Examples 9 in Polynomial Factorizations

Click this to start.

Examples 9

Factorize the polynomials below.

0. $\boldsymbol{v^2 + 15v + 36}$

1. $\boldsymbol{w^2 - 14w + 48}$

2. $\boldsymbol{3a^2 + 12a - 135}$

3. $\boldsymbol{t^2 + 5t + 5}$, and $\boldsymbol{t^2 + 3t + 5}$.

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{v^2 + 15v + 36}$.

Let' set first, $\boldsymbol{P = v^2 + 15v + 36}$.
Then, since $\boldsymbol{P}$ is quadratic, we may want to begin with checking to see if $\boldsymbol{P}$ can be put in a complete square. That is, we get the discriminant, and then, check to see if it is 0.

The discriminant of a polynomial $\boldsymbol{av^2 + bv + c}$ where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant is $\boldsymbol{b^2 - 4ac}$, and thus, that of $\boldsymbol{P}$ is $\boldsymbol{15^2 - 4 \cdot 1 \cdot 36}$, which is not 0, because $\boldsymbol{4 \cdot 1 \cdot 36}$ is even, but $\boldsymbol{15^2}$ is odd, since an odd number squared is odd. So $\boldsymbol{P}$ cannot be put in a complete square.

So let's factorize $\boldsymbol{P}$ in the usual manner.

To begin with, $\boldsymbol{P}$ is similar to $\boldsymbol{x^2 + (a + b)x + ab}$, which is factorized to $\boldsymbol{(x + a)(x + b)}$.

Thus, we can expect that $\boldsymbol{P}$ gets factorized to $\boldsymbol{(v + s)(v + t)}$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ are integers.

In other words, assuming that $\boldsymbol{s}$ and $\boldsymbol{t}$ are integers, we get to solve a system where $\boldsymbol{st = 36}$ and $\boldsymbol{s + t = 15}$. If any, we can find two integers adding up to 15 and getting multiplied to be 36 using divisors of 36 via trial and error.

If however, the values satisfying the system above cannot be integers, that is, $\boldsymbol{s}$ and $\boldsymbol{t}$ cannot be integers, we need to consult the quadratic formula, which is the solution to the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$, and the quadratic formula is as follows.

$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$, which is $\boldsymbol{x = \frac{-k \pm \sqrt{k^2 - ac}}{a}}$ if $\boldsymbol{b = 2k}$.

We can put it the way below, too.

$\boldsymbol{x = \frac{-b \pm \sqrt{D}}{2a}}$ where $\boldsymbol{D = b^2 - 4ac}$, or if $\boldsymbol{b = 2k}$, $\boldsymbol{x = \frac{-k \pm \sqrt{d}}{a}}$ where $\boldsymbol{d = k^2 - ac}$.

Now, finding the pair of integers satisfying the system where $\boldsymbol{st = 36}$, and $\boldsymbol{s + t = 15}$, we want to begin with divisors of 36, and then, check to see if the sum of the divisors is 15.

We have $\mathbf{36 = 12 \cdot 3}$, and $\mathbf{12 + 3 = 15}$. So $\boldsymbol{P}$ can be factorized to $\boldsymbol{(v + 12)(v + 3)}$.

In short:

$\boldsymbol{v^2 + 15v + 36 = v^2 + 12v + 3v + 36 = v(v + 12) + 3(v + 12) = (v + 12)(v + 3)}$.

Suggestions or Solutions
To the Problem in the Example 1

We have $\boldsymbol{w^2 - 14w + 48}$.

Factorizing it, we can get it the way below.

$\boldsymbol{w^2 - 14w + 48 = w^2 - 8w - 6w + 48 = w(w - 8) - 6(w - 8) = (w - 8)(w - 6)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P = w^2 - 14w + 48}$.

The polynomial $\boldsymbol{P}$ is quadratic, so we may want to begin with checking to see if $\boldsymbol{P}$ can be put in a complete square. So we want to see if the discriminant is 0. The discriminant is

$\boldsymbol{(-14)^2 - 4\cdot1\cdot48 = 2\cdot7\cdot2\cdot7 - 4\cdot48 = 4\cdot7\cdot7 - 4\cdot48 = 4\cdot49 - 4\cdot48 = 4(49 - 48) = 4 \neq 0}$.

