Examples F in Polynomial Factorizations

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Examples F

Factorize the polynomials below.

0. $\mathbf{x^4 - 3x^3 - 15x^2 + 19x + 30}$

1. $\mathbf{12x^4 + 46x^3 - 18x^2 - 36x + 16}$

Suggestions or Solutions
To the Problem in the Example 0

We have $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30}$.

Factorizing it, we can get it the way below.

Evaluating $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30}$ for $\boldsymbol{x = -1}$, we get $1 + 3 - 15 - 19 + 30 = 0$, so doing the synthetic division, we get

-1	1	-3	-15	19	30
		-1	4	11	-30
	1	-4	-11	30	0

So we get $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + 1)(x^3 - 4x^2 - 11x + 30)}$.

Next, evaluating $\boldsymbol{x^3 - 4x^2 - 11x + 30}$ for $\boldsymbol{x = 2}$, we get $8 - 16 - 22 + 30 = 0$.

Thus, by the synthetic division, we get

2	1	-4	-11	30
		2	-4	-30
	1	-2	-15	0

So we get $\boldsymbol{x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15) = (x - 2)(x + 3)(x - 5)}$.

Therefore, $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + 1)(x - 2)(x + 3)(x - 5)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting first, $\boldsymbol{P = x^4 - 3x^3 - 15x^2 + 19x + 30}$, we can say that the polynomial $\boldsymbol{P}$ is of degree 4. So if $\boldsymbol{P}$ can get factorized, it will probably be a product of polynomials of degree 3 or less.

A polynomial of degree 3 can eventually get factorized to a product of polynomials of degree 1. So eventually, $\boldsymbol{P}$ can get factorized to a product of polynomials of degree 1.

And thus, we can try factorizing $\boldsymbol{P}$ the way we factorize polynomials of degree 3 as

$\boldsymbol{x^3 + ux^2 + vx + w}$.

So assuming that $\boldsymbol{x + e}$ is a divisor of $\boldsymbol{P}$, and setting

$\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + e)(ax^3 + bx^2 + cx + d)}$, we get $\boldsymbol{ed = 30}$.

We can thus, assume that $\boldsymbol{e}$ and $\boldsymbol{d}$ are divisors of 30. And we know

$\boldsymbol{e^4 + 3e^3 - 15e^2 - 19e + 30 = (-e + e)(-ae^3 + be^2 - ce + d) = 0}$ if $\boldsymbol{x = -e}$, which is a divisor of 30, too, since $\boldsymbol{e}$ is a divisor of 30. And we know $\boldsymbol{e}$ is the negative of $\boldsymbol{-e}$.

So putting a divisor of 30 into $\boldsymbol{x}$ in $\boldsymbol{P}$, and getting $\boldsymbol{P = 0}$, the negative of the divisor is the value of $\boldsymbol{e}$. By trial and error, we can try getting the divisor that makes $\boldsymbol{P = 0}$.

Then, we get a factor of $\boldsymbol{P}$. What is the factor though?

The factor is $\boldsymbol{x + e}$. So dividing $\boldsymbol{P}$ by $\boldsymbol{x + e}$, we get as the quotient a polynomial of degree 3 as $\boldsymbol{ax^3 + bx^2 + cx + d}$.

Then, we can try factorizing it using the method described in the previous examples.

So let's now try factorizing $\boldsymbol{P = x^4 - 3x^3 - 15x^2 + 19x + 30}$ using the idea above.

To begin with, taking -1 as a divisor of 30, and putting it into $\boldsymbol{x}$ in $\boldsymbol{P}$, that is, evaluating

$\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30}$ for $\boldsymbol{x = -1}$, we get $1 + 3 - 15 - 19 + 30 = 0$.

So we can see that $\boldsymbol{x + 1}$ is a factor of $\boldsymbol{P}$.

And thus, applying to $\boldsymbol{P}$ the synthetic division by the factor $\boldsymbol{x + 1}$, we get

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & -15 & 19 & 30 \\ & & -1 & 4 & 11 & -30 \\ \hline & 1 & -4 & -11 & 30 & 0 \end{array}$$

So we get $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + 1)(x^3 - 4x^2 - 11x + 30)}$.

Next, we want to check to see if the quotient $\boldsymbol{x^3 - 4x^2 - 11x + 30}$ can still be factorized.

Let's try this time, factorizing it a bit differently.

