Examples B in Polynomial Factorizations

Click this to start.

Examples B

Why polynomial factorizations?

They are for your algebra skill. So knowing them very well and doing them very well, you can do very well manipulate, that is, change, convert, or modify math expressions.

So when studying this material, too, pay particular attention to the processes math expressions can be changed or modified through. Then, you can see better how to manipulate, that is, change or modify math expressions.

- It is in fact, your algebra that can actually connect the problems to the solutions.

While learning polynomial factorizations, you can improve your skill of algebra a lot. And those factorizations are great tools, and can be powerful if you know them very well, and do practices on them often enough.

Cliché but always true that knowing is one thing and doing it is another.

So make your own examples, too, and do them yourselves.

Then, you can do better your algebra as well as see better the principles working behind the tools called polynomial factorizations.

Factorize the polynomials below.

0. $\mathbf{6x^2 + 23x + 20}$

1. $\mathbf{28x^2 + 71x + 18}$

Suggestions or Solutions

To the Problem in the Example 0

We have $\mathbf{6x^2 + 23x + 20}$.

Factorizing it, we can get it the way below.

$$6x^2 + 23x + 20 = 2{\cdot}3x^2 + (2{\cdot}4 + 3{\cdot}5)x + 4{\cdot}5 = 2{\cdot}3x^2 + 2{\cdot}4x + 3{\cdot}5x + 4{\cdot}5$$

$$= 2x(3x + 4) + 5(3x + 4) = (3x + 4)(2x + 5).$$

If not quite sure of the idea behind the processes above, follow the steps below.

In this example, we have several things to discuss besides the solution.
And the things are about growing your algebra, which is the purpose of this book.
That is to say that they are for the enhancement of your algebra skill.

Now, setting first, $P = 6x^2 + 23x + 20$, we can say that the polynomial P is quadratic, and is in the form of $ux^2 + vx + w$. So?

The polynomial $ux^2 + vx + w$ is similar to a particular polynomial, which is factorize to $(ax + b)(cx + d)$.

What then, is the particular polynomial?

It is $acx^2 + (ad + bc)x + bd$.

Taking ac as u, $ad + bc$ as v, and bd as w, we get $ux^2 + vx + w$, which P is in the form of.
So P can be said to be similar to $acx^2 + (ad + bc)x + bd$, factorize to $(ax + b)(cx + d)$.
Thus, setting $ac = 6$, $ad + bc = 23$, and $bd = 20$, and then, finding a, b, c, and d, we can

get the factorization of $\boldsymbol{P}$. How come?

First, setting $\boldsymbol{ac = 6}$, $\boldsymbol{ad + bc = 23}$, and $\boldsymbol{bd = 20}$, we get

$\boldsymbol{acx^2 + (ad + bc)x + bd = 6x^2 + 23x + 20}$, which is the very $\boldsymbol{P}$.

Next, we know $\boldsymbol{acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)}$, so finding $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$, we can get the factorization of $\boldsymbol{P}$.

That is, we can set $\boldsymbol{6x^2 + 23x + 20 = (ax + b)(cx + d)}$.

That is to say that solving the system of equations below, we can get it done.

$\boldsymbol{ac = 6}$, $\boldsymbol{ad + bc = 23}$, and $\boldsymbol{bd = 20}$. Are they all the equations we can make, though?

In other words, is there any other equation we can add to the system?

That seems to be it. Then, we've got a problem. What problem then, is it?

The number of equations has to be the same as the number of unknowns.
We have four unknowns, yet the system has three equations only.
So the system lacks one equation, and thus, is not complete.
So we've got a situation here. What else then, can we do?

We can set up a reasonable constraint, which is on the unknowns, of course, and then, make use of the constraint. Doing math in fact, we often do so.

That is to say that we can set up a reasonable condition that the unknowns have to obey.
Satisfying the condition, along with the three equations, we can find the four unknowns.

That is because an equation is in fact, a condition to be met.
What do we mean by a reasonable condition though?

We can restrict the unknowns to be integers. How?

We want $\boldsymbol{P = 6x^2 + 23x + 20}$ to be in the form of $\boldsymbol{acx^2 + (ad + bc)x + bd}$, which is factorized to $\boldsymbol{(ax + b)(cx + d)}$.

So setting $\boldsymbol{ac = 6}$, $\boldsymbol{bd = 20}$, and $\boldsymbol{ad + bc = 23}$, we can put $\boldsymbol{P}$ into the form above.

And we know divisors of an integer are integers, $\boldsymbol{a}$ and $\boldsymbol{c}$ are divisors of $\boldsymbol{ac}$, that is, 6, and $\boldsymbol{b}$ and $\boldsymbol{d}$ are divisors of $\boldsymbol{bd}$, that is, 20.

