Examples 8 in Polynomial Factorizations

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Examples 8

Factorize the polynomials below.

0. $\boldsymbol{a^2 + 10a + 9}$

1. $\boldsymbol{\frac{4}{25}h^2 - \frac{17}{10}h + \frac{289}{64}}$

2. $\boldsymbol{48q^2 - 168q + 147}$

3. $\boldsymbol{\frac{2}{3}p^2 - 6p + \frac{27}{2}}$

Suggestions or Solutions
To the Problem in the Example 0

We have $a^2 + 10a + 9$.

If a polynomial is a complete square, the factorization of it is easy.
That's because putting it into a complete square, we get the factorization of it.

Suppose now, $P = a^2 + 10a + 9$. How then, can we see if P is a complete square?

Assuming for instance, $Q = ax^2 + bx + c$, we can say that the quadratic polynomial Q can be put in a complete square if the discriminant D of Q is 0, and $D = b^2 - 4ac$.

So getting the discriminant D of P, we get $D = 10^2 - 4{\cdot}1{\cdot}9 = 100 - 36 \neq 0$.
So P cannot be put in a complete square. What then?

We have a factorization identity, $x^2 + (u + v)x + uv = (x + u)(x + v)$, which has the same structure as the one the polynomial P has. So?

So assuming the polynomial P is $a^2 + (u + v)a + uv$, we can expect that P can be factorized to $(a + u)(a + v)$, where u and v are integers, $uv = 10$, and $u + v = 9$.

And if P is factorable within the scope of integers, using the divisors of 10, we can find by trial and error, a pair of integers that can be multiplied to be 10, and add up to 9.

We have $10 = 1 + 9$, and $9 = 1{\cdot}9$. So we can see that u and v are 1 and 9 respectively.
That is, $u = 1$ and $v = 9$, or $u = 9$ and $v = 1$. Thus, we get
$P = a^2 + 10a + 9 = a^2 + (1 + 9)a + 1{\cdot}9 = a^2 + a + 9a + 9$
$= a(a + 1) + 9(a + 1) = (a + 1)(a + 9)$.

In short:

$a^2 + 10a + 9 = a^2 + (1 + 9)a + 1{\cdot}9 = a^2 + a + 9a + 9 = a(a + 1) + 9(a + 1)$
$= (a + 1)(a + 9)$.

Suggestions or Solutions

To the Problem in the Example 1

We have $\frac{4}{25}\boldsymbol{h}^2-\frac{17}{10}\boldsymbol{h}+\frac{289}{64}$.

Let's set first, $\boldsymbol{p}=\frac{4}{25}\boldsymbol{h}^2-\frac{17}{10}\boldsymbol{h}+\frac{289}{64}$.

The polynomial $\boldsymbol{P}$ looks quite messy, doesn't it?
However, the coefficient of $\boldsymbol{h}^2$ is $(\frac{2}{5})^2$, and the denominator of $\frac{289}{64}$ is 8^2. So?

So we can reasonably expect that $\boldsymbol{P}$ can be put in a complete square.
So we may want to first, get the discriminant of $\boldsymbol{P}$, and then, see if it is 0.
The discriminant is as follows.

$$(-\frac{17}{10})^2-4\cdot\frac{4}{25}\cdot\frac{289}{64}=(\frac{17}{10})^2-4\cdot\frac{4}{25}\cdot\frac{17^2}{8\cdot 8}=(\frac{17}{10})^2-\frac{1}{25}\cdot\frac{17^2}{2\cdot 2}=(\frac{17}{10})^2-\frac{17^2}{100}=\mathbf{0}.$$

So the factorization can go the way below.

$$\boldsymbol{p}=\frac{4}{25}\boldsymbol{h}^2-\frac{17}{10}\boldsymbol{h}+\frac{289}{64}=(\frac{2}{5}\boldsymbol{h})^2-2\cdot\frac{17}{20}\boldsymbol{h}+(\frac{17}{8})^2=(\frac{2}{5}\boldsymbol{h})^2-2\cdot\frac{2\cdot 17}{2\cdot 20}\boldsymbol{h}+(\frac{17}{8})^2$$

$$=(\frac{2}{5}\boldsymbol{h})^2-2\cdot\frac{2}{5}\cdot\frac{17}{8}\boldsymbol{h}+(\frac{17}{8})^2=(\frac{2}{5}\boldsymbol{h}-\frac{17}{8})^2.$$