Thus, $\boldsymbol{P}$ cannot be put in a complete square.

So let's factorize $\boldsymbol{P}$ in the usual manner.

To begin with, $\boldsymbol{P}$ is similar to $\boldsymbol{x^2 - (a + b)x + ab}$, which is factorized to $\boldsymbol{(x - a)(x - b)}$.

However, $\boldsymbol{P}$ can be said to be similar to $\boldsymbol{x^2 + (a + b)x + ab}$, too, which gets factorized to $\boldsymbol{(x + a)(x + b)}$.

That is because changing $\boldsymbol{a}$ and $\boldsymbol{b}$ with $\boldsymbol{-a}$ and $\boldsymbol{-b}$ respectively in $\boldsymbol{(x - a)(x - b)}$, we get

$\boldsymbol{(x - a)(x - b) = \{x - (-a)\}\{x - (-b)\} = (x + a)(x + b)}$.

So $\boldsymbol{x^2 - (a + b)x + ab}$ is just about the same as $\boldsymbol{x^2 + (a + b)x + ab}$.

Thus, we can expect that $\boldsymbol{P}$ gets factorized to a polynomial in a form of $\mathbf{(\boldsymbol{w} + \boldsymbol{s})(\boldsymbol{w} + \boldsymbol{t})}$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ are integers.

In other words, assuming that $\boldsymbol{s}$ and $\boldsymbol{t}$ are integers, we get to solve a system where $\boldsymbol{st} = \mathbf{48}$ and $\boldsymbol{s} + \boldsymbol{t} = \mathbf{-14}$.

And if any, we can find two integers adding up to -14 and getting multiplied to be 48 using divisors of 48 via trial and error.

If however, no integers satisfy the system above, that is, $\boldsymbol{s}$ and $\boldsymbol{t}$ cannot be integers, we want to consult the quadratic formula.

We have $\mathbf{48 = 6 \cdot 8}$, but $\mathbf{6 + 8 \neq -14}$, yet have this, too: $\mathbf{48 = (-6) \cdot (-8)}$, and $\mathbf{-6 + (-8) = -14}$.

So the factorization of $\boldsymbol{P}$ can proceed the way below.

$$\boldsymbol{P} = \boldsymbol{w}^2 - 14\boldsymbol{w} + 48 = \boldsymbol{w}^2 - 8\boldsymbol{w} - 6\boldsymbol{w} + 48 = \boldsymbol{w}(\boldsymbol{w} - 8) - (6\boldsymbol{w} - 48) = \boldsymbol{w}(\boldsymbol{w} - 8) - 6(\boldsymbol{w} - 8)$$
$$= (\boldsymbol{w} - 8)(\boldsymbol{w} - 6).$$

In short:

$$\boldsymbol{w}^2 - 14\boldsymbol{w} + 48 = \boldsymbol{w}^2 - 8\boldsymbol{w} - 6\boldsymbol{w} + 48 = \boldsymbol{w}(\boldsymbol{w} - 8) - 6(\boldsymbol{w} - 8) = (\boldsymbol{w} - 8)(\boldsymbol{w} - 6).$$

Suggestions or Solutions
To the Problem in the Example 2

We have $\mathbf{3a^2 + 12a - 135}$.

Factorizing it, we can get it the way below.

$\mathbf{3a^2 + 12a - 135 = 3(a^2 + 4a - 45) = 3(a^2 - 5a + 9a - 45) = 3\{a(a - 5) + 9(a - 5)\}}$

$\mathbf{= 3(a - 5)(a + 9)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P} = \mathbf{3a^2 + 12a - 135}$, we can say that the polynomial $\boldsymbol{P}$ is quadratic, so if the discriminant is 0, $\boldsymbol{P}$ can be put in a complete square.

Examining however, the coefficients and the numeric term, we can notice that 3 is a divisor common to all the terms in the polynomial $\boldsymbol{P}$.

So we may want to first, take out 3 from the polynomial.

How do we know though, if 3 divides 135 without actually doing the divisions?

We can see, of course, if 3 divides 135 by doing the actual division.

We can also use though, such a tool as follows.

- If the sum of all the digits in an integer is a multiple of 3, 3 divides the integer.