Assuming $\boldsymbol{K}$ is the quotient, and $\boldsymbol{(x + d)}$ is a factor of $\boldsymbol{K}$, we can set $\boldsymbol{K}$ this way too:
$\boldsymbol{K = x^3 - 4x^2 - 11x + 30 = (x + d)Q}$, where $\boldsymbol{Q}$ is a polynomial of degree 2 as $\boldsymbol{ax^2 + bx + c}$.

And actually setting $\boldsymbol{x^3 - 4x^2 - 11x + 30 = (x + d)(ax^2 + bx + c)}$, and expanding the right hand side, we can get $\boldsymbol{dc = 30}$.

What then, can we say about $\boldsymbol{d}$ and $\boldsymbol{c}$?

We can assume that $\boldsymbol{c}$ and $\boldsymbol{d}$ are divisors of 30. Then, $\boldsymbol{-c}$ and $\boldsymbol{-d}$ are divisors of 30, too.

And if we get $\boldsymbol{K = 0}$ when $\boldsymbol{x = -d}$, we can say that $\boldsymbol{(x + d)}$ is a factor of the polynomial $\boldsymbol{K}$.

So putting a divisor of 30 into $\boldsymbol{x}$ in $\boldsymbol{K}$, and getting $\boldsymbol{K = 0}$, we can say that the negative of the divisor can be the value of $\boldsymbol{d}$.

Then, we can say that $\boldsymbol{(x + d)}$ is a factor of $\boldsymbol{K}$.

What if however, we get $\boldsymbol{K = 0}$ when $\boldsymbol{x = d}$?

We have $\boldsymbol{K = x^3 - 4x^2 - 11x + 30}$. So we get $\boldsymbol{K = d^3 - 4d^2 - 11d + 30}$ if $\boldsymbol{x = d}$.

So if $\boldsymbol{K = 0}$ when $\boldsymbol{x = d}$, that is, if $\boldsymbol{d^3 - 4d^2 - 11d + 30 = 0}$, we can say that $\boldsymbol{x - d}$ is a divisor of $\boldsymbol{x^3 - 4x^2 - 11x + 30}$, and thus, is a factor of $\boldsymbol{K}$. How come?

We can put $\boldsymbol{K}$ this way, too: $\boldsymbol{K = x^3 - 4x^2 - 11x + 30 = (x - d)(ax^2 + bx + c)}$.

Then, we get $\boldsymbol{K = 0}$ when $\boldsymbol{x = d}$. So $\boldsymbol{x - d}$ divides $\boldsymbol{K}$, and thus, is a factor of $\boldsymbol{K}$.

So it does not matter whether we use as a factor $\boldsymbol{x + d}$ or $\boldsymbol{x - d}$.

If $\boldsymbol{K = 0}$ when $\boldsymbol{x = d}$, that is, if $\boldsymbol{d^3 - 4d^2 - 11d + 30 = 0}$, we can say that $\boldsymbol{x - d}$ is a divisor of $\boldsymbol{x^3 - 4x^2 - 11x + 30}$, and thus, is a factor of $\boldsymbol{K}$.

If $\boldsymbol{K = 0}$ when $\boldsymbol{x = -d}$, that is, if $\boldsymbol{-d^3 - 4d^2 + 11d + 30 = 0}$, we can say that $\boldsymbol{x + d}$ is a divisor of $\boldsymbol{x^3 - 4x^2 - 11x + 30}$, and thus, is a factor of $\boldsymbol{K}$.

And in fact, both of the two cases above are equivalent to each other.

• Let's now factorize $\boldsymbol{K = x^3 - 4x^2 - 11x + 30}$ using the idea stated above.

To begin with, taking 2 as a divisor of 30, and putting it into $\boldsymbol{x}$ in $\boldsymbol{K}$, that is, evaluating $\boldsymbol{x^3 - 4x^2 - 11x + 30}$ for $\boldsymbol{x = 2}$, we get $2^3 - 4{\cdot}2^2 - 11{\cdot}2 + 30 = 8 - 16 - 22 + 30 = 0$.