We can put it this way, too:

$$\boldsymbol{p}=\frac{4}{25}\boldsymbol{h}^2-\frac{17}{10}\boldsymbol{h}+\frac{289}{64}=(\frac{2}{5}\boldsymbol{h})^2-\frac{17}{20}\boldsymbol{h}-\frac{17}{20}\boldsymbol{h}+(\frac{17}{8})^2=\frac{2}{5}\boldsymbol{h}(\frac{2}{5}\boldsymbol{h}-\frac{17}{8}\boldsymbol{h})+\frac{17}{8}(-\frac{8}{20}\boldsymbol{h}+\frac{17}{8})$$

$$=\frac{2}{5}\boldsymbol{h}(\frac{2}{5}\boldsymbol{h}-\frac{17}{8}\boldsymbol{h})-\frac{17}{8}(\frac{8}{20}\boldsymbol{h}-\frac{17}{8})=\frac{2}{5}\boldsymbol{h}(\frac{2}{5}\boldsymbol{h}-\frac{17}{8}\boldsymbol{h})-\frac{17}{8}(\frac{2}{5}\boldsymbol{h}-\frac{17}{8})=(\frac{2}{5}\boldsymbol{h}-\frac{17}{8})^2.$$

In short:

$$\frac{4}{25}\boldsymbol{h}^2-\frac{17}{10}\boldsymbol{h}+\frac{289}{64}=(\frac{2}{5}\boldsymbol{h})^2-2\cdot\frac{17}{20}\boldsymbol{h}+(\frac{17}{8})^2=(\frac{2}{5}\boldsymbol{h})^2-2\cdot\frac{2\cdot 17}{2\cdot 20}\boldsymbol{h}+(\frac{17}{8})^2$$

$$=(\frac{2}{5}\boldsymbol{h})^2-2\cdot\frac{2}{5}\cdot\frac{17}{8}\boldsymbol{h}+(\frac{17}{8})^2=(\frac{2}{5}\boldsymbol{h}-\frac{17}{8})^2.$$

Suggestions or Solutions
To the Problem in the Example 2

Suppose $\boldsymbol{P} = \mathbf{48}\boldsymbol{q}^2 - \mathbf{168}\boldsymbol{q} + \mathbf{147}$.

Norammly, factorizing a polynomial, we begin with a divisor common to all the terms in the polynomial. Then, the divisor is called a common divisor, and is a factor of the polynomial.

And checking to see if an integer divides another integer, we can use a tool as below.

- If 3 divides the sum of all the digits in an integer, 3 divides the integer, too.

Now, in the polynomial $\boldsymbol{P}$, the coefficients are 48, and –168, and the numeric term is 147.

So we may want to begin with a divisor common to all those integers.
Then, such a divisor can divide $\boldsymbol{P}$, of course.
We can readily see that 2 cannot be a common factor since 147 is odd.
So let's see now, if 3 can do it.

For 48, we get $4 + 8 = 12 = 3 \cdot 4$.

For 168, we get $1 + 6 + 8 = 15 = 3 \cdot 5$.

For 147, we get $1 + 4 + 7 = 12 = 3 \cdot 4$.

Thus, we can take 3 for the common factor, and we can reduce the polynomial to such a product form as follows. $\boldsymbol{P} = \mathbf{48}\boldsymbol{q}^2 - \mathbf{168q} + \mathbf{147} = \mathbf{3(16}\boldsymbol{q}^2 - \mathbf{56}\boldsymbol{q} + \mathbf{49)}$.

Next, we can quckly notice that $\mathbf{16 = 4^2}$, **-56** is even, and $\mathbf{49 = 7^2}$.
Thus, we can expect that the smaller polynomial, which is inside the parentheses, can be put in a complete square.

So keeping that in mind, let's factorize the polynomial.
Then, we get $\mathbf{16}\boldsymbol{q}^2 - \mathbf{56}\boldsymbol{q} + \mathbf{49} = \mathbf{(4}\boldsymbol{q}\mathbf{)}^2 - \mathbf{2 \cdot 7 \cdot 4}\boldsymbol{q} + \mathbf{7}^2 = \mathbf{(4}\boldsymbol{q} - \mathbf{7)}^2$.
Thus, $\boldsymbol{P}$ gets factorized to $\mathbf{3(4}\boldsymbol{q} - \mathbf{7)}^2$.

In short:

$\mathbf{8}\boldsymbol{q}^2 - \mathbf{168}\boldsymbol{q} + \mathbf{147} = \mathbf{3(16}\boldsymbol{q}^2 - \mathbf{56}\boldsymbol{q} + \mathbf{49)} = \mathbf{(4}\boldsymbol{q}\mathbf{)}^2 - \mathbf{2 \cdot 7 \cdot 4}\boldsymbol{q} + \mathbf{7}^2 = \mathbf{(4}\boldsymbol{q} - \mathbf{7)}^2$.