The sum of all the digits in 135 is $\mathbf{1 + 3 + 5 = 9}$, which is a multiple of 3, so 3 divides it.

Thus, we get $\boldsymbol{P} = \mathbf{3a^2 + 12a - 135 = 3(a^2 + 4a - 45)}$.

Suppose now, Q is $a^2 + 4a - 45$, and let's see if Q can be put in a complete square.

The discriminant of Q is $2^2 - 1\cdot(-45)$, which is not 0, and therefore, Q cannot be put in a complete square.

So let's factorize Q in the usual manner.

The polynomial Q is similar to $x^2 + (a + b)x + ab$, which is factorized to $(x + a)(x + b)$.

Thus, we can expect that Q gets factorized to $(a + s)(a + t)$, where s and t are integers.

That is to say that assuming that s and t are integers, we get to solve a system of two equations where $st = -45$ and $s + t = 4$. If any, we can find two integers adding up to 4 and getting multiplied to be -45 using divisors of -45 via trial and error.

If however, no integers satisfy the system above, that is, s and t cannot be integers, we want to consult the quadratic formula.

Now, to begin with, we can get $-45 = 5\cdot(-9)$, but we get $5 + (-9) \neq 4$.

And next, we get $-45 = (-5)\cdot 9$, and $-5 + 9 = 4$.

So the factorization of P can proceed as follows.

$$P = 3a^2 + 12a - 135 = 3(a^2 + 4a - 45)$$

$$= 3(a^2 - 5a + 9a - 45) = 3\{a(a - 5) + 9(a - 5)\} = 3(a - 5)(a + 9).$$

Suggestions or Solutions

To the Problem in the Example 3

We have $\boldsymbol{t^2 + 5t + 5}$, and $\boldsymbol{t^2 + 3t + 5}$.

To begin with, setting $\boldsymbol{t^2 + 5t + 5 = 0}$, we get $\boldsymbol{t = \frac{-5 \pm \sqrt{D}}{2}}$, where $\boldsymbol{D = 5^2 - 4 \cdot 1 \cdot 5 = 5}$.

Thus, $\boldsymbol{t^2 + 5t + 5 = \left(t - \frac{-5+\sqrt{5}}{2}\right)\left(t - \frac{-5-\sqrt{5}}{2}\right) = \left(t + \frac{5-\sqrt{5}}{2}\right)\left(t + \frac{5+\sqrt{5}}{2}\right)}$.

Next, the discriminant of $\boldsymbol{t^2 + 3t + 5}$ is $\boldsymbol{3^2 - 4 \cdot 1 \cdot 5 = 9 - 20 = -11}$.

Thus, $\boldsymbol{t^2 + 3t + 5}$ is not factorable (within the scope of real numbers).

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P = t^2 + 5t + 5}$, and $\boldsymbol{Q = t^2 + 3t + 5}$.

The polynomials $\boldsymbol{P}$ and $\boldsymbol{Q}$ are quadratic, so we may want to begin with checking to see if they can be put in complete squares. So we want to see if the discriminants of both are 0.

The discriminant of $\boldsymbol{P}$ is $\boldsymbol{5^2 - 4 \cdot 1 \cdot 5 = 5(5 - 4) = 5}$, which is not 0.

The discriminant of $\boldsymbol{Q}$ is $\boldsymbol{3^2 - 4 \cdot 1 \cdot 5 = 9 - 20 = -11}$, which is not 0 either.

So we can put neither of $\boldsymbol{P}$ and $\boldsymbol{Q}$ in a complete square, and therefore, we may want to factorize both of those in the usual manner.

First, both are similar to $\boldsymbol{x^2 + (a + b)x + ab}$, which we can factorize to $\boldsymbol{(x + a)(x + b)}$.

So next, we can expect each polynomial can get factorized to $\boldsymbol{(t + u)(t + v)}$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are integers, since the coefficients and the numeric term are integers.

Thus, next, beginning with $\boldsymbol{P}$, we can set $\boldsymbol{uv} = \mathbf{5}$, and $\boldsymbol{u} + \boldsymbol{v} = \mathbf{5}$, which make a system of equations for $\boldsymbol{u}$ and $\boldsymbol{v}$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ cannot be integers satisfying the system of equations above, we need to use the quadratic formula. In fact, there is no pair of integers satisfying the system. How come?

