Examples 8 in Polynomial Factorizations

Click this to start.

Examples 7

Factorize the polynomials below.

0. $\boldsymbol{y^2 + 14y + 49}$

1. $\boldsymbol{t^2 + 7t + \frac{49}{4}}$

2. $\boldsymbol{9t^2 + 48t + 64}$

3. $\boldsymbol{\frac{4}{9}t^2 + \frac{20}{3}t + 25}$

4. $\boldsymbol{49r^2 + \frac{112}{3}r + \frac{64}{9}}$

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{y^2 + 14y + 49}$.

Factorizing it, we get $\boldsymbol{y^2 + 14y + 49 = y + 7y + 7y + 49 = y(y + 7) + 7(y + 7) = (y + 7)^2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Frequently, a quaratic polynomial can be converted to a complete square. So for instance, we can put $\boldsymbol{u^2x^2 + 2uvx + v^2}$ in a complete square this way: $\boldsymbol{u^2x^2 + 2uvx + v^2 = (ux + v)^2}$.

And the simplest complete square is $\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$.

Now, setting first, $\boldsymbol{P = y^2 + 14y + 49}$, we can say that the polynomial $\boldsymbol{P}$ is quadratic, and is quite close to the simplest complete square. So?

So putting $\boldsymbol{P}$ in the same structure as the simplest one has, we can see if it can be put in a complete square.

$\boldsymbol{P = y^2 + 14y + 49 = y^2 + 2 \cdot 7 \cdot y + 7^2 = (y + 7)^2}$.

Thus, the polynomia $\boldsymbol{P}$ is factorized to $\boldsymbol{(y + 7)^2}$.

And the actual factorization processes can be

$\boldsymbol{y^2 + 14y + 49 = y^2 + 7y + 7y + 49 = y(y + 7) + 7(y + 7) = (y + 7)(y + 7) = (y + 7)^2}$.

What if however, a polynomial cannot be put in a complete square?

Even if a quadratic polynomial cannot be put in a complete square, we can still use the idea of the complete square. In such a case, we reconstruct the polynomial so that it can be expressed in terms of a complete square and a nonzero numeric term.

For instance, $\boldsymbol{y^2 + 14y + 50}$ can be put in $\boldsymbol{(y + 7)^2 + 1}$, which is made of a complete square and a nonzero numeric term.

By the way, any real number squard (1 multiplied by the same number twice) cannot be negative. In other words, a real number squared is greater than or equal to 0.

Therefore, we get $\boldsymbol{(y + 7)^2 \geq 0 \Rightarrow y^2 + 14y + 50 = (y + 7)^2 + 1 \geq 1}$.

So we get $\boldsymbol{y^2 + 14y + 50 \geq 1}$.

In short:

$\boldsymbol{y^2 + 14y + 49 = y + 7y + 7y + 49 = y(y + 7) + 7(y + 7) = (y + 7)^2}$.

Suggestions or Solutions
To the Problem in the Example 1

We have $t^2+7t+\frac{49}{4}$.

Factorizing it, we get can get it the way below.

$$t^2+7t+\frac{49}{4}=t^2+\frac{7}{2}\cdot t+\frac{7}{2}\cdot t+(\frac{7}{2})^2=t(t+\frac{7}{2})(\frac{7}{2})(t+\frac{7}{2})=(t+\frac{7}{2})(t+\frac{7}{2})=(t+\frac{7}{2})^2.$$

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $P=t^2+7t+\frac{49}{4}$, we can say that P is quadratic, but could expect that P is not factorable within the scople of integers, becuase the numeric term is not an intger.

It can be the case though, P can be factorable within the scople of integers.

Many quaratic polynomials can be put in complete squares, and we can factorize readily such a polynomial. And we can quickly check to see if a quadratic polynomial can be put in a complete square evaluating the discriminant of such a polynomial.

Suppose for instance, $Q = ax^2 + bx + c$, and D is the discriminant.

Then, $D = b^2 - 4ac$, which is in fact, a part of the quadratic formula as follows.

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. So the formula can be put this way, too: $x=\frac{-b\pm\sqrt{D}}{2a}$ where $D = b^2 - 4ac$.

Thus, we have

$$ax^2+bx+c=a(x-\frac{-b+\sqrt{b^2-4ac}}{2a})(x-\frac{-b-\sqrt{b^2-4ac}}{2a})=a(x-\frac{-b+\sqrt{D}}{2a})(x-\frac{-b-\sqrt{D}}{2a}).$$

(If not sure of how to get it, refer to the example 0 in **Examples 6**.)

