Examples E in Polynomial Factorizations

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Examples E

Factorize the polynomials below.

0. $\boldsymbol{x^3 + 2x^2 - 7x + 4}$

1. $\boldsymbol{x^3 - 3x^2 + 3x - 1}$.

2. $\boldsymbol{6x^3 - 5x^2 - 12x - 4}$

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{x^3 + 2x^2 - 7x + 4}$.

Factorizing it, we can get it the way below.

Setting first, $\boldsymbol{P = x^3 + 2x^2 - 7x + 4}$, and evaluating $\boldsymbol{P}$ for $\boldsymbol{x = 1}$, we get $1 + 2 - 7 + 4 = 0$, so $\boldsymbol{x - 1}$ is a factor, and thus, applying to $\boldsymbol{P}$ the synthetic division by $\boldsymbol{x - 1}$, we get

$$
\begin{array}{r|rrrr}
1 & 1 & 2 & -7 & 4 \\
  &   & 1 & 3 & -4 \\
\hline
  & 1 & 3 & -4 & 0
\end{array}
$$

Thus, we get

$\boldsymbol{x^3 + 2x^2 - 7x + 4 = (x - 1)(x^2 + 3x - 4) = (x - 1)(x - 1)(x + 4) = (x - 1)^2(x + 4)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = x^3 + 2x^2 - 7x + 4}$, we can say that $\boldsymbol{P}$ is similar to the polynomial $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$, factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$.

So we can expect that $\boldsymbol{P}$ gets factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are integers.

Setting thus, $\boldsymbol{P = x^3 + 2x^2 - 7x + 4 = (x + a)(x + b)(x + c)}$, we can try finding $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, assuming that they are integers, that is, $\boldsymbol{P}$ gets factorized within the scope of integers.

Now, expanding $\boldsymbol{(x + a)(x + b)(x + c)}$, we get $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$.

So we can set $\boldsymbol{P = x^3 + 2x^2 - 7x + 4 = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$.

Comparing therefore, the terms in both sides of the equal sign, we get $\mathbf{4 = abc}$. So we can assume that a, b, and c are divisors of 4, and so are the negatives of a, b, and c.

Now, getting back to $x^3 + 2x^2 - 7x + 4 = (x + a)(x + b)(x + c)$, we can see this:

$(-a)^3 + 2(-a)^2 - 7(-a) + 4 = (-a + a)(-a + b)(-a + c) = 0$ if $x = -a$.

And we assume that a is a divisor of 4, and so is $-a$.

So putting a divisor of 4 into x in P, and getting $P = 0$, we can say that the negative of the divisor can be a. How then, can we get the other two, which are b and c?

We know $(x + a)$, $(x + b)$, and $(x + c)$ are factors of P. So $x + a$ can divide P.

And dividing P by $x + a$, we get $(x + b)(x + c)$, which is thus, the quotient.

The quotient will be however, in the form of of $x^2 + mx + n$, which will be though, factorized to $(x + b)(x + c)$.

So let's now get the factorization of P using the method above.

First, the divisors of 4 are ±1, ±2, and ±4.

Next, evaluating $P = x^3 + 2x^2 - 7x + 4$ for $x = 1$, we get 1 + 2 – 7 + 4 = 0.

So $x - 1$ is a factor of P, and applying to P the synthetic division by $x - 1$, we get

$$
\begin{array}{r|rrrr}
1 & 1 & 2 & -7 & 4 \\
  &   & 1 & 3 & -4 \\
\hline
  & 1 & 3 & -4 & 0
\end{array}
$$

So we get

$x^3 + 2x^2 - 7x + 4 = (x - 1)(x^2 + 3x - 4) = (x - 1)(x - 1)(x + 4) = (x - 1)^2(x + 4)$.

Suggestions or Solutions

To the Problem in the Example 1

We have $\boldsymbol{x^3 - 3x^2 + 3x - 1}$.

Factorizing it, we can get it the way below.

Setting first, $\boldsymbol{P = x^3 - 3x^2 + 3x - 1}$, and evaluating $\boldsymbol{P}$ for $\boldsymbol{x = 1}$, we get $1 - 3 + 3 - 1 = 0$, so $\boldsymbol{x - 1}$ is a factor, and thus, applying to $\boldsymbol{P}$ the synthetic division by $\boldsymbol{x - 1}$, we get

$$
\begin{array}{r|rrrr}
1 & 1 & -3 & 3 & -1 \\
 & & 1 & -2 & 1 \\
\hline
 & 1 & -2 & 1 & 0
\end{array}
$$

Thus, we get $\boldsymbol{x^3 - 3x^2 + 3x - 1 = (x - 1)(x^2 - 2x + 1) = (x - 1)(x - 1)^2 = (x - 1)^3}$.

In fact, we have factorization identities as follows.

$\boldsymbol{x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3}$.

$\boldsymbol{x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3}$.

And putting both together, we can set $\boldsymbol{x^3 \pm 3x^2y + 3xy^2 \pm y^3 = (x \pm y)^3}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = x^3 - 3x^2 + 3x - 1}$, we can say that the polynomial $\boldsymbol{P}$ is similar to $\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$, which gets factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$.

