Examples D in Polynomial Factorizations

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Examples D

Factorize the polynomials below.

0. $x^3 + 10x^2 + 31x + 30$

1. $x^3 - 10x^2 + 31x - 30$

2. $x^3 + 4x^2 - x - 4$

Suggestions or Solutions

To the Problem in the Example 0

We have $\boldsymbol{x^3 + 10x^2 + 31x + 30}$.

Factorizing it, we can get it the way below.

Setting first, $\boldsymbol{P = x^3 + 10x^2 + 31x + 30}$, and evaluating $\boldsymbol{P}$ for $\boldsymbol{x = -2}$, we get

$(-2)^3 + 10(-2)^2 + 31(-2) + 30 = -8 + 40 - 62 + 30 = 0$.

So $\boldsymbol{x + 2}$ is a divisor, that is, a factor of $\boldsymbol{P}$.

Thus next, applying to $\boldsymbol{P}$ the synthetic division by $\boldsymbol{(x + 2)}$, we get

$$\begin{array}{r|rrrr} -2 & 1 & 10 & 31 & 30 \\ & & -2 & -16 & -30 \\ \hline & 1 & 8 & 15 & 0 \end{array}$$

So we get $\boldsymbol{P = x^3 + 10x^2 + 31x + 30 = (x + 2)(x^2 + 8x + 15) = (x + 2)(x + 3)(x + 5)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P = x^3 + 10x^2 + 31x + 30}$.

Unlike the polynomials in the previous examples, the polynomial $\boldsymbol{P}$ is of degree 3, and doesn't seem to have any divisor that we can quickly see.

If $\boldsymbol{P}$ can get factorized though, it will probably be a product of polynomials of degree 2 or 1. And we have covered factorizations of polynomials of degree 2.

So let's go over factorizations of polynomials of degree 2.

To begin with, a polynomial $\boldsymbol{x^2 + 2xy + y^2}$ gets factorized to $\boldsymbol{(x + y)^2}$, which is called a complete square.

Next, $\boldsymbol{x^2 + ux + v}$ can be factorized to $\boldsymbol{(x + m)(x + n)}$ where $\boldsymbol{u = m + n}$ and $\boldsymbol{v = mn}$. Next, $\boldsymbol{sx^2 + tx + w}$ can be factorized to $\boldsymbol{(ax + b)(cx + d)}$ where $\boldsymbol{s = ac}$, $\boldsymbol{t = ad + bc}$, and $\boldsymbol{w = bd}$.

So we can see a pattern that a polynomial of degree 2 can get factorized to a product of two factors, each of which is of degree 1.

What then, about the factorizations of a polynomial of degree 3?

We can expect that a polynomial of degree 3 can get factorized to the form of a polynomial as $\boldsymbol{(x + a)(x + b)(x + c)}$.

More specifically, we can expect the polynomial to get factorized first, to a product of a polynomial of degree 1 as $\boldsymbol{x + d}$ and a polynomial of degree 2 as $\boldsymbol{x^2 + ux + v}$, which in tern, can be factorized to $\boldsymbol{(x + m)(x + n)}$, so eventually, the polynomial of degree 3 can be factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$.

In fact, we have a factorization identity where $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$

$\boldsymbol{= (x + a)(x + b)(x + c)}$.

So if factorizing a polynomial in the form of $\boldsymbol{x^3 + ux^2 + vx + w}$, assuming that it can be factorized to a form of $\boldsymbol{(x + a)(x + b)(x + c)}$, we can try finding the values of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

How come?

We know that the polynomial $\boldsymbol{P}$ is similar to $\boldsymbol{x^3 + ux^2 + vx + w}$, which is again, similar to $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$, which gets factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$.

So we can expect that $\boldsymbol{P}$ gets factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are integers if $\boldsymbol{P}$ gets factorized within the scope of integers.

Then, $\boldsymbol{(x + a)}$, $\boldsymbol{(x + b)}$, and $\boldsymbol{(x + c)}$ are factors of $\boldsymbol{P}$.

If setting thus, $\boldsymbol{P = x^3 + 10x^2 + 31x + 30 = (x + a)(x + b)(x + c)}$, we can try finding $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, assuming they are integers, i.e., $\boldsymbol{P}$ gets factorized within the scope of integers.