Taking advantage of the discriminant, usually designated by $\boldsymbol{D}$, we can readily check to see if it is the case. Where is the discriminant $\boldsymbol{D}$ from, though?

It is from the quadratic formula, isn't it?
So let's take a closer look at the formula, which is $\boldsymbol{x} = \frac{-b\pm\sqrt{D}}{2a}$, where $\boldsymbol{D} = \boldsymbol{b}^2 - \mathbf{4}\boldsymbol{ac}$.

The formula is for a quadratic equation $\boldsymbol{ax}^2 + \boldsymbol{bx} + \boldsymbol{c} = \mathbf{0}$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant.

However, the two polynomials $\boldsymbol{P}$ and $\boldsymbol{Q}$ are in a form of $\boldsymbol{x}^2 + \boldsymbol{mx} + \boldsymbol{n}$, where $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers.

So applying the formula to a quadratic equation $\boldsymbol{x}^2 + \boldsymbol{mx} + \boldsymbol{n} = \mathbf{0}$, we can set first, $\boldsymbol{a} = \mathbf{1}$, $\boldsymbol{b} = \boldsymbol{m}$, and $\boldsymbol{c} = \boldsymbol{n}$, and thus next, we can get

$\boldsymbol{x} = \frac{-b\pm\sqrt{D}}{2a} = \frac{-m\pm\sqrt{D}}{2\cdot 1} = \frac{-m\pm\sqrt{D}}{2}$, where $\boldsymbol{D} = \boldsymbol{b}^2 - \mathbf{4}\boldsymbol{ac} = \boldsymbol{m}^2 - \mathbf{4}\cdot\mathbf{1}\cdot\boldsymbol{n} = \boldsymbol{m}^2 - \mathbf{4}\boldsymbol{n}$.

Thus, we get $\boldsymbol{x} = \frac{-m\pm\sqrt{D}}{2}$, where $\boldsymbol{D} = \boldsymbol{m}^2 - \mathbf{4}\boldsymbol{n}$.

So we get $x^2 + mx + n = (x - \frac{-m+\sqrt{D}}{2})(x - \frac{-m-\sqrt{D}}{2}) = (x + \frac{m-\sqrt{D}}{2})(x + \frac{m+\sqrt{D}}{2}) = 0$.

Thus, we get $x^2 + mx + n = (x + \frac{m-\sqrt{D}}{2})(x + \frac{m+\sqrt{D}}{2}) = 0$, where $\boldsymbol{D} = \boldsymbol{m}^2 - \mathbf{4}\boldsymbol{n}$.

Now, our expectation is that each of P and Q can get factorized to a form of $(t + u)(t + v)$, where u and v are integers.

The two polynomials P and Q are in a form of $x^2 + mx + n$, where m and n are integers.

We have $x^2 + mx + n = (x + \frac{m-\sqrt{D}}{2})(x + \frac{m+\sqrt{D}}{2}) = 0$, where $D = m^2 - 4n$.

So replacing x with t, we get $t^2 + mt + n = (t + \frac{m-\sqrt{D}}{2})(t + \frac{m+\sqrt{D}}{2}) = 0$, where $D = m^2 - 4n$.

Thus, we can see that $u = \frac{m-\sqrt{D}}{2}$, and $v = \frac{m+\sqrt{D}}{2}$, where $D = m^2 - 4n$.

Now, we know m and n are integers.

So the discriminant D is an integer because m and n are integers.

That is because we get an integer doing a multiplication with integers and doing a subtraction with integers as well as doing an addition with them. So a multiplication, a subtraction, or an addition with integers produces an integer.

Now, if the values of u and v are integers, D has to be an integer squared as 3^2 since D is inside the square root sign. So let's have a look at what D the polynomial P has.

The discriminant D of P is $5^2 - 4 \cdot 1 \cdot 5 = 5(5 - 4) = 5$, which is not an integer squared.

So the values of u and v cannot be integers.

Thus, there is no pair of integers satisfying the system where $uv = 5$, and $u + v = 5$.

Therefore, factorizing P, we need to use the quadratic formula.

