Examples G in Polynomial Factorizations

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Examples G

0. Factorize the polynomial below.

$x^3 + 2x^2y + xy^2 + 4xy + 2x^2z + xz^2 + 4xz + 2xyz + 2x^2 + 2y^2 + 2z^2 + 4yz$

1. Find the value of $a^2 + b^2 + c^2$ in each case below.

1.0. $a + b + c = 25$, and $ab + bc + ca = 37$.

1.1. $a - b - c = 32$, and $bc - ab - ca = 82$.

1.2. $c - a - b = 27$, and $ab - bc - ca = 98$.

1.3. $a - b - c = 12$, and $ab - bc + ca = 21$.

Suggestions or Solutions
To the Problem in the Example 0

We have $x^3 + 2x^2y + xy^2 + 4xy + 2x^2z + xz^2 + 4xz + 2xyz + 2x^2 + 2y^2 + 2z^2 + 4yz$.

Factorizing it, we can get it the way below.

$$P = x^3 + 2x^2y + xy^2 + 4xy + 2x^2z + xz^2 + 4xz + 2xyz + 2x^2 + 2y^2 + 2z^2 + 4yz$$

$$= x^3 + 2(y + z + 1)x^2 + (y^2 + z^2 + 2yz + 4y + 4z)x + 2(y^2 + z^2 + 2yz)$$

$$= x^3 + 2(y + z + 1)x^2 + \{(y + z)^2 + 4(y + z)\}x + 2(y + z)^2$$

$$= x^3 + 2(y + z + 1)x^2 + (y + z)(y + z + 4)x + 2(y + z)^2$$

So setting $t = y + z$, we get $P = x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2$.

Next, evaluating P when $x = -2$, we get

$$(-2)^3 + 2(t + 1)(-2)^2 + t(t + 4)(-2) + 2t^2 = -8 + 8t + 8 - 2t^2 - 8t + 2t^2 = 0.$$

So $x + 2$ is a factor, and thus, doing the synthetic division, we get

$$\begin{array}{r|llll} -2 & 1 & 2(t + 1) & t(t + 4) & 2t^2 \\ & & -2 & -4t & -2t^2 \\ \hline & 1 & 2t & t^2 & 0 \end{array}$$

Thus, we get $P = (x + 2)(x^2 + 2tx + t^2) = (x + 2)(x + t)^2$.

We know $t = y + z$. So we get $P = (x + 2)(x + y + z)^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose the polynomials given is P. Though the polynomial P looks quite complicated, yet factorizing it, we will get to see it's fairly simple.

Examining $\boldsymbol{P}$, we can notice that we can take $\boldsymbol{P}$ for a polynomial of degree 3 with respect to $\boldsymbol{x}$. So first, we may want to put $\boldsymbol{P}$ in terms of $\boldsymbol{x}$. Then, we get

$$\begin{aligned}
\boldsymbol{P} &= x^3 + 2x^2y + xy^2 + 4xy + 2x^2z + xz^2 + 4xz + 2xyz + 2x^2 + 2y^2 + 2z^2 + 4yz \\
&= x^3 + 2x^2y + 2x^2z + 2x^2 + xy^2 + xz^2 + 2xyz + 4xy + 4xz + 2y^2 + 2z^2 + 4yz \\
&= x^3 + 2x^2(y + z + 1) + x(y^2 + z^2 + 2yz + 4y + 4z) + 2y^2 + 2z^2 + 4yz \\
&= x^3 + 2(y + z + 1)x^2 + (y^2 + z^2 + 2yz + 4y + 4z)x + 2(y^2 + z^2 + 2yz).
\end{aligned}$$

Now, setting $\boldsymbol{u} = 2(y + z + 1)$, $\boldsymbol{v} = y^2 + z^2 + 2yz + 4y + 4z$, and $\boldsymbol{w} = 2(y^2 + z^2 + 2yz)$, we get $\boldsymbol{P} = x^3 + ux^2 + vx + w$.

So we can take $\boldsymbol{P}$ for a polynomial of degree 3 with respect to $\boldsymbol{x}$, and therefore, we can consider factorizing $\boldsymbol{P}$ the way we do to the polynomial $x^3 + ux^2 + vx + w$, which can be factorized to $(x + a)(x + b)(x + c)$.