So $\boldsymbol{x - 2}$ is a factor of $\boldsymbol{x^3 - 4x^2 - 11x + 30}$, and by the synthetic division, we get

$$\begin{array}{c|cccc}
2 & 1 & -4 & -11 & 30 \\
 & & 2 & -4 & -30 \\
\hline
 & 1 & -2 & -15 & 0
\end{array}$$

Thus, dividing $\boldsymbol{x^3 - 4x^2 - 11x + 30}$ by $\boldsymbol{x - 2}$, we get $\boldsymbol{x^2 - 2x - 15}$ as the quotient.

So we get $\boldsymbol{x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)}$.

Next, we want to check to see if the quotient $\boldsymbol{x^2 - 2x - 15}$ can still be factorized.

Then, by the same token, we can set

$\boldsymbol{x^2 - 2x - 15 = (x - c)Q}$, where $\boldsymbol{Q}$ is a polynomials of degree 1 as $\boldsymbol{ax + b}$.

So if $\boldsymbol{c^2 - 2c - 15 = 0}$, $\boldsymbol{x - c}$ is a divisor of $\boldsymbol{x^2 - 2x - 15}$.

And in fact, we can get $\boldsymbol{x^2 - 2x - 15 = 0}$ for $\boldsymbol{x = -3}$, which is a divisor of -15.

That is, evaluating $\boldsymbol{x^2 - 2x - 15}$ for $\boldsymbol{x = -3}$, we get $(-3)^2 - 2\cdot(-3) - 15 = 9 + 6 - 15 = 0$.

So $\boldsymbol{x + 3}$ is a divisor, that is, a factor of $\boldsymbol{x^2 - 2x - 15}$, and doing the synthetic division, we get this:

$$
\begin{array}{r|rrr}
-3 & 1 & -2 & -15 \\
 & & -3 & 15 \\
\hline
 & 1 & -5 & 0
\end{array}
$$

Thus, dividing $\boldsymbol{x^2 - 2x - 15}$ by $\boldsymbol{x + 3}$, we get $\boldsymbol{x - 5}$ as the quotient.

So we get $\boldsymbol{x^2 - 2x - 15 = (x + 3)(x - 5)}$.

Thus, putting threads together, we get

$\boldsymbol{P = x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + 1)(x^3 - 4x^2 - 11x + 30)}$

$\boldsymbol{= (x + 1)(x - 2)(x^2 - 2x - 15) = (x + 1)(x - 2)(x + 3)(x - 5)}$.

In short:

Evaluating $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30}$ for $\boldsymbol{x = -1}$, we get 1 + 3 – 15 – 19 + 30 = 0, so doing the synthetic division, we get

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & -15 & 19 & 30 \\ & & -1 & 4 & 11 & -30 \\ \hline & 1 & -4 & -11 & 30 & 0 \end{array}$$

So we get $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + 1)(x^3 - 4x^2 - 11x + 30)}$.

Next, evaluating $\boldsymbol{x^3 - 4x^2 - 11x + 30}$ for $\boldsymbol{x = 2}$, we get 8 – 16 – 22 + 30 = 0.

Thus, by the synthetic division, we get

$$\begin{array}{r|rrrr} 2 & 1 & -4 & -11 & 30 \\ & & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

So we get $\boldsymbol{x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15) = (x - 2)(x + 3)(x - 5)}$.

Therefore, $\boldsymbol{x^4 - 3x^3 - 15x^2 + 19x + 30 = (x + 1)(x - 2)(x + 3)(x - 5)}$.

Suggestions or Solutions

To the **Problem E** in the Example **0**

We have $\mathbf{12x^4 + 46x^3 - 18x^2 - 36x + 16}$.

Factorizing it, we can get it the way below:

To begin with, $\mathbf{12x^4 + 46x^3 - 18x^2 - 36x + 16 = 2(6x^4 + 23x^3 - 9x^2 - 18x + 8)}$.

So next, evaluating $\mathbf{6x^4 + 23x^3 - 9x^2 - 18x + 8}$ for $x = \frac{1}{2}$, we get

$\mathbf{6(\tfrac{1}{2})^4 + 23(\tfrac{1}{2})^3 - 9(\tfrac{1}{2})^2 - 18(\tfrac{1}{2}) + 8 = \tfrac{6}{2^4} + \tfrac{23}{2^3} - \tfrac{9}{2^2} - 9 + 8 = \tfrac{6+46-36}{2^4} - 1 = \tfrac{16}{2^4} - 1 = 0}$. So we get

$$
\begin{array}{c|ccccc}
\frac{1}{2} & 6 & 23 & -9 & -18 & 8 \\
 & & 3 & 13 & 2 & -8 \\
\hline
 & 6 & 26 & 4 & -16 & 0
\end{array}
$$

So we get $\mathbf{6x^4 + 23x^3 - 9x^2 - 18x + 8 = (x - \tfrac{1}{2})(6x^3 + 26x^2 + 4x - 16)}$.