How can we find them though?

Expanding $\boldsymbol{(x + a)(x + b)(x + c)}$, we get $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$.

And we have set $\boldsymbol{x^3 + 10x^2 + 31x + 30 = (x + a)(x + b)(x + c)}$.

So we can set $\boldsymbol{P = x^3 + 10x^2 + 31x + 30 = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$.

Comparing thus, both sides of the equality above term by term, we get $\boldsymbol{30 = abc}$. So we can notice that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ can be divisors of 30, and so can the negatives of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Now, getting back to $\boldsymbol{x^3 + 10x^2 + 31x + 30 = (x + a)(x + b)(x + c)}$, we can see these:

$\boldsymbol{(-a)^3 + 10(-a)^2 + 31(-a) + 30 = (-a + a)(-a + b)(-a + c) = 0}$ if $\boldsymbol{x = -a}$.

$\boldsymbol{(-b)^3 + 10(-b)^2 + 31(-b) + 30 = (-b + a)(-b + b)(-b + c) = 0}$ if $\boldsymbol{x = -b}$.

$\boldsymbol{(-c)^3 + 10(-c)^2 + 31(-c) + 30 = (-c + a)(-c + b)(-c + c) = 0}$ if $\boldsymbol{x = -c}$.

And we assume that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are divisors of 30, and so are the negatives of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

So putting a divisor of 30 into $\boldsymbol{x}$ in $\boldsymbol{P}$, and getting $\boldsymbol{P = 0}$, we can say that the ***negative*** of the divisor is one of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

For instance, we can take it as $\boldsymbol{a}$. Then, $\boldsymbol{x + a}$ is a factor of $\boldsymbol{P}$, since it divides $\boldsymbol{P}$, and it is in fact, one of the three factors of $\boldsymbol{P}$.

And the three are $\boldsymbol{(x + a)}$, $\boldsymbol{(x + b)}$, and $\boldsymbol{(x + c)}$, since each divides $\boldsymbol{P}$.

How then, can we get the other two factors?

We know $(x + a)$, $(x + b)$, and $(x + c)$ are factors of P.

And factors are divisors. So $x + a$ can divide P. What then, is the quotient?

We have $P = x^3 + 10x^2 + 31x + 30 = (x + a)(x + b)(x + c)$.

So dividing P by $x + a$, we get $(x + b)(x + c)$, which is thus, the quotient.

The quotient will not be however, in the form of $(x + b)(x + c)$, of course.
It will be in the form of $x^2 + mx + n$, which will be though, factorized to $(x + b)(x + c)$.

So let's now get the factorization of P using the method above.

First, the divisors of 30 are ±1, ±2, ±3, ±6, ±10, ±15, and ±30.

Next, putting a divisor of 30 into x in $x^3 + 10x^2 + 31x + 30$, and getting 0, we can say that the *negative* of the divisor is the value of one of a, b, and c.

So if P can be factorized within the scope of integers, we will get to find an integer that makes P be $\mathbf{0}$, and then, the integer is one of the divisors of 30 in this case, and the *negative* of the integer is one of a, b, and c.

Now, evaluating $P = x^3 + 10x^2 + 31x + 30$ for $x = -2$, we get
$(-2)^3 + 10(-2)^2 + 31(-2) + 30 = -8 + 40 – 62 + 30 = 0$.

We know that we have set $P = x^3 + 10x^2 + 31x + 30 = (x + a)(x + b)(x + c)$.
So at least one of a, b, and c is 2.

And thus, we can now see that $x + 2$ is a factor of $P = x^3 + 10x^2 + 31x + 30$.
So next, applying to P the synthetic division by $(x + 2)$, we get

$$
\begin{array}{r|rrrr}
-2 & 1 & 10 & 31 & 30 \\
 & & -2 & -16 & -30 \\
\hline
 & 1 & 8 & 15 & 0
\end{array}
$$

Thus, dividing $x^3 + 10x^2 + 31x + 30$ by $x + 2$, we get $x^2 + 8x + 15$ as the quotient.

So we get $P = x^3 + 10x^2 + 31x + 30 = (x + 2)(x^2 + 8x + 15)$.