Then, setting $P = t^2 + 5t + 5 = 0$, we get $t = \frac{-5\pm\sqrt{D}}{2}$, where $D = 5^2 - 4 \cdot 1 \cdot 5 = 5$.

So we get $P = t^2 + 5t + 5 = (t - \frac{-5+\sqrt{5}}{2})(t - \frac{-5-\sqrt{5}}{2}) = (t + \frac{5-\sqrt{5}}{2})(t + \frac{5+\sqrt{5}}{2})$.

Thus, we get $\boldsymbol{P = t^2 + 5t + 5 = (t + \frac{5-\sqrt{5}}{2})(t + \frac{5+\sqrt{5}}{2})}$.

So we can see that $\boldsymbol{P}$ can get factorized to $\boldsymbol{(t + \frac{5-\sqrt{5}}{2})(t + \frac{5+\sqrt{5}}{2})}$.

• Let's next, move on to the polynomial $\boldsymbol{Q = t^2 + 3t + 5}$.

Either checking to see if $\boldsymbol{Q}$ can be put in a complete square, or checking to see if $\boldsymbol{Q}$ can get factorized to $\boldsymbol{(t + u)(t + v)}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are integers, we can take advantage of the discriminant $\boldsymbol{D}$.

The discriminant $\boldsymbol{D}$ of $\boldsymbol{Q}$ is $\boldsymbol{3^2 - 4 \cdot 1 \cdot 5 = 9 - 20 = -11}$, which is neither 0 nor an integer squared.

So $\boldsymbol{Q}$ cannot be put in a complete square, and cannot be factorized to $\boldsymbol{(t + u)(t + v)}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are integers. That's not it, though.

The polynomial $\boldsymbol{Q}$ is not factorable. That is to say that $\boldsymbol{Q}$ has no divisor.

In other words, $\boldsymbol{Q}$ itself is a prime polynomial. How come?

We know the solution to $\boldsymbol{ax^2 + bx + c = 0}$ is $\boldsymbol{x = \frac{-b \pm \sqrt{D}}{2a}}$, where $\boldsymbol{D}$ is the discriminant.

Now, if the discriminant $\boldsymbol{D}$ is negative, $\boldsymbol{D}$ cannot be inside a square root sign.

In real number space, no negative number can be inside a square root sign.

In other words, the equation $\boldsymbol{Q = t^2 + 3t + 5 = 0}$ can have no solution.

Therefore, $\boldsymbol{Q}$ is not factorable.

More precisely, $\boldsymbol{Q}$ cannot be factorable in real number space, yet it can be factorable in complex number space, because a square root of a negative number is an imaginary number, which can exist not in real number space but in complex number space.

So we have

- If a quadratic polynomial is factorable, the discriminant is greater than or equal to 0.

- If a polynomial $\boldsymbol{x^2 + mx + n}$ where $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers is factorized to $\boldsymbol{(x + u)(x + v)}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are integers, the discriminant is an integer squared as 3^2.

What is the real number space though?

It's a space where all real numbers exist, and real numbers only can exist.
So it is a set of all real numbers.

Real numbers include rational numbers and irrational numbers as $\pm\sqrt{2}, \pm\pi$, where π is the circular ratio, which is 3.141592…, and rational numbers include integers and fractional numbers as 0.2, -0.3, 3.5, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{-9}{2}$, etc.

However, the complex number space is a space where any number can exist, and thus, is the entire number space.

In short:

To begin with, setting $\boldsymbol{t^2 + 5t + 5 = 0}$, we get $\boldsymbol{t = \frac{-5 \pm \sqrt{D}}{2}}$, where $\boldsymbol{D = 5^2 - 4 \cdot 1 \cdot 5 = 5}$.

Thus, $\boldsymbol{t^2 + 5t + 5 = (t - \frac{-5+\sqrt{5}}{2})(t - \frac{-5-\sqrt{5}}{2}) = (t + \frac{5-\sqrt{5}}{2})(t + \frac{5+\sqrt{5}}{2})}$.

Next, the discriminant of $\boldsymbol{t^2 + 3t + 5}$ is $\boldsymbol{3^2 - 4 \cdot 1 \cdot 5 = 9 - 20 = -11}$.

Thus, $\boldsymbol{t^2 + 3t + 5}$ is not factorable (within the scope of real numbers).