.

To begin with, we are going to get to the solution first.

Doing a factorization, we find all the prime factors applicable, and put them in a form of a product. How then do we find factors?

Finding factors, we find divisors, because factor are divisors.
So factorizing a polynomial, we want to find the divisors of the polynomial.
How then can we find a divisor of a polynomial?

When finding a divisor of a polynomial, we want to find a divisor that can divide each and every term in the polynomial.
That is, finding a divisor of a polynomial, we find a divisor common to all the terms in the polynomial. So how do we call such a divisor?

Such a divisor is called a common divisor, and does divide the polynomial.
So the common divisor is a factor of the polynomial.

Thus, finding a factor of a polynomial, we want to find a divisor common to all the terms in the polynomial.

In short, finding a factor of a polynomial, we want to find a common divisor.

It's not the case though, every polynomial shows us such a common divisor.
And the polynomial in this example, does not show it to us.
What then do we do?

In such a case, we want to try modifying the polynomial so that we can find a common divisor.

So setting first, $\boldsymbol{P = u^2 + 15u + 56}$, we can modify $\boldsymbol{P}$ the way below.

$\boldsymbol{P = u^2 + 15u + 56 = (u^2 + 8u) + (7u + 56)}$.

Then, we can take $\boldsymbol{P}$ as a polynomial made of two terms.
One is $\boldsymbol{(u^2 + 8u)}$, and the other is $\boldsymbol{(7u + 56)}$.

What then can we notice in the each of the two terms above?

We can notice that $\boldsymbol{u}$ is common in the first term $\boldsymbol{(u^2 + 8u)}$, and that 7 is common in the second term $\boldsymbol{(7u + 56)}$, since $56 = 7 \cdot 8$. So what?

So next, we can get $\boldsymbol{P = (u^2 + 8u) + (7u + 56) = u(u + 8) + 7(u + 8)}$. What then?

Then, we can see that $\boldsymbol{(u + 8)}$ is common to $\boldsymbol{u(u + 8)}$ and $\boldsymbol{7(u + 8)}$, and thus, is the common divisor we want. That is, $\boldsymbol{(u + 8)}$ is a factor of the polynomial $\boldsymbol{P}$.

Thus, we get $\boldsymbol{P = u(u + 8) + 7(u + 8) = (u + 8)(u + 7)}$.

In short, $\boldsymbol{u^2 + 15u + 56 = u^2 + (8 + 7)u + 7 \cdot 8 = u^2 + 7u + 8u + 7 \cdot 8 = u(u + 7) + 8(u + 7)}$

$\boldsymbol{= (u + 7)(u + 8)}$.

How can we quickly see though, if the polynomial $\boldsymbol{P}$ can be modified the way above?

• Fast Factorization

Examining the polynomial $\boldsymbol{P = u^2 + 15u + 56}$, we can notice that the polynomial $\boldsymbol{P}$ is similar to a polynomial as follows.

$\boldsymbol{x^2 + (a + b)x + ab}$, which is factorized to $\boldsymbol{(x + a)(x + b)}$.

Then in $\boldsymbol{x^2 + (a + b)x + ab}$, replacing $\boldsymbol{x}$ with $\boldsymbol{u}$, and assuming $\boldsymbol{a + b = 15}$, and $\boldsymbol{ab = 56}$, we get $\boldsymbol{u^2 + 15u + 56}$, which is the polynomial $\boldsymbol{P}$.

So finding two integers, the sum of which 15, and the product of which is 56, we get $\boldsymbol{a}$ and $\boldsymbol{b}$. Then, we can get the factorization of $\boldsymbol{P}$, because $\boldsymbol{P}$ is factorized to $\boldsymbol{(u + a)(u + b)}$.

How then can we find such two integers, that is, $\boldsymbol{a}$ and $\boldsymbol{b}$?

First, either of the two integers is $\boldsymbol{a}$, and the other is $\boldsymbol{b}$.
And next, by trial-and-error, we can find the two.
We don't just try random integers though.   What then do we try?