So we can expect $\boldsymbol{P}$ to get factorized to $\boldsymbol{(x + a)(x + b)(x + c)}$.

Setting thus, $\boldsymbol{P = x^3 - 3x^2 + 3x - 1 = (x + a)(x + b)(x + c)}$, we can try finding $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, assuming they are integers, that is, $\boldsymbol{P}$ gets factorized within the scope of integers.

Now, expanding $(x + a)(x + b)(x + c)$, we get $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

So we can set $P = x^3 - 3x^2 + 3x - 1 = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

Comparing therefore, both sides of the equal sign term by term, we get $-1 = abc$. So we can assume that a, b, and c are divisors of -1, and so are the negatives of a, b, and c.

Now, getting back to $x^3 - 3x^2 + 3x - 1 = (x + a)(x + b)(x + c)$, we can see this:

$(-a)^3 + 2(-a)^2 - 7(-a) + 4 = (-a + a)(-a + b)(-a + c) = 0$ if $x = -a$.

And we assume that a is a divisor of -1, and so is $-a$.

So putting a divisor of -1 into x in P, and getting $P = 0$, we can say that the negative of the divisor can be a.

And we know $(x + a)$, $(x + b)$, and $(x + c)$ are factors of P. So $x + a$ can divide P.

And dividing P by $x + a$, we get $(x + b)(x + c)$, which is thus, the quotient.

The quotient will be however, in the form of of $x^2 + mx + n$, which will be though, factorized to $(x + b)(x + c)$.
So let's now get the factorization of P using the method above.

First, the divisors of -1 are ±1.

Next, evaluating $P = x^3 - 3x^2 + 3x - 1$ for $x = 1$, we get $1 - 3 + 3 - 1 = 0$.

So $x - 1$ is a factor of P, and applying to P the synthetic division by $x - 1$, we get

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

So we get

$x^3 - 3x^2 + 3x - 1 = (x - 1)(x^2 - 2x + 1) = (x - 1)(x - 1)(x - 1) = (x - 1)^3$.

Suggestions or Solutions
To the Problem in the Example 2

We have $\mathbf{6x^3 - 5x^2 - 12x - 4}$.

Factorizing it, we can get the way below.

Evaluating $\mathbf{6x^3 - 5x^2 - 12x - 4}$ for $x = -\frac{2}{-1} = \mathbf{2}$, we get 48 – 20 – 24 – 4 = 0, so we get

2	6	-5	-12	-4
		12	14	4
	6	7	2	0

Thus, we get $\mathbf{6x^3 - 5x^2 - 12x - 4 = (x - 2)(6x^2 + 7x + 2) = (x - 2)(2x + 1)(3x + 2)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P} = \mathbf{6x^3 - 5x^2 - 12x - 4}$.

The polynomial $\boldsymbol{P}$ is a bit different from the previous one.

Unlike the preceding examples, the coefficient in the highest term is not 1 but 6.
So this time, it's not the case where we just go ahead and use the identity below.

$\boldsymbol{x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc = (x + a)(x + b)(x + c)}$.

So let's now look for some other way we can try.

We know a polynomial $\boldsymbol{acx^2 + (ad + bc)x + bd}$ gets factorize to $\boldsymbol{(ax + b)(cx + d)}$.

So we can reasonably expect that a polynomial $\boldsymbol{sx^3 + tx^2 + ux + v}$ can get factorized to $\boldsymbol{(ax + b)(cx + d)(ex + g)}$.

In fact, expanding $(ax + b)(cx + d)(ex + g)$, we get

$acex^3 + (acg + \text{etc.})x^2 + (adg + \text{etc.})x + bdg$, which is quite similar to $sx^3 + tx^2 + ux + v$.

So assuming $sx^3 + tx^2 + ux + v$ gets factorized to $(ax + b)(cx + d)(ex + g)$, we can set

$sx^3 + tx^2 + ux + v = acex^3 + (acg + \text{etc.})x^2 + (adg + \text{etc.})x + bdg$.

What then, do we get?

We can get $s = ace$, and $v = bdg$.   So?

Assuming a, c, and e are divisors of s, and b, d, and g are divisors of v, and finding a, b, c, d, e, and g by the similar method applied to $x^3 + ux^2 + vx + w$, we could get the factorization of $sx^3 + tx^2 + ux + v$.

How then, can we find a, b, c, d, e, and g?

Setting $A = sx^3 + tx^2 + ux + v = (ax + b)(cx + d)(ex + g)$, we can see this:

$A = 0$ when $x = -\frac{b}{a}$, $-\frac{d}{c}$, or $-\frac{g}{e}$.

Thus, if $m = \frac{b}{a}$, $\frac{d}{c}$, or $\frac{g}{e}$, we can use $x + m$ as a factor of A.   How come?

We can put $(ax + b)(cx + d)(ex + g)$ the way below.