Next, evaluating $\mathbf{6x^3 + 26x^2 + 4x - 16}$ for $x = \frac{2}{3}$, we get

$\mathbf{6(\tfrac{2}{3})^3 + 26(\tfrac{2}{3})^2 + 4(\tfrac{2}{3}) - 16 = \tfrac{6\cdot 8}{3^3} + \tfrac{26\cdot 4}{3^2} + \tfrac{8}{3} - 16 = \tfrac{48+26\cdot 4\cdot 3+8\cdot 9}{27} - 16 = \tfrac{48+26\cdot 12+72}{27}}$

$\mathbf{= \tfrac{12(4+26+6)}{27} - 16 = \tfrac{12\cdot 36}{27} - 16 = \tfrac{12\cdot 36-16\cdot 27}{27} = \tfrac{4(3\cdot 36-4\cdot 27)}{27} = \tfrac{12(36-4\cdot 9)}{27} = 0}$. So we get

$$
\begin{array}{c|cccc}
\frac{2}{3} & 6 & 26 & 4 & -16 \\
 & & 4 & 20 & 16 \\
\hline
 & 6 & 30 & 24 & 0
\end{array}
$$

So we get $\mathbf{6x^3 + 26x^2 + 4x - 16 = (x - \tfrac{2}{3})(6x^2 + 30x + 24) = 6(x - \tfrac{2}{3})(x^2 + 5x + 4)}$

$\mathbf{= 6(x - \tfrac{2}{3})(x + 1)(x + 4)}$.

Thus, $\mathbf{12x^4 + 46x^3 - 18x^2 - 36x + 16 = 2(x - \tfrac{1}{2})6(x - \tfrac{2}{3})(x + 1)(x + 4)}$

$\mathbf{= (2x - 1)2(3x - 2)(x + 1)(x + 4) = 2(2x - 1)(3x - 2)(x + 1)(x + 4)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P} = \mathbf{12}\boldsymbol{x}^4 + \mathbf{46}\boldsymbol{x}^3 - \mathbf{18}\boldsymbol{x}^2 - \mathbf{36}\boldsymbol{x} + \mathbf{16}$.

To begin with, 2 is a divisor common to all the terms in $\boldsymbol{P}$, so 2 is a factor of $\boldsymbol{P}$.

Thus, we get $\boldsymbol{P} = \mathbf{2(6}\boldsymbol{x}^4 + \mathbf{23}\boldsymbol{x}^3 - \mathbf{9}\boldsymbol{x}^2 - \mathbf{18}\boldsymbol{x} + \mathbf{8)}$.

So next, we want to get $\mathbf{6}\boldsymbol{x}^4 + \mathbf{23}\boldsymbol{x}^3 - \mathbf{9}\boldsymbol{x}^2 - \mathbf{18}\boldsymbol{x} + \mathbf{8}$ factorized.

We know a polynomial $\boldsymbol{sx}^3 + \boldsymbol{tx}^2 + \boldsymbol{ux} + \boldsymbol{v}$ can get factorized to $\mathbf{(}\boldsymbol{ax} + \boldsymbol{b}\mathbf{)(}\boldsymbol{cx} + \boldsymbol{d}\mathbf{)(}\boldsymbol{ex} + \boldsymbol{g}\mathbf{)}$.

So we can reasonably expect that a polynomial $\boldsymbol{sx}^4 + \boldsymbol{tx}^3 + \boldsymbol{ux}^2 + \boldsymbol{vx} + \boldsymbol{w}$ can eventually get factorized to $\mathbf{(}\boldsymbol{ax} + \boldsymbol{b}\mathbf{)(}\boldsymbol{cx} + \boldsymbol{d}\mathbf{)(}\boldsymbol{ex} + \boldsymbol{g}\mathbf{)(}\boldsymbol{hx} + \boldsymbol{k}\mathbf{)}$.