We know $\boldsymbol{ab = 56}$. So $\boldsymbol{a}$ and $\boldsymbol{b}$ are divisors of 56.

So we can set $\boldsymbol{ab = 1 \cdot 56 = 2 \cdot 28 = 4 \cdot 14 = 8 \cdot 7}$.
Also, we can set $\boldsymbol{ab = (-1) \cdot (-56) = (-2) \cdot (-28) = (-4) \cdot (-14) = (-8) \cdot (-7)}$.

Thus, $\boldsymbol{(a, b)}$ is one of the eight cases as follows.

(1, 56), **(2, 28)**, **(4, 14)**, **(8, 7)**, **(-1, -56)**, **(-2, -28)**, **(-4, -14)**, and **(-8, -7)**.

And of the eight cases above, the fourth one, that is, **(8, 7)** satisfies $\boldsymbol{a + b = 15}$. Also, it should satisfy $\boldsymbol{ab = 56}$, because none of the other seven cases can satisfy $\boldsymbol{a + b = 15}$.
So we get $\boldsymbol{P = u^2 + 15u + 56 = (u + 8)(u + 7)}$.
And we know $\boldsymbol{u + 7}$ is prime, and so is $\boldsymbol{u + 8}$.
So we can now say that $\boldsymbol{u^2 + 15u + 56}$ is fully factorized to $\boldsymbol{(u + 7)(u + 8)}$.

Thus, together with the fact that $\boldsymbol{a}$ and $\boldsymbol{b}$ are divisors of 56, by trial-and-error, we can find $\boldsymbol{a}$ and $\boldsymbol{b}$ so that the sum of $\boldsymbol{a}$ and $\boldsymbol{b}$ is 15.

What if however, no case satisfies $\boldsymbol{a + b = 15}$?

Then, the polynomial is not factorable within the scope of integers.
If a polynomial is factorable within the scope of integers, we can get its factors where all the coefficients are integers.

For instance, if a polynomial has as factors $\mathbf{2}\boldsymbol{x}+\mathbf{1}$, $\boldsymbol{x}-\mathbf{1}$, and $\mathbf{3}\boldsymbol{x}-\mathbf{2}$, the polynomial is factorable within the scope of integers, since integers are used as all the numeric terms and all the coefficients in all the factors.

If however, a factor of a polynomial is $\sqrt{\mathbf{2}}\boldsymbol{x}-\mathbf{2}$ or $\boldsymbol{x}+\sqrt{\mathbf{2}}$, the polynomial is not factorable within the scope of integers, since integers cannot be used as all the coefficients and all the numeric terms.

What then, about a polynomial that has as factors $\boldsymbol{x}+\frac{2}{3}$, $\frac{3}{2}\boldsymbol{x}+\mathbf{1}$, or $\frac{2}{3}\boldsymbol{x}+\frac{1}{2}$?

We can put the factors the way below.

$\boldsymbol{x}+\frac{2}{3}=\frac{1}{3}(\mathbf{3}\boldsymbol{x}+\mathbf{2})$, $\frac{3}{2}\boldsymbol{x}+\mathbf{1}=\frac{1}{2}(\mathbf{3}\boldsymbol{x}+\mathbf{2})$, and $\frac{2}{3}\boldsymbol{x}+\frac{1}{2}=\frac{1}{6}(\mathbf{4}\boldsymbol{x}+\mathbf{3})$.

Then, all the coefficients and the numeric terms are all integers.
And in high school math, covered are factorizations within the scope of integers.

Now, the purpose of doing polynomial factorizations is to develop and enhance your algebra skill. So you may want to do some more algebra with this factorization.

Finding in fact, $\boldsymbol{a}$ and $\boldsymbol{b}$ algebraically, we want to solve a system of the two equations where $\boldsymbol{a + b = 15}$, and $\boldsymbol{ab = 56}$.