Therefore, if the discriminant $D = 0$, we get

$$ax^2+bx+c=a(x-\frac{-b}{2a})(x-\frac{-b}{2a})=a(x+\frac{b}{2a})(x+\frac{b}{2a})=a(x+\frac{b}{2a})^2.$$

So $\boldsymbol{Q}$ can be put in a complete square if the discriminant $\boldsymbol{D}$ is 0.
That is, a quadratic polnomial can be put in a complete square if its discriminant is 0.
What then about $\frac{b}{2a}$, which is not an integer?

We can put it this way, too: $\frac{b}{2a} = \frac{1}{2a} \cdot b$. So we can put $a(x+\frac{b}{2a})^2$ the way below, too.

$$a(x+\tfrac{b}{2a})^2 = a(\tfrac{2a}{2a}x+\tfrac{1}{2a}b)^2 = a\{\tfrac{1}{2a}(2ax+b)\}^2 = a(\tfrac{1}{2a})^2(2ax+b)^2 = \tfrac{1}{4a}(2ax+b)^2.$$

Now, the discriminant of $\boldsymbol{P}$ is $7^2 - 4 \cdot 1 \cdot \frac{49}{4} = 0$. So $\boldsymbol{P}$ can be put in a complete square.

And we can get $ax^2+bx+c = a(x+\frac{b}{2a})^2$ if the discriminant $\boldsymbol{D}$ is 0.

Since $P = t^2+7t+\frac{49}{4}$, we can take 1 as a, can take 7 as b, and can take $\frac{49}{4}$ as c.

So we can get $t^2+7t+\frac{49}{4} = (t+\frac{7}{2})^2$.

In fact, $t^2+7t+\frac{49}{4} = t^2+2\cdot\frac{7}{2}\cdot t+(\frac{7}{2})^2 = (t+\frac{7}{2})^2$.

The factorization process can be as follows.

$$t^2+7t+\tfrac{49}{4} = t^2+\tfrac{7}{2}\cdot t+\tfrac{7}{2}\cdot t+(\tfrac{7}{2})^2 = t(t+\tfrac{7}{2})(\tfrac{7}{2})(t+\tfrac{7}{2}) = (t+\tfrac{7}{2})(t+\tfrac{7}{2}) = (t+\tfrac{7}{2})^2.$$

And if necessary, we can put it the way below, too.

$$(t+\tfrac{7}{2})^2 = (\tfrac{2}{2}t+\tfrac{1}{2}\cdot 7)^2 = \{\tfrac{1}{2}(2t+7)\}^2 = \tfrac{1}{4}(2t+7)^2.$$

In short:

$$t^2+7t+\tfrac{49}{4} = t^2+\tfrac{7}{2}\cdot t+\tfrac{7}{2}\cdot t+(\tfrac{7}{2})^2 = t(t+\tfrac{7}{2})(\tfrac{7}{2})(t+\tfrac{7}{2}) = (t+\tfrac{7}{2})(t+\tfrac{7}{2}) = (t+\tfrac{7}{2})^2.$$

Suggestions or Solutions
To the Problem in the Example 2

We have $\mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{48}\boldsymbol{t} + \mathbf{64}$.

If a quaratic polynomial can be put in a complete square, we can readily factorize it. The discriminant can tell us immediately if it is the case.

Assumig for instance, $\boldsymbol{Q = ax^2 + bx + c}$, and $\boldsymbol{D}$ is the discriminant, we get $\boldsymbol{D = b^2 - 4ac}$. And if $\boldsymbol{D} = \mathbf{0}$, we can say that $\boldsymbol{Q}$ can be put in a complete square.

Now, setting first, $\boldsymbol{P} = \mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{48}\boldsymbol{t} + \mathbf{64}$, we can say that the polynomial $\boldsymbol{P}$ is quadratic, so checking the discriminant of $\boldsymbol{P}$, we can see if $\boldsymbol{P}$ can be put in a complete square.

So finding $\boldsymbol{D}$ for $\boldsymbol{P}$, we get $\mathbf{48^2 - 4 \cdot 9 \cdot 64 = 48^2 - (2 \cdot 3 \cdot 8)^2 = 48^2 - 48^2 = 0}$.

Thus, $\boldsymbol{D} = \mathbf{0}$, and $\boldsymbol{P}$ can be put in a complete square.

And we can put $\boldsymbol{P}$ in a complete square the way below, too.

$\boldsymbol{P} = \mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{48}\boldsymbol{t} + \mathbf{64} = \mathbf{(3}\boldsymbol{t}\mathbf{)^2} + \mathbf{2 \cdot 8 \cdot 3}\boldsymbol{t} + \mathbf{8^2} = \mathbf{(3}\boldsymbol{t} + \mathbf{8)^2}$, to which $\boldsymbol{P}$ gets factorized, of course.