Suggestions or Solutions

To the Problem in the Example 3

We have $\frac{2}{3}p^2 - 6p + \frac{27}{2}$.

Factorizing it, we can get it the way below.

$$\frac{2}{3}p^2 - 6p + \frac{27}{2} = \frac{1}{6}\cdot 6(\frac{2}{3}p^2 - 6p + \frac{27}{2}) = \frac{1}{6}(4p^2 - 36p + 81).$$

Meanwhile, $4p^2 - 36p + 81 = 4p^2 - 2\cdot 9p - 2\cdot 9p + 81 = 2p(2p - 9) - 9(2p - 9) = (2p - 9)^2$.

Therefore, $\frac{2}{3}p^2 - 6p + \frac{27}{2} = \frac{1}{6}(2p - 9)^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Running into a quadratic polynomial where coefficients or the constant term are not integers, we tend to think such a polynomial cannot be put in a complete square.

It is often the case though, such a polynomial can be put in a complete square.

We have already taken care of such polynomials earlier in this set of examples.
In such a polynomial, the coefficient of the quadratic term is a number squared as $\frac{4}{25}$, and so is the numeric term. A quadratic term is a term where a variable is squared as $2x^2$.

Now, setting $Q = \frac{2}{3}p^2 - 6p + \frac{27}{2}$, and examining the coefficients and the numeric term in Q, we can see that they are not numbers squared, so it seems that Q cannot be put in a complete square.

Nevertheless, Q can still be put in a complete square.

So since Q is quadratic, we may want to check to see if its discriminant is 0. If so, it can be put in a complete square.   What is the discriminant though?

Assuming for instance, $\boldsymbol{D}$ is the discriminant of $\boldsymbol{ax^2 + bx + c}$, we get $\boldsymbol{D = b^2 - 4ac}$.

And if $\boldsymbol{D = 0}$, the polynomial $\boldsymbol{ax^2 + bx + c}$ can be put in a complete square. How?

The polynomial $\boldsymbol{ax^2 + bx + c}$ is factorized to $\boldsymbol{a(x+\frac{b+\sqrt{D}}{2a})(x+\frac{b-\sqrt{D}}{2a})}$, where $\boldsymbol{D = b^2 - 4ac}$.

So if $\boldsymbol{D = b^2 - 4ac = 0}$, we get $\boldsymbol{ax^2 + bx + c = a(x+\frac{b}{2a})(x+\frac{b}{2a}) = a(x+\frac{b}{2a})^2}$.

And thus, we may want to check first, to see if the discriminant $\boldsymbol{D}$ of $\boldsymbol{Q}$ is 0.

Then, $\boldsymbol{D}$ of $\boldsymbol{Q}$ is $\boldsymbol{(-6)^2 - 4\cdot\frac{2}{3}\cdot\frac{27}{2} = 36 - 36 = 0}$, so $\boldsymbol{Q}$ can be put in a complete square.

What then is the next?

Next, we have $\boldsymbol{ax^2 + bx + c = a(x+\frac{b}{2a})^2}$.

And in the case of $\boldsymbol{Q}$, we have $\boldsymbol{a = \frac{2}{3}}$, and $\boldsymbol{b = -6}$, so we get $\boldsymbol{\frac{b}{2a} = \frac{-6}{2\cdot\frac{2}{3}} = -\frac{18}{4} = -\frac{9}{2}}$.

And thus, $\boldsymbol{Q}$ can be factorized to $\boldsymbol{\frac{2}{3}(p-\frac{9}{2})^2}$.

Running into a polynomial where the coefficients and numeric term are messy as in the case of the polynomial $\boldsymbol{Q}$, we can begin with simplifying such a polynomial.

We can take this approach when the coefficients and numeric term are rational numbers.

Suppose for instance, we want to factorize $\boldsymbol{U = \frac{1}{6}x^2 - \frac{2.3}{12}x - \frac{0.7}{4}}$.

Then first, multiplying $\boldsymbol{U}$ by 10, we get $\boldsymbol{10U = \frac{10}{6}x^2 - \frac{23}{12}x - \frac{7}{4}}$.