The system is though, no other than factorizing the polynomial $\boldsymbol{P}$. Why?

Solving the system of equations above, we will end up with a quadratic equation, which is no other than setting the polynomial $\boldsymbol{P = 0}$. Why?

First, we have $\boldsymbol{a + b = 15}$, so we get $\boldsymbol{b = 15 - a}$. Thus, we get $\boldsymbol{ab = a(15 - a) = 56}$.
So we get $\boldsymbol{a(15 - a) = 15a - a^2 = 56 \Rightarrow a^2 - 15a + 56 = 0}$.
And we can factorize $\boldsymbol{a^2 - 15a + 56}$ to $\boldsymbol{(a - 7)(a - 8)}$.
That is, we get $\boldsymbol{a^2 - 15a + 56 = (a - 7)(a - 8)}$.
So we get $\boldsymbol{a^2 - 15a + 56 = 0 \Rightarrow (a - 7)(a - 8) = 0 \Rightarrow a = 7}$ or $\boldsymbol{8}$.
And we have $\boldsymbol{b = 15 - a}$. So we get
$\boldsymbol{a = 7 \Rightarrow b = 15 - a = 15 - 7 = 8}$.
$\boldsymbol{a = 8 \Rightarrow b = 15 - a = 15 - 8 = 7}$.

Thus, the solution to the system is $\boldsymbol{a = 7}$ and $\boldsymbol{b = 8}$, or $\boldsymbol{a = 8}$ and $\boldsymbol{b = 7}$.

Next, we get $\boldsymbol{P = 0 \Rightarrow u^2 + 15u + 56 = 0}$.
And we can factorize $\boldsymbol{u^2 + 15u + 56}$ to $\boldsymbol{(u + 7)(u + 8)}$.
That is, we get $\boldsymbol{u^2 + 15u + 56 = (u + 7)(u + 8)}$.
So we get $\boldsymbol{P = 0 \Rightarrow u^2 + 15u + 56 = 0 \Rightarrow (u + 7)(u + 8) = 0 \Rightarrow u = -7}$ or $\boldsymbol{-8}$.
That is, $\boldsymbol{P = 0 \Rightarrow u = -7}$ or $\boldsymbol{-8}$.

Thus, solving the system of equations where $\boldsymbol{a + b = 15}$, and $\boldsymbol{ab = 56}$, we just get back to the very problem in this example, that is, factorizing the polynomial $\boldsymbol{P}$.
That's because the values of $\boldsymbol{-a}$ and $\boldsymbol{-b}$ are the solution to $\boldsymbol{P = 0}$. Why?

The equation $\boldsymbol{P = 0}$ is $\boldsymbol{u^2 + 15u + 56 = 0}$, and can be put this way: $\boldsymbol{u^2 + (a + b)u + ab = 0}$.
And we have $\boldsymbol{u^2 + (a + b)u + ab = (u + a)(u + b)}$.
So the equation $\boldsymbol{u^2 + 15u + 56 = 0}$ is now the same as $\boldsymbol{(u + a)(u + b) = 0}$.
And solving $\boldsymbol{(u + a)(u + b) = 0}$, we get $\boldsymbol{u = -a}$ or $\boldsymbol{-b}$, because $\boldsymbol{u + a = 0}$ or $\boldsymbol{u + b = 0}$.
Thus, the values of $\boldsymbol{-a}$ and $\boldsymbol{-b}$ are the solution to the equation $\boldsymbol{P = 0}$.

Now, we have two typical approaches to a factorization.

One is a logical guessing, which has been covered above, and the other is a straight factorization, which is in this case, using this identity: $(s + t)^2 = s^2 + 2st + t^2$.
How then can we use the identity above?

Referring to the identity, we can put first, $P = u^2 + 15u + 56$ into the form of $s^2 + 2st + t^2$.