And thus, setting $\boldsymbol{sx}^4 + \boldsymbol{tx}^3 + \boldsymbol{ux}^2 + \boldsymbol{vx} + \boldsymbol{w} = \mathbf{(}\boldsymbol{ax} + \boldsymbol{b}\mathbf{)(}\boldsymbol{cx} + \boldsymbol{d}\mathbf{)(}\boldsymbol{ex} + \boldsymbol{g}\mathbf{)(}\boldsymbol{hx} + \boldsymbol{k}\mathbf{)}$, and expanding the right hand side, we get a polynomial as below.

$\boldsymbol{acehx}^4 + \mathbf{(}\boldsymbol{acek} + \text{etc.}\mathbf{)}\boldsymbol{x}^3 + \mathbf{(}\boldsymbol{acgk} + \text{etc.}\mathbf{)}\boldsymbol{x}^2 + \mathbf{(}\boldsymbol{adgk} + \text{etc.}\mathbf{)}\boldsymbol{x} + \boldsymbol{bdgk}$.

Then, we can see that $\boldsymbol{s} = \boldsymbol{aceh}$, and that $\boldsymbol{w} = \boldsymbol{bdgk}$.

So assuming that $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{e}$, and $\boldsymbol{h}$ are divisors of $\boldsymbol{s}$, and $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{g}$, and $\boldsymbol{k}$ are divisors of $\boldsymbol{w}$, and finding $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{k}$ by the similar method applied to $\boldsymbol{x}^4 + \boldsymbol{ux}^3 + \boldsymbol{vx}^2 + \boldsymbol{wx} + \boldsymbol{s}$, we could get the factorization of $\boldsymbol{sx}^4 + \boldsymbol{tx}^3 + \boldsymbol{ux}^2 + \boldsymbol{vx} + \boldsymbol{w}$.

How can we find $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{k}$, though?

Setting $\boldsymbol{A} = \boldsymbol{sx}^4 + \boldsymbol{tx}^3 + \boldsymbol{ux}^2 + \boldsymbol{vx} + \boldsymbol{w} = \mathbf{(}\boldsymbol{ax} + \boldsymbol{b}\mathbf{)(}\boldsymbol{cx} + \boldsymbol{d}\mathbf{)(}\boldsymbol{ex} + \boldsymbol{g}\mathbf{)(}\boldsymbol{hx} + \boldsymbol{k}\mathbf{)}$, we can see this:

$\boldsymbol{A} = \mathbf{0}$ when $\boldsymbol{x} = -\frac{b}{a}, -\frac{d}{c}, -\frac{g}{e}$, or $-\frac{k}{h}$.

Thus, if $\boldsymbol{m} = \frac{b}{a}, \frac{d}{c}, \frac{g}{e}$, or $\frac{k}{h}$, then $\boldsymbol{x} + \boldsymbol{m}$ is a factor of $\boldsymbol{A}$.

And we can put it the way below, too.

If $\boldsymbol{m} = -\frac{b}{a}$, $-\frac{d}{c}$, $-\frac{g}{e}$, or $-\frac{k}{h}$, then $\boldsymbol{x - m}$ is a factor of $\boldsymbol{A}$.

So first, we can assume that $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{e}$, and $\boldsymbol{h}$ are divisors of $\boldsymbol{s}$, and $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{g}$, and $\boldsymbol{k}$ are divisors of $\boldsymbol{w}$.
And by trial and error, coming up with, for instance, $-\frac{b}{a}$ that makes $\boldsymbol{A = 0}$, we can get a factor of $\boldsymbol{A}$. And the factor is $\boldsymbol{x + \frac{b}{a}}$.

And next, we can get the other factors factorizing the quotient we get after applying to $\boldsymbol{A}$ the synthetic division by the factor.

And of course, if the quotient is prime, we can use the quotient itself as the other factor.

So let's now, set $\boldsymbol{Q = 6x^4 + 23x^3 - 9x^2 - 18x + 8}$, and factorize $\boldsymbol{Q}$ using the idea above.

Then, first, we want to find a value of $\boldsymbol{x}$ that makes $\boldsymbol{Q = 0}$. And the value is a fraction where the numerator is a divisor of 8, and the denominator is a divisor of 6.

(The divisors of 8 are ±1, ±2, ±4, and ±8, and the divisors of 6 are ±1, ±2, ±3, and ±6.)

So for instance, using as a numerator, 1 (a divisor of 8), and as a denominator, -2 (a divisor of 6), we get $\boldsymbol{x = -\frac{1}{-2} = \frac{1}{2}}$.