And thus, $x + 2$ is a factor of P, and is one of the three factors of P.

What then, is the quotient $x^2 + 8x + 15$?

The quotient $x^2 + 8x + 15$ is the product of the other two factors of P.

And in fact, $x^2 + 8x + 15$ is factorized to $(x + 3)(x + 5)$.

So we get $P = x^3 + 10x^2 + 31x + 30 = (x + 2)(x^2 + 8x + 15) = (x + 2)(x + 3)(x + 5)$.

And each of $x + 2$, $x + 3$, and $x + 5$ is prime.

Therefore, $P = x^3 + 10x^2 + 31x + 30$ gets (fully) factorized to $(x + 2)(x + 3)(x + 5)$.

In short:

Setting first, $P = x^3 + 10x^2 + 31x + 30$, and evaluating P for $x = -2$, we get

$(-2)^3 + 10(-2)^2 + 31(-2) + 30 = -8 + 40 – 62 + 30 = 0$.

So $x + 2$ is a divisor, that is, a factor of P.

Thus next, applying to P the synthetic division by $(x + 2)$, we get

$$
\begin{array}{r|rrrr}
-2 & 1 & 10 & 31 & 30 \\
 & & -2 & -16 & -30 \\
\hline
 & 1 & 8 & 15 & 0
\end{array}
$$

So we get $P = x^3 + 10x^2 + 31x + 30 = (x + 2)(x^2 + 8x + 15) = (x + 2)(x + 3)(x + 5)$.

Suggestions or Solutions
To the **Problem** in the Example 1

We have $\boldsymbol{x^3 - 10x^2 + 31x - 30}$.

Factorizing it, we can get it the way below.

Setting first, $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$, and evaluating $\boldsymbol{P}$ for $\boldsymbol{x = 2}$, we get

$2^3 - 10 \cdot 2^2 + 31 \cdot 2 - 30 = 8 - 40 + 62 - 30 = 0$.

So $\boldsymbol{x - 2}$ is a divisor, that is, a factor of $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$.

Next, applying to $\boldsymbol{P}$ the synthetic division by ($\boldsymbol{x - 2}$), we get

$$
\begin{array}{r|rrrr}
2 & 1 & -10 & 31 & -30 \\
  &   & 2 & -16 & 30 \\
\hline
  & 1 & -8 & 15 & 0
\end{array}
$$

Thus, we get $\boldsymbol{x^3 - 10x^2 + 31x - 30 = (x - 2)(x^2 - 8x + 15) = (x - 2)(x - 3)(x - 5)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$, we can say that the polynomial $\boldsymbol{P}$ is similar to $\boldsymbol{x^3 + ux^2 + vx + w}$, which is again, similar to $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$, which gets factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$.

So we can expect that $\boldsymbol{P}$ gets factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are integers.

Setting thus, $\boldsymbol{P = x^3 - 10x^2 + 31x - 30 = (x + a)(x + b)(x + c)}$, we can try finding $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, assuming they are integers, i.e., $\boldsymbol{P}$ gets factorized within the scope of integers.

How can we find them though?

Expanding $\boldsymbol{(x + a)(x + b)(x + c)}$, we get $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$.

And we have set $\boldsymbol{P = x^3 - 10x^2 + 31x - 30 = (x + a)(x + b)(x + c)}$.

So we can set $\boldsymbol{P = x^3 - 10x^2 + 31x - 30 = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$.

Comparing therefore, both sides of the equal sign term by term, we get $\boldsymbol{-30 = abc}$. So we can say that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ can be divisors of -30, and so can the negatives of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Now, getting back to $\boldsymbol{x^3 - 10x^2 + 31x - 30 = (x + a)(x + b)(x + c)}$, we can see these:

$\boldsymbol{(-a)^3 - 10(-a)^2 + 31(-a) - 30 = (-a + a)(-a + b)(-a + c) = 0}$ if $\boldsymbol{x = -a}$.

$\boldsymbol{(-b)^3 - 10(-b)^2 + 31(-b) - 30 = (-b + a)(-b + b)(-b + c) = 0}$ if $\boldsymbol{x = -b}$.