And next, we can convert P into a polynomial with a binomial squared as $(u + v)^2$, where v is a number. And a binomial is a polynomial that is made of two terms as $x + 1$, $2x + c$, or $x + 2y$.

And the conversion can proceed the way as follows.

To begin with, we can change P the way below.

$$(1)\ P = u^2 + 15u + 56 = u^2 + 2\cdot\tfrac{15}{2}\cdot u + (\tfrac{15}{2})^2 - (\tfrac{15}{2})^2 + 56.$$

Next, we can change the first three terms the way below.

$$u^2 + 2\cdot\tfrac{15}{2}\cdot u + (\tfrac{15}{2})^2 = (u + \tfrac{15}{2})^2.$$

And next, in the expression **(1)** above, we can change the last two terms the way below.

$$56 - (\tfrac{15}{2})^2 = 56 - \tfrac{1}{4}(150 + 50 + 25) = 56 - \tfrac{225}{4} = \tfrac{56\cdot 4 - 225}{4} = \tfrac{1}{4}(200 + 24 - 225) = -\tfrac{1}{4}.$$

So we get $P = u^2 + 15u + 56 = (u + \tfrac{15}{2})^2 - \tfrac{1}{4}$, which has a binomial squared. So what?

We have this identity, too: $s^2 - t^2 = (s - t)(s + t)$.

So next, using a fact that $\tfrac{1}{4} = (\pm\tfrac{1}{2})^2$, together with the identity above, we get

$$P = (u + \tfrac{15}{2})^2 - \tfrac{1}{4} = (u + \tfrac{15}{2})^2 - (\pm\tfrac{1}{2})^2 = \{(u + \tfrac{15}{2}) + (\pm\tfrac{1}{2})\}\{(u + \tfrac{15}{2}) - (\pm\tfrac{1}{2})\}.$$

We can however, just set $P = \{(u + \tfrac{15}{2}) + \tfrac{1}{2}\}\{(u + \tfrac{15}{2}) - \tfrac{1}{2}\}$. Why?

It’s because

$$P=\{(u+\tfrac{15}{2})+(\pm\tfrac{1}{2})\}\{(u+\tfrac{15}{2})-(\pm\tfrac{1}{2})\}=\{(u+\tfrac{15}{2})\pm\tfrac{1}{2}\}\{(u+\tfrac{15}{2})\mp\tfrac{1}{2})\}$$

$=\{(u+\tfrac{15}{2})+\tfrac{1}{2}\}\{(u+\tfrac{15}{2})-\tfrac{1}{2})\}$ or $\{(u+\tfrac{15}{2})-\tfrac{1}{2}\}\{(u+\tfrac{15}{2})+\tfrac{1}{2})\}$, both of which are the same.

So next, we get $P=\{(u+\tfrac{15}{2})+\tfrac{1}{2}\}\{(u+\tfrac{15}{2})-\tfrac{1}{2}\}=(u+\tfrac{16}{2})(u+\tfrac{14}{2})=(u+8)(u+7)$.

Thus, we get $P = u^2 + 15u + 56 = (u + 7)(u + 8)$.

In short, we get

$$P=u^2+15u+56=u^2+2\cdot\tfrac{15}{2}\cdot u+(\tfrac{15}{2})^2-(\tfrac{15}{2})^2+56=(u+\tfrac{15}{2})^2-\tfrac{1}{4}$$

$$\{(u+\tfrac{15}{2})+\tfrac{1}{2}\}\{(u+\tfrac{15}{2})-\tfrac{1}{2}\}=(u+\tfrac{16}{2})(u+\tfrac{14}{2})=(u+8)(u+7).$$

- So what else can we get doing the factorization the way above?

We can solve for x the quadratic equation $ax^2 + bx + c = 0$ the way below.