Thus next, putting $\frac{1}{2}$ into $\boldsymbol{x}$ in the polynomial $\boldsymbol{Q}$, we get

$$\boldsymbol{Q = 6(\tfrac{1}{2})^4 + 23(\tfrac{1}{2})^3 - 9(\tfrac{1}{2})^2 - 18(\tfrac{1}{2}) + 8 = \frac{6}{2^4} + \frac{23}{2^3} - \frac{9}{2^2} - 9 + 8 = \frac{6+46-36}{2^4} - 1 = \frac{16}{2^4} - 1 = 0}.$$

So we can see that $\boldsymbol{x - \frac{1}{2}}$ is a factor of $\boldsymbol{Q}$. Thus next, applying to $\boldsymbol{Q}$ the synthetic division by the factor $\boldsymbol{x - \frac{1}{2}}$, we get

$$\begin{array}{c|ccccc} \frac{1}{2} & 6 & 23 & -9 & -18 & 8 \\ & & 3 & 13 & 2 & -8 \\ \hline & 6 & 26 & 4 & -16 & 0 \end{array}$$

So we get $\boldsymbol{Q = 6x^4 + 23x^3 - 9x^2 - 18x + 8 = (x - \frac{1}{2})(6x^3 + 26x^2 + 4x - 16)}$.

Next, we want to check to see if $\mathbf{6x^3 + 26x^2 + 4x - 16}$ can be factorized.

Suppose $\boldsymbol{T} = \mathbf{6x^3 + 26x^2 + 4x - 16}$.

Then, by the same token, first, we want to find a value of $\boldsymbol{x}$ that makes $\boldsymbol{T} = \mathbf{0}$. And the value is a fraction where the numerator is a divisor of -16, and the denominator is a divisor of 6.

(The divisors of -16 are ±1, ±2, ±4, ±8, and ±16, and those of 6 are ±1, ±2, ±3, and ±6.)

So for instance, using as a numerator, 2 (a divisor of -16), and as a denominator, -3 (a divisor of 6), we get $\boldsymbol{x} = -\frac{2}{-3} = \frac{2}{3}$.

Thus next, putting $\frac{2}{3}$ into $\boldsymbol{x}$ in the polynomial $\boldsymbol{T}$, we get

$$\boldsymbol{T} = 6(\tfrac{2}{3})^3 + 26(\tfrac{2}{3})^2 + 4(\tfrac{2}{3}) - 16 = \tfrac{6\cdot 8}{3^3} + \tfrac{26\cdot 4}{3^2} + \tfrac{8}{3} - 16 = \tfrac{48+26\cdot 4\cdot 3+8\cdot 9}{27} - 16 = \tfrac{48+26\cdot 12+72}{27}$$

$$= \tfrac{12(4+26+6)}{27} - 16 = \tfrac{12\cdot 36}{27} - 16 = \tfrac{12\cdot 36-16\cdot 27}{27} = \tfrac{4(3\cdot 36-4\cdot 27)}{27} = \tfrac{12(36-4\cdot 9)}{27} = \mathbf{0}.$$

So we can see that $x - \frac{2}{3}$ is a factor of $\boldsymbol{T}$. Thus, doing the synthetic division, we get

$\frac{2}{3}$	6	26	4	-16
		4	20	16
	6	30	24	0

So we get $\mathbf{6x^3 + 26x^2 + 4x - 16} = (x - \frac{2}{3})(6x^2 + 30x + 24) = 6(x - \frac{2}{3})(x^2 + 5x + 4)$

$= 6(x - \frac{2}{3})(x+1)(x+4)$

Now, putting threads together, we get

$$\boldsymbol{P} = 2(6x^4 + 23x^3 - 9x^2 - 18x + 8) = 2(x - \tfrac{1}{2})(6x^3 + 26x^2 + 4x - 16)$$

$$= 2(x - \tfrac{1}{2})6(x - \tfrac{2}{3})(x+1)(x+4) = (2x-1)2(3x-2)(x+1)(x+4).$$

So $\mathbf{12x^4 + 46x^3 - 18x^2 - 36x + 16}$ gets factorized to $\mathbf{2(2x - 1)(3x - 2)(x + 1)(x + 4)}$.

And we can try factorizing $\boldsymbol{Q = 6x^4 + 23x^3 - 9x^2 - 18x + 8}$ the way below, too.

