Examples A in Polynomial Factorizations

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Examples A

Find the values of $\boldsymbol{a}$ and $\boldsymbol{b}$ in each of the cases below.

0. $\boldsymbol{a^2 + b^2 = 108}$, and $\boldsymbol{ab = 18}$.

1. $\boldsymbol{a - b = 10}$, and $\boldsymbol{a^2 + b^2 = 58}$.

2. $\boldsymbol{a + b = 12}$, and $\boldsymbol{a^2 - b^2 = 168}$.

Suggestions or Solutions
To the Problem 0 in the Example 1

Find the values of a and b for which $a^2 + b^2 = 85$, and $ab = 42$.

$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab = 85 + 2\cdot 42 = 85 + 84 = 169 = 13^2$.
So we get $a + b = \pm 13$, and $ab = 42$.

$a + b = 13 \Rightarrow a = 13 - b \Rightarrow ab = (13 - b)b = 42 \Rightarrow 13b - b^2 = 42 \Rightarrow b^2 - 13b + 42 = 0$.
$b^2 - 13b + 42 = (b - 6)(b - 7) = 0 \Rightarrow b = 6$ or 7.

$b = 6 \Rightarrow a = 13 - b = 13 - 6 = 7$. $b = 7 \Rightarrow a = 13 - b = 13 - 7 = 6$.
Thus, we get $(a, b) = (6, 7)$ or $(7, 6)$.

$a + b = -13 \Rightarrow a = -13 - b \Rightarrow ab = (-13 - b)b = -13b - b^2 = 42 \Rightarrow b^2 + 13b + 42 = 0$.
$b^2 + 13b + 42 = (b + 6)(b + 7) = 0 \Rightarrow b = -6$ or -7.

$b = -6 \Rightarrow a = -13 - b = -13 + 6 = -7$. $b = -7 \Rightarrow a = -13 - b = -13 + 7 = -6$.
Thus, we get $(a, b) = (-6, -7)$ or $(-7, -6)$.

Therefore, $(a, b) = (6, 7)$, $(7, 6)$, $(-6, -7)$, or $(-7, -6)$.

If not quite sure of the idea behind the processes above, follow the steps below.

We are given a system of equations, and the equations are $a^2 + b^2 = 85$, and $ab = 42$.
And the solution can be made in such an ordinary way as follows.

Assuming $b \neq 0$, and expressing a in terms of b using $ab = 42$, we can get $a = \frac{42}{b}$.

Then, putting it into $a^2 + b^2 = 85$, we get $(\frac{42}{b})^2 + b^2 = 85$.

And then, we can solve for b, the equation above.

Approaching however, the solution the way above, we get to do calculations quite messy and involved. In fact, the last equation for b above is of degree 4, which is pretty high.
So we may want to take some other way.

Studying polynomial factorizations, we want to pay attention to not only the results but the structures of polynomials being factorized, together with factorization processes, too.

Having done a substantial amount of practice on polynomial factorizations paying much attention to such structures and processes, we can notice that the two equations given are elements of a particular polynomial. What polynomial then, is it?

It is the polynomial that is factorized to $(a + b)^2$, and thus, is $a^2 + 2ab + b^2$.
What then, are the elements stated above?

They are $a^2 + b^2$ and ab, of which we have the values. So we get

$$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab = 85 + 2 \cdot 42 = 85 + 84 = 169 = 13^2.$$

Thus, we can see that $(a + b) = 13^2$, and therefore, we get $a + b = \pm 13$.

So we now have a new system where $a + b = \pm 13$, and $ab = 42$, which looks much simpler than the given system where $a^2 + b^2 = 85$, and $ab = 42$.

In other words, we just have reduced the system where $a^2 + b^2 = 85$, and $ab = 42$ to the new system where $a + b = \pm 13$, and $ab = 42$.

And solving the new system, since we have $a + b = \pm 13$, we want to consider two cases where $a + b = 13$, and $a + b = -13$, together with $ab = 42$.

So we get two systems as follows.

One of the two has $a + b = 13$, and $ab = 42$, and the other has $a + b = -13$, and $ab = 42$.

Beginning with the system where $\boldsymbol{a + b = 13}$, and $\boldsymbol{ab = 42}$, we can get first,

$$\boldsymbol{a + b = 13 \Rightarrow a = 13 - b \Rightarrow ab = (13 - b)b = 42 \Rightarrow 13b - b^2 = 42 \Rightarrow b^2 - 13b + 42 = 0},$$

which is noting but a simple quadratic equation for $\boldsymbol{b}$. So?

We can get the solution by a factorization (or by the quadratic formula, of course).