Let's next, look at the coefficients and the constant term in $\boldsymbol{P} = x^3 + ux^2 + vx + w$.

Then first, looking closely at $\boldsymbol{w} = 2(y^2 + z^2 + 2yz)$, we can notice that $\boldsymbol{w} = 2(y + z)^2$.

Moving next, on to $\boldsymbol{v} = y^2 + z^2 + 2yz + 4y + 4z$, we can notice that it can be put this way, too: $y^2 + z^2 + 2yz + 4y + 4z = (y + z)^2 + 4(y + z) = (y + z)\{(y + z) + 4\} = (y + z)(y + z + 4)$.

So we get $\boldsymbol{v} = (y + z)(y + z + 4)$.

Thus, we now have $\boldsymbol{P} = x^3 + 2(y + z + 1)x^2 + (y + z)(y + z + 4)x + 2(y + z)^2$, which looks a bit simpler, and is still in the form of $x^3 + ux^2 + vx + w$.

What about the coefficient $\boldsymbol{u} = 2(y + z + 1)$, though?

We can notice that $y + z$ is common to the two coefficients $\boldsymbol{u}$ and $\boldsymbol{v}$, and the constant term $\boldsymbol{w}$. That is, it is common to $2(y + z + 1)$, $(y + z)(y + z + 4)$, and $2(y + z)^2$.

So we may want to set it to another constant t, for instance.

Thus, setting $t = y + z$, we get $P = x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2$, which now looks much simpler.

Also of course, P is still in the form of $x^3 + ux^2 + vx + w$.

Thus, we can readily apply to P above the method we use factorizing polynomials of the form $x^3 + ux^2 + vx + w$, which we know, can be factorized to $(x + a)(x + b)(x + c)$.

So first, we want to collect the divisors of $2t^2$, and then, for instance, find a divisor called d that makes $x - d$ a factor of P.

The divisors of $2t^2$ are ±1, ±2, $\pm t$, $2t$, $\pm t^2$, and $\pm 2t^2$.

So let's first check to see if 1 can work. Then, putting 1 into x, we get

$1 + 2(t + 1) + t(t + 4) + 2t^2 = 3t^2 + 6t + 3 = 3(t^2 + 2t + 1) = 3(t + 1)^2$, which can be 0 when $t = -1$ only, and thus, is not 0 for every value of t.

If $x - 1$ is a factor of P, P has to be 0 for all values of t when $x = 1$. So 1 doesn't work.

So let's next, see if -2 works. Then, we get

$(-2)^3 + 2(t + 1)(-2)^2 + t(t + 4)(-2) + 2t^2 = -8 + 8t + 8 - 2t^2 - 8t + 2t^2 = 0$.

So we get $P = 0$ for all values of t when $x = -2$, and thus, -2 works.

So $x + 2$ is a factor of $P = x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2$.

Thus, doing the synthetic division, we get

$$
\begin{array}{r|llll}
-2 & 1 & 2(t+1) & t(t+4) & 2t^2 \\
 & & -2 & -4t & -2t^2 \\
\hline
 & 1 & 2t & t^2 & 0
\end{array}
$$

So dividing $x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2$ by $x + 2$, we get $x^2 + 2tx + t^2$ as the quotient.

Thus, we get $\boldsymbol{P = x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2 = (x + 2)(x^2 + 2tx + t^2)}$.

Next, the quotient $\boldsymbol{x^2 + 2tx + t^2}$ can be factorized to $\boldsymbol{(x + t)^2}$.

So we get $\boldsymbol{P = x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2 = (x + 2)(x + t)^2}$.

We know $\boldsymbol{t = y + z}$. So we get $\boldsymbol{P = (x + 2)(x + y + z)^2}$.

In short:

$$\boldsymbol{P = x^3 + 2x^2y + xy^2 + 4xy + 2x^2z + xz^2 + 4xz + 2xyz + 2x^2 + 2y^2 + 2z^2 + 4yz}$$

$$\boldsymbol{= x^3 + 2(y + z + 1)x^2 + (y^2 + z^2 + 2yz + 4y + 4z)x + 2(y^2 + z^2 + 2yz)}$$

$$\boldsymbol{= x^3 + 2(y + z + 1)x^2 + \{(y + z)^2 + 4(y + z)\}x + 2(y + z)^2}$$

$$\boldsymbol{= x^3 + 2(y + z + 1)x^2 + (y + z)(y + z + 4)x + 2(y + z)^2}$$

So setting $\boldsymbol{t = y + z}$, we get $\boldsymbol{P = x^3 + 2(t + 1)x^2 + t(t + 4)x + 2t^2}$.