Assuming first, $\boldsymbol{x + e}$ is a divisor of $\boldsymbol{Q}$, and setting
$\boldsymbol{6x^4 + 23x^3 - 9x^2 - 18x + 8 = (x + e)(ax^3 + bx^2 + cx + d)}$, we get $\boldsymbol{ed = 8}$.

Thus next, we can assume that $\boldsymbol{e}$ and $\boldsymbol{d}$ are divisors of 8. And we know

$\boldsymbol{6e^4 + 23e^3 - 9e^2 - 18e + 8 = (-e + e)(-ae^3 + be^2 - ce + d) = 0}$ if $\boldsymbol{x = -e}$, which is a divisor of 8, too, since $\boldsymbol{e}$ is a divisor of 8. And we know $\boldsymbol{e}$ is the negative of $\boldsymbol{-e}$.

So putting a divisor of 8 into $\boldsymbol{x}$ in $\boldsymbol{Q}$, and getting $\boldsymbol{Q = 0}$, we can say that the negative of the divisor is the value of $\boldsymbol{e}$.

By trial and error, we can try getting the divisor that makes $\boldsymbol{Q = 0}$.

Then, we get a factor of $\boldsymbol{Q}$. And the factor is $\boldsymbol{x + e}$. So dividing $\boldsymbol{Q}$ by $\boldsymbol{x + e}$, we get as the quotient a polynomial of degree 3 as $\boldsymbol{ax^3 + bx^2 + cx + d}$.

Then, we can try factorizing it using the method described in the previous examples.

In short:

To begin with, $\boldsymbol{12x^4 + 46x^3 - 18x^2 - 36x + 16 = 2(6x^4 + 23x^3 - 9x^2 - 18x + 8)}$.

So next, evaluating $\boldsymbol{6x^4 + 23x^3 - 9x^2 - 18x + 8}$ for $\boldsymbol{x = \frac{1}{2}}$, we get

$\boldsymbol{6(\frac{1}{2})^4 + 23(\frac{1}{2})^3 - 9(\frac{1}{2})^2 - 18(\frac{1}{2}) + 8 = \frac{6}{2^4} + \frac{23}{2^3} - \frac{9}{2^2} - 9 + 8 = \frac{6+46-36}{2^4} - 1 = \frac{16}{2^4} - 1 = 0}$. So we get

$\frac{1}{2}$	6	23	-9	-18	8
		3	13	2	-8
	6	26	4	-16	0

So we get $\mathbf{6x^4 + 23x^3 - 9x^2 - 18x + 8 = (x - \frac{1}{2})(6x^3 + 26x^2 + 4x - 16)}$.

Next, evaluating $\mathbf{6x^3 + 26x^2 + 4x - 16}$ for $x = \frac{2}{3}$, we get

$$\mathbf{6(\tfrac{2}{3})^3 + 26(\tfrac{2}{3})^2 + 4(\tfrac{2}{3}) - 16 = \tfrac{6\cdot 8}{3^3} + \tfrac{26\cdot 4}{3^2} + \tfrac{8}{3} - 16 = \tfrac{48+26\cdot 4\cdot 3+8\cdot 9}{27} - 16 = \tfrac{48+26\cdot 12+72}{27}}$$

$$\mathbf{= \tfrac{12(4+26+6)}{27} - 16 = \tfrac{12\cdot 36}{27} - 16 = \tfrac{12\cdot 36-16\cdot 27}{27} = \tfrac{4(3\cdot 36-4\cdot 27)}{27} = \tfrac{12(36-4\cdot 9)}{27} = 0}.$$

So we get

$\frac{2}{3}$	6	26	4	-16
		4	20	16
	6	30	24	0

So we get $\mathbf{6x^3 + 26x^2 + 4x - 16 = (x - \frac{2}{3})(6x^2 + 30x + 24) = 6(x - \frac{2}{3})(x^2 + 5x + 4)}$

$\mathbf{= 6(x - \frac{2}{3})(x+1)(x+4)}$.

Thus, $\mathbf{12x^4 + 46x^3 - 18x^2 - 36x + 16 = 2(x - \frac{1}{2})6(x - \frac{2}{3})(x+1)(x+4)}$

$\mathbf{= (2x-1)2(3x-2)(x+1)(x+4) = 2(2x-1)(3x-2)(x+1)(x+4)}$.