$\boldsymbol{(-c)^3 - 10(-c)^2 + 31(-c) - 30 = (-c + a)(-c + b)(-c + c) = 0}$ if $\boldsymbol{x = -c}$.

And we assume that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are divisors of -30, and so are the negatives of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

So putting a divisor of -30 into $\boldsymbol{x}$ in $\boldsymbol{P}$, and getting $\boldsymbol{P = 0}$, we can say that the negative of the divisor is one of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. For instance, we can take it as $\boldsymbol{a}$.

How then, can we get the other two integers $\boldsymbol{b}$ and $\boldsymbol{c}$?

We know $\boldsymbol{(x + a)}$, $\boldsymbol{(x + b)}$, and $\boldsymbol{(x + c)}$ are factors of $\boldsymbol{P}$.

And factors are divisors. So $\boldsymbol{x + a}$ can divide $\boldsymbol{P}$. What then, is the quotient?

We have $\boldsymbol{P = x^3 - 10x^2 + 31x - 30 = (x + a)(x + b)(x + c)}$.

So dividing $\boldsymbol{P}$ by $\boldsymbol{x + a}$, we get $\boldsymbol{(x + b)(x + c)}$, which is thus, the quotient.

The quotient will be however, in the form of $\boldsymbol{x^2 + mx + n}$, which will be though, factorized to $\boldsymbol{(x + b)(x + c)}$.

So let's now get the factorization of $\boldsymbol{P}$ using the method above.

First, the divisors of -30 are ±1, ±2, ±3, ±6, ±10, ±15, and ±30.

Next, putting one of the divisors into $\boldsymbol{x}$ in $\boldsymbol{x^3 - 10x^2 + 31x - 30}$, and getting 0, we can say that the one is the value of one of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

So if $\boldsymbol{P}$ can be factorized within the scope of integers, we will get to find an integer that can give us $\boldsymbol{P = 0}$, and the integer is one of the divisors of -30 in this case.

Now, evaluating $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$ for $\boldsymbol{x = 2}$, we get

$2^3 - 10 \cdot 2^2 + 31 \cdot 2 - 30 = 8 - 40 + 62 - 30 = 0$.

And we know that we have set $\boldsymbol{P = x^3 - 10x^2 + 31x - 30 = (x + a)(x + b)(x + c)}$.

So at least one of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is -2.

And thus, we can now see that $\boldsymbol{x - 2}$ is a factor of $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$.

So next, applying to $\boldsymbol{P}$ the synthetic division by $(\boldsymbol{x - 2})$, we get

$$
\begin{array}{r|rrrr}
2 & 1 & -10 & 31 & -30 \\
 & & 2 & -16 & 30 \\
\hline
 & 1 & -8 & 15 & 0
\end{array}
$$

Thus, dividing $\boldsymbol{x^3 - 10x^2 + 31x - 30}$ by $\boldsymbol{x - 2}$, we get $\boldsymbol{x^2 - 8x + 15}$ as the quotient.

So we get $\boldsymbol{x^3 - 10x^2 + 31x - 30 = (x - 2)(x^2 - 8x + 15)}$.

Next, we want to factorize the quotient $\boldsymbol{x^2 - 8x + 15}$.

In fact, $\boldsymbol{x^2 - 8x + 15}$ is factorized to $\boldsymbol{(x - 3)(x - 5)}$.

So we get $\boldsymbol{x^3 - 10x^2 + 31x - 30 = (x - 2)(x^2 - 8x + 15) = (x - 2)(x - 3)(x - 5)}$.

And each of $\boldsymbol{x - 2}$, $\boldsymbol{x - 3}$, and $\boldsymbol{x - 5}$ is prime.

Therefore, $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$ gets (fully) factorized to $\boldsymbol{(x - 2)(x - 3)(x - 5)}$.

Notice that replacing -2, -3, and -5 above with 2, 3, and 5 respectively, and using the identity where $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc = (x + a)(x + b)(x + c)}$, we get

$\boldsymbol{(x + 2)(x + 3)(x + 5) = x^3 + (2 + 3 + 5)x^2 + (6 + 15 + 10)x + 30 = x^3 + 10x^2 + 31x + 30}$.

So we can see at once, $\boldsymbol{x^3 + 10x^2 + 31x + 30}$ gets fully factorized to $\boldsymbol{(x + 2)(x + 3)(x + 5)}$.