First, we get $ax^2+bx+c=a(x^2+\tfrac{b}{a}x)+c=a\{x^2+2\cdot\tfrac{b}{2a}\cdot x+(\tfrac{b}{2a})^2-(\tfrac{b}{2a})^2\}+c$

$$=a\{x^2+2\cdot\tfrac{b}{2a}\cdot x+(\tfrac{b}{2a})^2\}-a(\tfrac{b}{2a})^2+c=a(x+\tfrac{b}{2a})^2-\tfrac{b^2}{4a}+c=0.$$

Next, taking advantage of the identity $p^2 - q^2 = (p - q)(p + q)$, we get

$$a(x+\tfrac{b}{a})^2-\tfrac{b^2}{4a}+c=0\Rightarrow(x+\tfrac{b}{a})^2-\tfrac{b^2}{4a^2}+\tfrac{c}{a}=(x+\tfrac{b}{2a})^2-\tfrac{b^2-4ac}{4a^2}=0$$

$$\Rightarrow\{(x+\tfrac{b}{2a})+\tfrac{\sqrt{b^2-4ac}}{2a}\}\{(x+\tfrac{b}{2a})-\tfrac{\sqrt{b^2-4ac}}{2a}\}=(x+\tfrac{b}{2a}+\tfrac{\sqrt{b^2-4ac}}{2a})(x+\tfrac{b}{2a}-\tfrac{\sqrt{b^2-4ac}}{2a})$$

$$\Rightarrow\{(x+\tfrac{b+\sqrt{b^2-4ac}}{2a})(x+\tfrac{b-\sqrt{b^2-4ac}}{2a})=0.$$

In other words, we get $(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a^2}=(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})=0$.

Therefore, the solution to the equation $ax^2 + bx + c = 0$ is as follows.

$x=-\frac{b+\sqrt{b^2-4ac}}{2a}$ or $-\frac{b-\sqrt{b^2-4ac}}{2a}$, that is, $x=-\frac{b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which is often called the quadratic formula.

So we just now have checked how to get the quadratic formula.

And the quadratic equation $ax^2 + bx + c = 0$ is one of the major equations we have to solve or work with in high school math and college math, too.

- So let's next, take a look at some nature of quadratic equations.

Knowing the nature and going through the processes below, we can get stronger the foundation of algebra.

Now, the equation $ax^2 + bx + c = 0$ is no other than $(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a^2}=0$.

However, we want to note that $ax^2+bx+c\neq(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})$, and that

$ax^2+bx+c=a(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})$. Why?

That's because dividing by a both sides of $ax^2 + bx + c = 0$, we get

$$x^2+\frac{b}{a}x+\frac{c}{a}=0\Rightarrow(x+\frac{b}{a})^2-\frac{b^2}{4a^2}+\frac{c}{a}=(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a^2}=(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})=0.$$

So multiplying $(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})$ by a, we get $ax^2 + bx + c = 0$.

Thus, $ax^2 + bx + c$ gets factorized to $a(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})$.

Meanwhile, we have

$$a(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})=(ax+\frac{b+\sqrt{b^2-4ac}}{2})(x+\frac{b-\sqrt{b^2-4ac}}{2a}).$$

$$a(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})=(x+\frac{b+\sqrt{b^2-4ac}}{2a})(ax+\frac{b-\sqrt{b^2-4ac}}{2}).$$

So ax^2+bx+c can be said to get factorized to either of the two below.

$(ax+\frac{b+\sqrt{b^2-4ac}}{2})(x+\frac{b-\sqrt{b^2-4ac}}{2a})$ and $(x+\frac{b+\sqrt{b^2-4ac}}{2a})(ax+\frac{b-\sqrt{b^2-4ac}}{2})$.

In short, ax^2+bx+c gets factorized to $a(x+\frac{b+\sqrt{b^2-4ac}}{2a})(x+\frac{b-\sqrt{b^2-4ac}}{2a})$.

So factorizing a polynomial ax^2+bx+c, we can use the quadratic formula.

- Let's for instance, factorize $3x^2+25x+28$ using the formula.