So we can set up a condition that $\boldsymbol{a}$ and $\boldsymbol{c}$ are two integers the product of which is 6, and also, $\boldsymbol{b}$ and $\boldsymbol{d}$ are two integers multiplied to be 20.

So *if any*, we can find the four integers $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ satisfying the equations below.

$\boldsymbol{ac = 6}$, $\boldsymbol{bd = 20}$, and $\boldsymbol{ad + bc = 23}$.

That is to say that the system is now, $\boldsymbol{ac = 6}$, $\boldsymbol{bd = 20}$, $\boldsymbol{ad + bc = 23}$, and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are integers. So the system has 4 equations. How come though, the system has 4 equations?

An equation is in fact, a condition to be satisfied.
And the system above can be said to have four conditions. And the four are as follows.

- $\boldsymbol{ac = 6}$.
- $\boldsymbol{bd = 20}$.
- $\boldsymbol{ad + bc = 23}$.
- $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are integers.

So the system is well defined. What do we mean by a system well defined, though?

If a problem or a system of equations is well defined, it is workable.

What do we mean by though, "*If any*" in the statement below?

If any, we can find the four integers $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ satisfying the equations below.
$\boldsymbol{ac = 6}$, $\boldsymbol{bd = 20}$, and $\boldsymbol{ad + bc = 23}$.

It means that it can be the case where there do not exist four integers that can be $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$. Then, we cannot get the factorization of $\boldsymbol{P}$ within the scope of integers.
What then do we do?

We want to use the quadratic formula, which is $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$, which is the solution to the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$.

Now the system is $\boldsymbol{ac = 6}$, $\boldsymbol{ad + bc = 23}$, and $\boldsymbol{bd = 20}$, where $\boldsymbol{a}$ and $\boldsymbol{c}$ are divisors of 6, and $\boldsymbol{b}$ and $\boldsymbol{d}$ are divisors of 20.

That is to say that $\boldsymbol{a}$ and $\boldsymbol{c}$ are integers the product of which is 6, $\boldsymbol{b}$ and $\boldsymbol{d}$ are integers the product of which is 20, and $\boldsymbol{ad + bc = 23}$.

And we can get the integers that can be $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ via trial and error.

Suppose first, $\boldsymbol{a = 1}$, $\boldsymbol{c = 6}$, $\boldsymbol{b = 4}$, and $\boldsymbol{d = 5}$.

Then, $\boldsymbol{ad + bc = 1 \cdot 5 + 4 \cdot 6 = 29}$, which is not 23, so we want to make another choice.

Suppose next, $\boldsymbol{a = 1}$, $\boldsymbol{c = 6}$, $\boldsymbol{b = 5}$, and $\boldsymbol{d = 4}$.

Then, $\boldsymbol{ad + bc = 1 \cdot 4 + 5 \cdot 6 = 34}$, which is not 23, so we want to make another choice.

Suppose next, $\boldsymbol{a = 2}$, $\boldsymbol{c = 3}$, $\boldsymbol{b = 5}$, and $\boldsymbol{d = 4}$.

Then, $\boldsymbol{ad + bc = 2 \cdot 4 + 5 \cdot 3}$, which is 23, so this time, we've got the right choice.

Now, we have $\boldsymbol{P = 6x^2 + 23x + 20 = acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)}$, in which $\boldsymbol{a = 2}$, $\boldsymbol{c = 3}$, $\boldsymbol{b = 5}$, and $\boldsymbol{d = 4}$.

So we get $\boldsymbol{P = 6x^2 + 23x + 20 = (ax + b)(cx + d) = (2x + 5)(3x + 4)}$.

And each of ($\boldsymbol{2x + 5}$) and ($\boldsymbol{3x + 4}$) is prime.
If an integer or polynomial is prime, it has no divisor other than 1 and itself.
Therefore, $\boldsymbol{P = 6x^2 + 23x + 20}$ is (fully) factorized to $\boldsymbol{(2x + 5)(3x + 4)}$.

So such a factorization procedure looks quite complicated, doesn't it?
Don't get discouraged though. The entire process can take time and effort, but it is often the case the right choice of the integers will be made far much more quickly than we think. It's just a matter of practice.

Now, what if we cannot find the integers that can be $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ anyway?

It can happen, of course. In that case, we want to consult the quadratic formula.
Using the formula, we can get the solution to this:

$\boldsymbol{\frac{1}{6}(6x^2 + 23x + 20) = 0}$, which means $\boldsymbol{\frac{1}{6}P = 0}$, which means $\boldsymbol{x^2 + \frac{23}{6}x + \frac{20}{6} = 0}$.

And the solution is $\boldsymbol{x = -\frac{5}{2}}$ or $\boldsymbol{-\frac{4}{3}}$, which is the solution to $\boldsymbol{(x + \frac{5}{2})(x + \frac{4}{3}) = 0}$.