So factorizing $\boldsymbol{b^2 - 13b + 42}$, we can get the solution to $\boldsymbol{b^2 - 13b + 42 = 0}$.

Then, we want to find two integers getting multiplied to be 42, and adding up to –13.

By trial and error, using divisors of 42, we can get $\boldsymbol{42 = (-6)\cdot(-7)}$, and $\boldsymbol{-6 + (-7) = -13}$.

So we get $\boldsymbol{b^2 - 13b + 42 = (b - 6)(b - 7) = 0 \Rightarrow b = 6}$ or $\boldsymbol{7}$.

And thus, we now have $\boldsymbol{a = 13 - b}$, and $\boldsymbol{b = 6}$ or $\boldsymbol{7}$. So we get

$\boldsymbol{b = 6 \Rightarrow a = 13 - b = 13 - 6 = 7}$.

$\boldsymbol{b = 7 \Rightarrow a = 13 - b = 13 - 7 = 6}$.

Therefore, we get $\boldsymbol{(a, b) = (6, 7)}$ or $\boldsymbol{(7, 6)}$.

• Let's next, move on to the other system where $\boldsymbol{a + b = -13}$, and $\boldsymbol{ab = 42}$.

Then, we can get first,

$$\boldsymbol{a + b = -13 \Rightarrow a = -13 - b \Rightarrow ab = (-13 - b)b = -13b - b^2 = 42 \Rightarrow b^2 + 13b + 42 = 0},$$

which is no more than a simple quadratic equation for $\boldsymbol{b}$.

So factorizing $\boldsymbol{b^2 + 13b + 42}$, we can get the solution to $\boldsymbol{b^2 + 13b + 42 = 0}$.
Then, we want to find divisors of 42, of which the sum is 13.

By trial and error, we can get $\boldsymbol{42 = 6\cdot7}$, and $\boldsymbol{6 + 7 = 13}$.

So we get $\boldsymbol{b^2 + 13b + 42 = (b + 6)(b + 7) = 0 \Rightarrow b = -6}$ or -7.

And thus, we now have $\boldsymbol{a = -13 - b}$, and $\boldsymbol{b = -6}$ or -7. So we get

$\boldsymbol{b = -6 \Rightarrow a = -13 - b = -13 + 6 = -7}$.

$\boldsymbol{b = -7 \Rightarrow a = -13 - b = -13 + 7 = -6}$.

Therefore, we get $\boldsymbol{(a, b) = (-6, -7)}$ or $\boldsymbol{(-7, -6)}$.

And putting threads together now, we get $\boldsymbol{(a, b) = (6, 7)}$, $\boldsymbol{(7, 6)}$, $\boldsymbol{(-6, -7)}$, or $\boldsymbol{(-7, -6)}$.

So studying polynomial factorizations, we want to pay attention to not only results but the structures of polynomials being factorized, together with factorization processes, too.

In short:

$\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab = 85 + 2 \cdot 42 = 85 + 84 = 169 = 13^2}$.

So we get $\boldsymbol{a + b = \pm 13}$, and $\boldsymbol{ab = 42}$.

$\boldsymbol{a + b = 13 \Rightarrow a = 13 - b \Rightarrow ab = (13 - b)b = 42 \Rightarrow 13b - b^2 = 42 \Rightarrow b^2 - 13b + 42 = 0}$.

$\boldsymbol{b^2 - 13b + 42 = (b - 6)(b - 7) = 0 \Rightarrow b = 6}$ or 7.

$\boldsymbol{b = 6 \Rightarrow a = 13 - b = 13 - 6 = 7}$. $\boldsymbol{b = 7 \Rightarrow a = 13 - b = 13 - 7 = 6}$.

Thus, we get $\boldsymbol{(a, b) = (6, 7)}$ or $\boldsymbol{(7, 6)}$.

$\boldsymbol{a + b = -13 \Rightarrow a = -13 - b \Rightarrow ab = (-13 - b)b = -13b - b^2 = 42 \Rightarrow b^2 + 13b + 42 = 0}$.

$\boldsymbol{b^2 + 13b + 42 = (b + 6)(b + 7) = 0 \Rightarrow b = -6}$ or -7.

$\boldsymbol{b = -6 \Rightarrow a = -13 - b = -13 + 6 = -7}$. $\boldsymbol{b = -7 \Rightarrow a = -13 - b = -13 + 7 = -6}$.

Thus, we get $\boldsymbol{(a, b) = (-6, -7)}$ or $\boldsymbol{(-7, -6)}$.

Therefore, $\boldsymbol{(a, b) = (6, 7)}$, $\boldsymbol{(7, 6)}$, $\boldsymbol{(-6, -7)}$, or $\boldsymbol{(-7, -6)}$.