And also, the factorization process can be as follows.

$$\mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{48}\boldsymbol{t} + \mathbf{64} = \mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{24}\boldsymbol{t} + \mathbf{24}\boldsymbol{t} + \mathbf{64} = \mathbf{3}\boldsymbol{t}\mathbf{(3}\boldsymbol{t} + \mathbf{8)} + \mathbf{24}\boldsymbol{t} + \mathbf{64} = \mathbf{3}\boldsymbol{t}\mathbf{(3}\boldsymbol{t} + \mathbf{8)} + \mathbf{8(3}\boldsymbol{t} + \mathbf{8)}$$

$$= \mathbf{(3}\boldsymbol{t} + \mathbf{8)(3}\boldsymbol{t} + \mathbf{8)} = \mathbf{(3}\boldsymbol{t} + \mathbf{8)^2}.$$

In short:

$\mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{48}\boldsymbol{t} + \mathbf{64} = \mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{24}\boldsymbol{t} + \mathbf{24}\boldsymbol{t} + \mathbf{64} = \mathbf{3}\boldsymbol{t}\mathbf{(3}\boldsymbol{t} + \mathbf{8)} + \mathbf{24}\boldsymbol{t} + \mathbf{64} = \mathbf{3}\boldsymbol{t}\mathbf{(3}\boldsymbol{t} + \mathbf{8)} + \mathbf{8(3}\boldsymbol{t} + \mathbf{8)}$

$= \mathbf{(3}\boldsymbol{t} + \mathbf{8)(3}\boldsymbol{t} + \mathbf{8)} = \mathbf{(3}\boldsymbol{t} + \mathbf{8)^2}$. Another way: $\mathbf{9}\boldsymbol{t}^{\mathbf{2}} + \mathbf{48}\boldsymbol{t} + \mathbf{64} = \mathbf{(3}\boldsymbol{t}\mathbf{)^2} + \mathbf{2 \cdot 3 \cdot 8}\boldsymbol{t} + \mathbf{8^2} = \mathbf{(3}\boldsymbol{t} + \mathbf{8)^2}$.

Suggestions or Solutions
To the Problem in the Example 3

Suppose $\boldsymbol{P = \frac{4}{9}t^2 + \frac{20}{3}t + 25}$.

Factorizing it, we can get it the way below.

$$\frac{4}{9}t^2 + \frac{20}{3}t + 25 = (\frac{2}{3}t)^2 + \frac{10}{3}t + \frac{10}{3}t + 5^2 = \frac{2}{3}t(\frac{2}{3}t + 5) + 5(\frac{2}{3}t + 5) = (\frac{2}{3}t + 5)^2.$$

If not quite sure of the idea behind the processes above, follow the steps below.

Quite often, a quaratic polynomial can be put in a complete square, and we can readily factorize such a polynomial. How then, can we see if it can be put in a complete square?

If for instance, $\boldsymbol{D}$ is the discriminant of $\boldsymbol{Q = ax^2 + bx + c}$, then $\boldsymbol{D = b^2 - 4ac}$, which is in fact, a part of the quadratic formula as follows.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. So the formula can be put this way, too: $x = \frac{-b \pm \sqrt{D}}{2a}$ where $\boldsymbol{D = b^2 - 4ac}$.

Thus, we have

$$ax^2 + bx + c = a(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a})(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}) = a(x - \frac{-b + \sqrt{D}}{2a})(x - \frac{-b - \sqrt{D}}{2a}).$$

(If not sure of how to get it, refer to the example 0 in **Examples 6**.)

Therefore, if the discriminant $\boldsymbol{D = 0}$, we get

$$ax^2 + bx + c = a(x - \frac{-b}{2a})(x - \frac{-b}{2a}) = a(x + \frac{b}{2a})(x + \frac{b}{2a}) = a(x + \frac{b}{2a})^2.$$

That is, a quadratic polnomial can be put in a complete square if its discriminant is 0.

And we can put the polynomial $\boldsymbol{Q}$ above the way below, too.

$\boldsymbol{ax^2+bx+c=a(x+\frac{b}{2a})^2=(\sqrt{a}\cdot x+\frac{\sqrt{a}\cdot b}{2a})^2}$, where $\sqrt{\boldsymbol{a}}$, and $\frac{\sqrt{a}\cdot b}{2a}$ are constant.

Thus, setting $\boldsymbol{m=\sqrt{a}}$, and $\boldsymbol{n=\frac{\sqrt{a}\cdot b}{2a}}$, we can set $\boldsymbol{Q}$ this way, too: $\boldsymbol{ax^2+bx+c=(mx+n)^2}$.

So evaluating the discriminant $\boldsymbol{D}$ of $\boldsymbol{P}$, we can quickly check to see if $\boldsymbol{P}$ can be put in a complete square.

Now, $\boldsymbol{P}$'s $\boldsymbol{D}$ is $\boldsymbol{(\frac{20}{3})^2-4\cdot(\frac{4}{9})\cdot 25=(\frac{20}{3})^2-\frac{4^2\cdot 5^2}{3^2}=(\frac{20}{3})^2-(\frac{20}{3})^2=0}$.