Next, multiplying $\boldsymbol{10U}$ by 12, which is the common denominator of $\boldsymbol{\frac{1}{6}}$, $\boldsymbol{\frac{1}{12}}$, and $\boldsymbol{\frac{1}{4}}$, we get $\boldsymbol{120U = 20x^2 - 23x - 21}$, which is a polynomial without fractional numbers.

So in sum, multiplying and dividing $\boldsymbol{U}$ by 120, we get

$$\boldsymbol{U = \tfrac{1}{6}x^2 - \tfrac{2.3}{12}x - \tfrac{0.7}{4} = 120 \cdot \tfrac{1}{120}(\tfrac{1}{6}x^2 - \tfrac{2.3}{12}x - \tfrac{0.7}{4}) = \tfrac{1}{120}(20x^2 - 23x - 21)}.$$

Next, for instance, setting $\boldsymbol{V = 20x^2 - 23x - 21}$, we can factorize $\boldsymbol{V}$ easily, and factorizing it, we get $\boldsymbol{V = (5x + 3)(4x - 7)}$. What then is the next?

We want to scale $\boldsymbol{V}$ back to the original one, which is $\boldsymbol{U}$.

That is, we want to divide $\boldsymbol{V = (5x + 3)(4x - 7)}$ by 120, and get $\boldsymbol{U}$ back.

So we get $\boldsymbol{U = \tfrac{1}{120}(5x + 3)(4x - 7)}$.

How can we though, factorize such a polynomials as $\boldsymbol{20x^2 - 23x - 21}$?

We will cover such a polynomial factorization in the **Examples B**.

Now in this problem, we are factorizing $\boldsymbol{Q = \tfrac{2}{3}p^2 - 6p + \tfrac{27}{2}}$.

So taking the common denominator of $\boldsymbol{\tfrac{1}{3}}$ and $\boldsymbol{\tfrac{1}{2}}$, and multiplying and dividing $\boldsymbol{Q}$ by it, which is $\boldsymbol{\tfrac{1}{6}}$, we get $\boldsymbol{Q = \tfrac{2}{3}p^2 - 6p + \tfrac{27}{2} = \tfrac{1}{6} \cdot 6(\tfrac{2}{3}p^2 - 6p + \tfrac{27}{2}) = \tfrac{1}{6}(4p^2 - 36p + 81)}$.

So next, we want to factorize $\boldsymbol{4p^2 - 36p + 81}$.

Examining the coefficients and the numeric term, we can see that $\boldsymbol{4 = 2^2}$, $\boldsymbol{81 = 9^2}$, and $\boldsymbol{36 = 2 \cdot 2 \cdot 9}$, so we can expect that the polynomial above can be put in a complete square.

Actually getting the discriminant $\boldsymbol{D}$, we get $\boldsymbol{D = (-36)^2 - 4 \cdot 4 \cdot 81 = (4 \cdot 9)^2 - 4 \cdot 4 \cdot 9 \cdot 9 = 0}$.

So it is the case. And thus, factorizing $\mathbf{4p^2 - 36p + 81}$, we get

$$\mathbf{4p^2 - 36p + 81 = 4p^2 - 4{\cdot}9p + 81 = 4p^2 - 2{\cdot}9p - 2{\cdot}9p + 81 = 2p(2p - 9) - 2{\cdot}9p + 81}$$

$$\mathbf{= 2p(2p - 9) - 9(2p - 9) = (2p - 9)(2p - 9) = (2p - 9)^2.}$$

So we get

$$\boldsymbol{Q = \tfrac{2}{3}p^2 - 6p + \tfrac{27}{2} = \tfrac{1}{6}(4p^2 - 36p + 81) = \tfrac{1}{6}(2p - 9)^2 = \tfrac{1}{6}\{2(p - \tfrac{9}{2})\}^2 = \tfrac{4}{6}(p - \tfrac{9}{2})^2 = \tfrac{2}{3}(p - \tfrac{9}{2})^2.}$$

In short:

$$\boldsymbol{\tfrac{2}{3}p^2 - 6p + \tfrac{27}{2} = \tfrac{1}{6}\cdot 6(\tfrac{2}{3}p^2 - 6p + \tfrac{27}{2}) = \tfrac{1}{6}(4p^2 - 36p + 81).}$$

Meanwhile, $\mathbf{4p^2 - 36p + 81 = 4p^2 - 2{\cdot}9p - 2{\cdot}9p + 81 = 2p(2p - 9) - 9(2p - 9) = (2p - 9)^2}$.

Therefore, $\boldsymbol{\tfrac{2}{3}p^2 - 6p + \tfrac{27}{2} = \tfrac{1}{6}(2p - 9)^2}$.