• And let's now talk about some more algebra.

Once we have found a pair of values for a and b in the system where $a^2 + b^2 = 85$, and $ab = 42$, we can quickly find the other three pairs reflecting the two facts below.

• First, we can notice that the equation $a^2 + b^2 = 85$ remains the same even if a and b get exchanged. So we can see that a gets values b can get, and vice versa. It doesn't necessarily mean though, that a and b are the same.

• Next, the equation $ab = 42$ tells us the product of a and b is positive, so a and b have the same sign. That is, a and b both are positive, or are negative at the same time.
So once we have found a pair of values for a and b, we can quickly find the other three pairs using the two facts above.

Suppose for instance, we have found $(a, b) = (6, 7)$.

Then, we can immediately see that $(a, b) = (7, 6)$, $(-6, -7)$, or $(-7, -6)$, too.

That's because of these:

First, a gets values that b can get, and vice versa.

So a can be 7 since $b = 7$, and b can be 6 since $a = 6$.

Thus, we can get $(a, b) = (7, 6)$, too.

Next, a and b can be negative, too, at the same time.

Thus, we can have $(a, b) = (-6, -7)$ or $(-7, -6)$, too.

So if in a multiple-choice test, we get to choose the solution to a system of equations like the system where $a^2 + b^2 = 85$, and $ab = 42$, we can cut corners the way above.

Suggestions or Solutions

To the Problem 1 in the Example 1

Find the values of a and b for which $a^2 + b^2 = 58$, and $a - b = 10$.

$(a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab = 58 - 2ab = 10^2 \Rightarrow 29 - ab = 50 \Rightarrow ab = -21$.

So we get $a - b = 10$, and $ab = -21$.

Thus, we get $a - b = 10 \Rightarrow a = b + 10 \Rightarrow ab = (b + 10)b = -21 \Rightarrow b^2 + 10b + 21 = 0$.

So we get $b^2 + 10b + 21 = (b + 3)(b + 7) = 0 \Rightarrow b = -3$ or -7.

$b = -3 \Rightarrow a = b + 10 = -3 + 10 = 7$.

$b = -7 \Rightarrow a = b + 10 = -7 + 10 = 3$.

Therefore, we get $(a, b) = (7, -3)$ or $(3, -7)$.

If not quite sure of the idea behind the processes above, follow the steps below.

We are given a system of equations, which are $a - b = 10$, and $a^2 + b^2 = 58$.

Unlike the system in the problem 0, the system above can just be solved in a usual manner without much of complexity in calculation.

- Expressing a in terms of b using $a - b = 10$, we get $a = b + 10$, which we can put into $a^2 + b^2 = 58$, and then, we solve $(b + 10)^2 + b^2 = 58$. Solving it, we can factorize it or use the quadratic formula.

Let's for practice purpose though, approach the solution the way we get the solution to the previous system.

Examining the two equations in the system given here, we can notice that one of the two equations is an element of a particular polynomial, which is factorized to $(a - b)^2$.
What then, is the particular polynomial?

It is $a^2 - 2ab + b^2$. And we have $a^2 + b^2 = 58$, which is the element stated above.

What about ab, though?

We can find the value of ab using $a - b = 10$, together with $a^2 + b^2 = 58$.

And using the values of $a - b$ and $a^2 + b^2$, we get

$(a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab \Rightarrow 10^2 = 58 - 2ab \Rightarrow 2ab = -42 \Rightarrow ab = -21$.

So we just have reduced the given system where $a - b = 10$, and $a^2 + b^2 = 58$ to a new system where $a - b = 10$, and $ab = -21$, which is much simpler than the system given.

Now, solving the new system above, we can get first,

$a - b = 10 \Rightarrow a = b + 10 \Rightarrow ab = (b + 10)b = -21 \Rightarrow b^2 + 10b + 21 = 0$, which is noting but a simple quadratic equation for b.

So factorizing $b^2 + 10b + 21$, we can get the solution to $b^2 + 10b + 21 = 0$.

First, finding divisors of 21, of which the sum is 10, we can get $21 = 3 \cdot 7$, and $3 + 7 = 10$.

So we get $b^2 + 10b + 21 = (b + 3)(b + 7) = 0 \Rightarrow b = -3$ or -7.

Thus next, since we have $a = b + 10$, we get

$b = -3 \Rightarrow a = b + 10 = -3 + 10 = 7$.

$b = -7 \Rightarrow a = b + 10 = -7 + 10 = 3$.

Therefore, we get $(a, b) = (7, -3)$ or $(3, -7)$.