We have $a=3$, $b=25$, and $c=28$. So setting $3x^2+25x+28=0$, we get

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-25\pm\sqrt{25^2-4\cdot3\cdot28}}{2\cdot3}=\frac{-25\pm\sqrt{625-336}}{6}=\frac{-25\pm\sqrt{289}}{6}=\frac{-25\pm17}{6}\Rightarrow x=-\frac{4}{3}\text{ or }-7.$$

Thus, the solution to $3x^2+25x+28=0$ is $x=-\frac{4}{3}$ or -7.

So we get $3x^2+25x+28=0\Rightarrow(x+\frac{4}{3})(x+7)=0$.

However, the polynomial $3x^2+25x+28$ does get factorized to $3(x+\frac{4}{3})(x+7)$, and not to $(x+\frac{4}{3})(x+7)$.

That's because dividing $3x^2+25x+28=0$ by 3, we get $x^2+\frac{25}{3}x+\frac{28}{3}=0$, and then, we get $x^2+\frac{25}{3}x+\frac{28}{3}=(x+\frac{4}{3})(x+7)=0$.

Thus, we get $3x^2+25x+28=3(x+\frac{4}{3})(x+7)=(3x+4)(x+7)$.

Suggestions or Solutions
To the Problem 1 in the Example 0

We have $x^2 - 3x - 54$.

Factorizing it, we can get it the way below.

$x^2 - 3x - 54 = x^2 + 6x - 9x - 54 = x(x + 6) - 9(x + 6) = (x + 6)(x - 9)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first $P = x^2 - 3x - 54$, we can say that the polynomial P is quadratic and the coefficients and numeric term are integers.

Also, P is similar to $x^2 - (a + b)x + ab$, which is factorized to $(x - a)(x - b)$.

However, P is similar to $x^2 + (a + b)x + ab$, too, which gets factorized to $(x + a)(x + b)$.

That is because changing a and b with $-a$ and $-b$ respectively in $(x - a)(x - b)$, we get

$(x - a)(x - b) = \{x - (-a)\}\{x - (-b)\} = (x + a)(x + b)$.

So $x^2 - (a + b)x + ab$ is just about the same as $x^2 + (a + b)x + ab$.

Thus, in $x^2 + (a + b)x + ab$, assuming $a + b = -3$, and $ab = -54$, we get $x^2 - 3x - 54$, which is the polynomial P.

So finding two integers, the sum of which -3, and the product of which is -54, we get a and b. Then, we can get the factorization of P, because P gets factorized to $(x + a)(x + b)$, where a and b are integers.

How then can we find such two integers, that is, a and b?

First, either of the two integers is a, and the other is b.
And next, by trial-and-error, we can find the two.

First off, we know $ab = -54$. So a and b are divisors of -54.

Next, by trial and error, we can try finding two integers that can get multiplied to be -54, and also, add up to -3.

If such a pair gets found, either of the two integers is a and the other is b, so P can get factorized to $(x + a)(x + b)$.

That is, assuming that a and b are integers, we want to solve a system where $ab = -54$ and $a + b = -3$.

If any, we can find two integers adding up to -3 and getting multiplied to be -54 using divisors of -54 via trial and error. Then, the two integers found are a and b.

If however, no integers satisfy the system above, that is, a and b cannot be integers, we want to consult the quadratic formula.

By trial and error with divisors of -54, we can get
$6(-9) = -54$, and $6 + (-9) = -3$.

So either of a and b is 6, and the other is -9, and thus, P is factorized to $(x + 6)(x - 9)$.

The actual factorization process can be as follows.

$$x^2 - 3x - 54 = x^2 + (6 - 9)x + 6\cdot(-9) = x^2 + 6x - 9x + 6\cdot(-9) = x(x + 6) - 9(x + 6)$$

$$= (x + 6)(x - 9).$$

In short:

$$x^2 - 3x - 54 = x^2 + 6x - 9x - 54 = x(x + 6) - 9(x + 6) = (x + 6)(x - 9).$$