In short:

Setting first, $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$, and evaluating $\boldsymbol{P}$ for $\boldsymbol{x = 2}$, we get

$2^3 - 10 \cdot 2^2 + 31 \cdot 2 - 30 = 8 - 40 + 62 - 30 = 0$.

So $\boldsymbol{x - 2}$ is a divisor, that is, a factor of $\boldsymbol{P = x^3 - 10x^2 + 31x - 30}$.

Next, applying to $\boldsymbol{P}$ the synthetic division by ($\boldsymbol{x - 2}$), we get

$$\begin{array}{r|rrrr} 2 & 1 & -10 & 31 & -30 \\ & & 2 & -16 & 30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

Thus, we get $\boldsymbol{x^3 - 10x^2 + 31x - 30 = (x - 2)(x^2 - 8x + 15) = (x - 2)(x - 3)(x - 5)}$.

Suggestions or Solutions
To the Problem in the Example 2

We have $x^3 + 4x^2 - x - 4$.

Factorizing it, we can get it the way below.

Setting first, $P = x^3 + 4x^2 - x - 4$, and evaluating P for $x = 1$, we get $1 + 4 - 1 - 4 = 0$, so $x - 1$ is a factor, and thus, applying to P the synthetic division by $x - 1$, we get

$$\begin{array}{r|rrrr} 1 & 1 & 4 & -1 & -4 \\ & & 1 & 5 & 4 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

Thus, we get $x^3 + 4x^2 - x - 4 = (x - 1)(x^2 + 5x + 4) = (x - 1)(x + 1)(x + 4)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $P = x^3 + 4x^2 - x - 4$, we can say that the polynomial P is similar to the polynomial $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$, factorized to $(x + a)(x + b)(x + c)$.

So we can expect that P gets factorized to $(x + a)(x + b)(x + c)$, where a, b, and c are integers.

Setting thus, $P = x^3 + 4x^2 - x - 4 = (x + a)(x + b)(x + c)$, we can try finding a, b, and c, assuming they are integers, that is, P gets factorized within the scope of integers.

Next, we have $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

And we have set $P = x^3 + 4x^2 - x - 4 = (x + a)(x + b)(x + c)$.

So we can set $P = x^3 + 4x^2 - x - 4 = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

Comparing therefore, both sides of the equal sign term by term, we get $-4 = abc$. So we can say that a, b, and c can be divisors of -4, and so can the negatives of a, b, and c.

Now, getting back to $x^3 + 4x^2 - x - 4 = (x + a)(x + b)(x + c)$, we can see this:

$$(-a)^3 + 4(-a)^2 - (-a) - 4 = (-a + a)(-a + b)(-a + c) = 0 \text{ if } x = -a.$$

And we assume that a is a divisor of -4, and so is the negative of a.

So putting a divisor of -4 into x in P, and getting $P = 0$, we can say that the negative of the divisor can be a. How then can we get the other two, which are b and c?

We know $(x + a)$, $(x + b)$, and $(x + c)$ are factors of P.

And factors are divisors. So $x + a$ can divide P.

And dividing P by $x + a$, we get $(x + b)(x + c)$, which is thus, the quotient.

The quotient will be however, in the form of of $x^2 + mx + n$, which will be though, factorized to $(x + b)(x + c)$.

So let's now get the factorization of P using the method above.

First, the divisors of -4 are ±1, ±2, and ±4.

Next, evaluating $P = x^3 + 4x^2 - x - 4$ for $x = 1$, we get $1 + 4 - 1 - 4 = 0$.

So $x - 1$ is a factor of P, and applying to P the synthetic division by $x - 1$, we get

$$\begin{array}{r|rrrr} 1 & 1 & 4 & -1 & -4 \\ & & 1 & 5 & 4 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

So we get $x^3 + 4x^2 - x - 4 = (x - 1)(x^2 + 5x + 4) = (x - 1)(x + 1)(x + 4)$.

And each of $x - 1$, $x + 1$, and $x + 4$ is prime.

Therefore, $P = x^3 + 4x^2 - x - 4$ gets fully factorized to $(x - 1)(x + 1)(x + 4)$.