So $\boldsymbol{P}$ can be put in a complete square, and we get

$\boldsymbol{\frac{4}{9}t^2+\frac{20}{3}t+25=(\frac{2}{3}t)^2+2\cdot\frac{2}{3}t\cdot 5+5^2=(\frac{2}{3}t+5)^2}$.

And the factorization process can be as follows.

$$\boldsymbol{\frac{4}{9}t^2+\frac{20}{3}t+25=(\frac{2}{3}t)^2+\frac{10}{3}t+\frac{10}{3}t+5^2=\frac{2}{3}t(\frac{2}{3}t+5)+5(\frac{2}{3}t+5)=(\frac{2}{3}t+5)^2}.$$

And also, if necessary, we can put it the way below, too.

$$\boldsymbol{(\frac{2}{3}t+5)^2=(\frac{2}{3}t+\frac{3\cdot 5}{3})^2=\{\frac{1}{3}(2t+15)\}^2=\frac{1}{9}(2t+15)^2}.$$

In short:

$$\boldsymbol{\frac{4}{9}t^2+\frac{20}{3}t+25=(\frac{2}{3}t)^2+\frac{10}{3}t+\frac{10}{3}t+5^2=\frac{2}{3}t(\frac{2}{3}t+5)+5(\frac{2}{3}t+5)=(\frac{2}{3}t+5)^2}.$$

Suggestions or Solutions
To the Problem in the Example 4

We have $\boldsymbol{49r^2 + \frac{112}{3}r + \frac{64}{9}}$.

Factorizing it, we can get it the way below.

$$\boldsymbol{P = 49r^2 + \frac{112}{3}r + \frac{64}{9} = 49r^2 + \frac{56}{3}r + \frac{56}{3}r + \frac{64}{9} = 7r(7r + \frac{8}{3}) + \frac{8}{3}(7r + \frac{8}{3}) = (7r + \frac{8}{3})^2}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = 49r^2 + \frac{112}{3}r + \frac{64}{9}}$, we could expect that $\boldsymbol{P}$ is not factorable within the scope of integers, because in $\boldsymbol{P}$, all the coefficients and the numeric term are not integers.

It is often the case however, we can put a quadratic polynomial into a complete square.

For instance, $\boldsymbol{(x + \frac{1}{2})^2}$ is a quadratic polynomial, and is in a complete square, and is in fact, equal to $\boldsymbol{x^2 + x + \frac{1}{4}}$.

And we can see if $\boldsymbol{P}$ can be put in a complete square if we get the discriminant of $\boldsymbol{P}$, and see if its discriminant is 0.

Suppose for instance, $\boldsymbol{D}$ is the discriminat of a quadratic polynomial $\boldsymbol{Q = ax^2 + bx + c}$.

Then, taking the discriminat $\boldsymbol{D}$ of $\boldsymbol{Q}$, we get $\boldsymbol{D = b^2 - 4ac}$.

So $\boldsymbol{D}$ of $\boldsymbol{P}$ is as follows.

$$\boldsymbol{(\frac{112}{3})^2 - 4 \cdot 49 \cdot \frac{64}{9} = (\frac{112}{3})^2 - 2^2 \cdot 7^2 \cdot \frac{8^2}{3^2} = (\frac{112}{3})^2 - (2 \cdot 7 \cdot \frac{8}{3})^2 = (\frac{112}{3})^2 - (\frac{112}{3})^2 = 0}.$$

Thus, ***P*** can be put in a complete square, and can be factorized the way below.

$$\boldsymbol{P} = 49r^2 + \tfrac{112}{3}r + \tfrac{64}{9} = (7r)^2 + 2 \cdot \tfrac{8}{3} \cdot 7r + (\tfrac{8}{3})^2 = (7r + \tfrac{8}{3})^2.$$

And we can put it this way, too:

$$\boldsymbol{P} = 49r^2 + \tfrac{112}{3}r + \tfrac{64}{9} = 49r^2 + \tfrac{56}{3}r + \tfrac{56}{3}r + \tfrac{64}{9} = 7r(7r + \tfrac{8}{3}) + \tfrac{8}{3}(7r + \tfrac{8}{3}) = (7r + \tfrac{8}{3})^2.$$

In short:

$$\boldsymbol{P} = 49r^2 + \tfrac{112}{3}r + \tfrac{64}{9} = 49r^2 + \tfrac{56}{3}r + \tfrac{56}{3}r + \tfrac{64}{9} = 7r(7r + \tfrac{8}{3}) + \tfrac{8}{3}(7r + \tfrac{8}{3}) = (7r + \tfrac{8}{3})^2.$$