$(ax + b)(cx + d)(ex + g) = a(x + \frac{b}{a})c(x + \frac{d}{c})e(x + \frac{g}{e}) = ace(x + \frac{b}{a})(x + \frac{d}{c})(x + \frac{g}{e})$.

So we can now set $\boldsymbol{A = sx^3 + tx^2 + ux + v = ace(x+\frac{b}{a})(x+\frac{d}{c})(x+\frac{g}{e})}$.

And thus, we can say that $(x+\frac{b}{a})$, $(x+\frac{d}{c})$, and $(x+\frac{g}{e})$ are factors of $\boldsymbol{A}$.

Suppose now that we have found $\frac{b}{a}$, for instance.

Then, applying to $\boldsymbol{A}$ the synthetic division by $(x+\frac{b}{a})$, we get as the quotient a polynomial of degree 2, which is equivalent to $ace(x+\frac{d}{c})(x+\frac{g}{e})$, of course.

What then about the other two factors $(x+\frac{d}{c})$ and $(x+\frac{g}{e})$?

We have $ace(x+\frac{d}{c})(x+\frac{g}{e}) = ac(x+\frac{d}{c})e(x+\frac{g}{e}) = a(cx+d)(ex+g)$.

And we know how to get $a(cx + d)(ex + g)$ as a factorization.

So we will eventually get $(x+\frac{b}{a})a(cx+d)(ex+g) = (ax + b)(cx + d)(ex + g)$.

And thus, factorizing a polynomial as $sx^3 + tx^2 + ux + v$, we can begin with assuming $A = sx^3 + tx^2 + ux + v = (ax + b)(cx + d)(ex + g)$. What then is the next?

We want to find the value of x that makes $A = 0$. How then, can we find the value of x?

We know that setting $A = sx^3 + tx^2 + ux + v = (ax + b)(cx + d)(ex + g)$, we get

$A = 0$ if $x = -\frac{b}{a}, -\frac{d}{c}$, or $-\frac{g}{e}$. And we assume that $s = ace$, and $v = bdg$.
So first, we can assume that a, c, and e are divisors of s, and b, d, and g are divisors of v.

And by trial and error, coming up with, for instance, $-\frac{b}{a}$ that makes $\boldsymbol{A} = \mathbf{0}$, we can get a factor of $\boldsymbol{A}$. And the factor is $\boldsymbol{x + \frac{b}{a}}$.

And next, we can get the other factors if factorizing the quotient we get after applying to $\boldsymbol{A}$ the synthetic division by the factor.

And of course, if the quotient is prime, we can use the quotient itself as the other factor.

So let's now, factorize $\boldsymbol{P = 6x^3 - 5x^2 - 12x - 4}$ using the idea above.

Then, first, we want to find a value of $\boldsymbol{x}$ that makes $\boldsymbol{P = 0}$. And the value is a fraction where the numerator is a divisor of -4, and the denominator is a divisor of 6.

(The divisors of 6 are ±1, ±2, ±3, and ±6, and the divisors of -4 are ±1, ±2, and ±4.)

So for instance, using as a numerator, 2 (a divisor of -4), and as a denominator, -1 (a divisor of 6), we get $\boldsymbol{x = -\frac{2}{-1} = \frac{2}{1} = 2}$.

Thus next, putting **2** into $\boldsymbol{x}$ in the polynomial $\boldsymbol{P}$, we get

$$\boldsymbol{P = 6x^3 - 5x^2 - 12x - 4 = 6 \cdot 2^3 - 5 \cdot 2^2 - 12 \cdot 2 - 4 = 48 - 20 - 24 - 4 = 0}.$$

So we can see that $\boldsymbol{x - 2}$ is a factor of $\boldsymbol{P}$.

Thus next, applying to $\boldsymbol{P}$ the synthetic division by the factor $\boldsymbol{x - 2}$, we get

$$\begin{array}{r|rrrr} 2 & 6 & -5 & -12 & -4 \\ & & 12 & 14 & 4 \\ \hline & 6 & 7 & 2 & 0 \end{array}$$

So we get $\boldsymbol{6x^3 - 5x^2 - 12x - 4 = (x - 2)(6x^2 + 7x + 2)}$.

And in fact, $\boldsymbol{6x^2 + 7x + 2}$ gets factorized to $\boldsymbol{(2x + 1)(3x + 2)}$.

Thus, we get $\boldsymbol{6x^3 - 5x^2 - 12x - 4 = (x - 2)(6x^2 + 7x + 2) = (x - 2)(2x + 1)(3x + 2)}$.

In short:

Evaluating $\mathbf{6x^3 - 5x^2 - 12x - 4}$ for $\boldsymbol{x = -\frac{2}{-1} = 2}$, we get 48 – 20 – 24 – 4 = 0, so we get

$$
\begin{array}{c|cccc}
2 & 6 & -5 & -12 & -4 \\
 & & 12 & 14 & 4 \\
\hline
 & 6 & 7 & 2 & 0
\end{array}
$$

Thus, we get $\mathbf{6x^3 - 5x^2 - 12x - 4 = (x - 2)(6x^2 + 7x + 2) = (x - 2)(2x + 1)(3x + 2)}$.