Note that the solution above is to an equation $\boldsymbol{x^2 + \frac{23}{6}x + \frac{20}{6} = 0}$, which is $\boldsymbol{\frac{1}{6}P = 0}$.

Thus, we get

$\boldsymbol{(x + \frac{5}{2})(x + \frac{4}{3}) = \frac{1}{6}P}$, so $\boldsymbol{P = 6(x + \frac{5}{2})(x + \frac{4}{3}) = 2(x + \frac{5}{2})3(x + \frac{4}{3}) = (2x + 5)(3x + 4)}$.

In short:

$\boldsymbol{6x^2 + 23x + 20 = 2\cdot 3x^2 + (2\cdot 4 + 3\cdot 5)x + 4\cdot 5 = 2\cdot 3x^2 + 2\cdot 4x + 3\cdot 5x + 4\cdot 5}$

$\boldsymbol{= 2x(3x + 4) + 5(3x + 4) = (3x + 4)(2x + 5)}$.

Suggestions or Solutions

To the Problem in the Example 1

We have $\mathbf{28}\boldsymbol{x}^{\mathbf{2}} + \mathbf{71}\boldsymbol{x} + \mathbf{18}$.

Factorizing it, we can get it the way below.

$$28x^2 + 71x + 18 = 4\cdot7x^2 + (7\cdot9 + 2\cdot4)x + 2\cdot9 = 4\cdot7x^2 + 7\cdot9x + 2\cdot4x + 2\cdot9$$

$$= 7x(4x + 9) + 2(4x + 9) = (4x + 9)(7x + 2).$$

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $P = 28x^2 + 71x + 18$, we can say that the polynomial P takes the form of a polynomial $ux^2 + vx + w$, which is similar to $acx^2 + (ad + bc)x + bd$, which is factorize to $(ax + b)(cx + d)$.

So taking ac as u, $ad + bc$ as v, and bd as w, we get $ux^2 + vx + w$, which P is in the form of. So P can be said to be similar to $acx^2 + (ad + bc)x + bd$, factorize to $(ax + b)(cx + d)$.

And we want a, b, c, and d to be integers.
That is to say that we want to factorize P within the scope of integers.

Assuming thus, a, b, c, and d are integers, setting $ac = 28$, $bd = 18$, and $ad + bc = 71$, and then, finding a, b, c, and d, we can get the factorization of P.

So we want to solve a system where $ac = 28$, $ad + bc = 71$, and $bd = 18$, where a and c are divisors of 28, and b and d are divisors of 18.

That is to say that a and c are integers the product of which is 28, b and d are integers the product of which is 18, and $ad + bc = 71$.

So a and c are divisors of 28, and b and d are divisors of 18.

And if any, we can get the integers that can be $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ via trial and error.

Suppose first, $\boldsymbol{a = 1}$, $\boldsymbol{c = 28}$, $\boldsymbol{b = 2}$, and $\boldsymbol{d = 9}$.
Then, $\boldsymbol{ad + bc = 1 \cdot 9 + 2 \cdot 28 = 65}$, which is not 71, so we want to make another choice.

Suppose next, $\boldsymbol{a = 2}$, $\boldsymbol{c = 14}$, $\boldsymbol{b = 3}$, and $\boldsymbol{d = 6}$.
Then, $\boldsymbol{ad + bc = 2 \cdot 6 + 3 \cdot 14 = 54}$, which is not 71, so we want to make another choice.

Suppose next, $\boldsymbol{a = 7}$, $\boldsymbol{c = 4}$, $\boldsymbol{b = 2}$, and $\boldsymbol{d = 9}$.
Then, $\boldsymbol{ad + bc = 7 \cdot 9 + 2 \cdot 4}$, which is 71, so this time, we've made the right choice.

Now, we have this: $\boldsymbol{P = 28x^2 + 71x + 18 = acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)}$,
where $\boldsymbol{a = 7}$, $\boldsymbol{c = 4}$, $\boldsymbol{b = 2}$, and $\boldsymbol{d = 9}$.

So we get $\boldsymbol{P = 28x^2 + 71x + 18 = (ax + b)(cx + d) = (7x + 2)(4x + 9)}$.

And each of ($\boldsymbol{7x + 2}$) and ($\boldsymbol{4x + 9}$) is prime.
If an integer or polynomial is prime, it has no divisor other than 1 and itself.

Therefore, $\boldsymbol{P = 28x^2 + 71x + 18}$ is (fully) factorized to $\boldsymbol{(7x + 2)(4x + 9)}$.

In short:

$\boldsymbol{28x^2 + 71x + 18 = 4 \cdot 7x^2 + (7 \cdot 9 + 2 \cdot 4)x + 2 \cdot 9 = 4 \cdot 7x^2 + 7 \cdot 9x + 2 \cdot 4x + 2 \cdot 9}$

$\boldsymbol{= 7x(4x + 9) + 2(4x + 9) = (4x + 9)(7x + 2)}$.