Suggestions or Solutions

To the Problem 2 in the Example 1

Find the values of a and b for which $a^2 - b^2 = 168$, and $a + b = 12$.

$\boldsymbol{a + b = 12 \Rightarrow a = 12 - b \Rightarrow a^2 - b^2 = (12 - b)^2 - b^2 = 168}$

$\boldsymbol{\Rightarrow (12 - b)^2 - b^2 = 12^2 - 24b + b^2 - b^2 = 12^2 - 24b = 168 \Rightarrow 24b = 144 - 168 \Rightarrow b = -1}$

$\boldsymbol{\Rightarrow a + b = a - 1 = 12 \Rightarrow a = 13}$.

Therefore, $\boldsymbol{(a, b) = (13, -1)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

We are given a system of equations where $\boldsymbol{a + b = 12}$, and $\boldsymbol{a^2 - b^2 = 168}$.
As in the case of the problem 1, we can just directly approach the solution to the system above by an ordinary method. And the ordinary method is as follows.

- Putting $\boldsymbol{a}$ in terms of $\boldsymbol{b}$ using $\boldsymbol{a - b = 12}$, we get $\boldsymbol{a = 12 - b}$, which we can put into the other equation $\boldsymbol{a^2 - b^2 = 168}$, and then, we solve $\boldsymbol{(12 - b)^2 - b^2 = 168}$. Solving it though, we don't even need to bother factorizing it or using the quadratic formula.

That's because the quadratic ($\boldsymbol{b^2}$) terms will get canceled out.
Simplifying the final equation above, we get

$\boldsymbol{(12 - b)^2 - b^2 = 12^2 - 24b + b^2 - b^2 = 12^2 - 24b = 168 \Rightarrow 24b = 144 - 168 \Rightarrow b = -1}$.

So the final equation is not really quadratic, and is just a simple equation of degree 1.

Let's for practice purpose though, approach the solution the way we get the solution to the system in the Problem 0.

What polynomial factorization then, can we refer to?

We can refer to the factorization of $\boldsymbol{a^2 - b^2}$.

And factorizing it, we get $\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$, which is called a factorization identity. And in the identity, we can see two polynomials, which are in the system given.

What then, are the two?

The two are $\boldsymbol{a + b}$, and $\boldsymbol{a^2 - b^2}$, of which we have the values.

What about the value of $\boldsymbol{a - b}$, though?

We can find the value of $\boldsymbol{a - b}$ using $\boldsymbol{a + b = 12}$, along with $\boldsymbol{a^2 - b^2 = 168}$.

And taking advantage of the values of $\boldsymbol{a + b}$ and $\boldsymbol{a^2 - b^2}$, we get

$\boldsymbol{a^2 - b^2 = (a + b)(a - b) \Rightarrow 168 = 12(a - b) \Rightarrow a - b = 14}$.

So we now have reduced the given system to a system where $\boldsymbol{a - b = 14}$, and $\boldsymbol{a + b = 12}$, which is a simple system of equations of degree 1, which can be readily solved.

To begin with, adding up the two equations in the system, we get

$\boldsymbol{(a - b) + (a + b) = 14 + 12 \Rightarrow 2a = 26 \Rightarrow a = 13}$.

Next, subtracting one equation from the other, we can get

$\boldsymbol{(a - b) - (a + b) = 14 - 12 \Rightarrow -2b = 2 \Rightarrow b = -1}$.

Therefore, we get $\boldsymbol{(a, b) = (13, -2)}$.

In short:

$\boldsymbol{a + b = 12 \Rightarrow a = 12 - b \Rightarrow a^2 - b^2 = (12 - b)^2 - b^2 = 168}$

$\boldsymbol{\Rightarrow (12 - b)^2 - b^2 = 12^2 - 24b + b^2 - b^2 = 12^2 - 24b = 168 \Rightarrow 24b = 144 - 168 \Rightarrow b = -1}$

$\boldsymbol{\Rightarrow a + b = a - 1 = 12 \Rightarrow a = 13}$.

Therefore, $\boldsymbol{(a, b) = (13, -1)}$.

And we can get the same the way below, too.

$\boldsymbol{a^2 - b^2 = (a + b)(a - b) \Rightarrow 168 = 12(a - b) \Rightarrow a - b = 14}$.

$\boldsymbol{(a - b) + (a + b) = 14 + 12 \Rightarrow 2a = 26 \Rightarrow a = 13}$.

$\boldsymbol{(a - b) - (a + b) = 14 - 12 \Rightarrow -2b = 2 \Rightarrow b = -1}$.

Therefore, $\boldsymbol{(a, b) = (13, -2)}$.