Next, evaluating $\boldsymbol{P}$ when $\boldsymbol{x = -2}$, we get

$$\boldsymbol{(-2)^3 + 2(t + 1)(-2)^2 + t(t + 4)(-2) + 2t^2 = -8 + 8t + 8 - 2t^2 - 8t + 2t^2 = 0}.$$

So $\boldsymbol{x + 2}$ is a factor, and thus, doing the synthetic division, we get

$$\begin{array}{r|llll} -2 & 1 & \boldsymbol{2(t + 1)} & \boldsymbol{t(t + 4)} & \boldsymbol{2t^2} \\ & & -2 & \boldsymbol{-4t} & \boldsymbol{-2t^2} \\ \hline & 1 & \boldsymbol{2t} & \boldsymbol{t^2} & 0 \end{array}$$

Thus, we get $\boldsymbol{P = (x + 2)(x^2 + 2tx + t^2) = (x + 2)(x + t)^2}$.

We know $\boldsymbol{t = y + z}$. So we get $\boldsymbol{P = (x + 2)(x + y + z)^2}$.

Suggestions or Solutions
To the Problems in the Example 1

Find the value of $a^2 + b^2 + c^2$ in each case below.

1.0. $a + b + c = 25$, and $ab + bc + ca = 37$.

1.1. $a - b - c = 32$, and $bc - ab - ca = 82$.

1.2. $c - a - b = 27$, and $ab - bc - ca = 98$.

1.3. $a - b - c = 12$, and $ab - bc + ca = 21$.

1.0.

We have a factorization identity where $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

So setting $A = a^2 + b^2 + c^2$, since $a + b + c = 25$, and $ab + bc + ca = 37$, we get

$$25^2 = A + 2\cdot 37 \Rightarrow A = 625 - 74 = 551 \Rightarrow A = 551.$$

1.1.

This is just another algebra practice.
In the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we can notice that replacing b with $-b$, and c with $-c$, we get

$$(a - b - c)^2 = a^2 + b^2 + c^2 + 2\{-ab + (-b)(-c) - ca\} = a^2 + b^2 + c^2 + 2(bc - ab - ca).$$

Thus, we get $(a - b - c)^2 = a^2 + b^2 + c^2 + 2(bc - ab - ca)$.

So setting $A = a^2 + b^2 + c^2$, since $a - b - c = 32$, and $bc - ab - ca = 82$, we get

$$32^2 = A + 2\cdot 82 \Rightarrow A = 32^2 - 2\cdot 82 = (30 + 2)32 - 2\cdot 82 = 30\cdot 32 + 2\cdot 32 - 2\cdot 82$$

$$= 30\cdot 32 + 2(32 - 82) = 960 + 2(-50) = 960 - 100 = 860 \Rightarrow A = 860.$$

1.2. **Find the value of $a^2 + b^2 + c^2$ in the case below.**

$c - a - b = 27$, and $ab - bc - ca = 98$.

We have a factorization identity where $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

We can notice that in the identity above, replacing a with $-a$, and b with $-b$, we get

$(-a - b + c)^2 = a^2 + b^2 + c^2 + 2\{(-a)(-b) - bc - ca\} = a^2 + b^2 + c^2 + 2(ab - bc - ca)$.

Thus, we get $(c - a - b)^2 = a^2 + b^2 + c^2 + 2(ab - bc - ca)$.

So setting $A = a^2 + b^2 + c^2$, since $c - a - b = 27$, and $ab - bc - ca = 98$, we get

$32^2 = A + 2\cdot 82 \Rightarrow A = 27^2 - 2\cdot 98 = (20 + 7)27 - 2\cdot 98 = 20\cdot 27 + 7\cdot 27 - 2\cdot(100 - 2)$

$= 20(20 + 7) + 7(20 + 7) - 200 + 4 = 400 + 140 + 140 + 49 - 196$

$= 540 + 49 - 59 = 540 - 10 = 530 \Rightarrow A = 530$.

1.3. Find the value of $a^2 + b^2 + c^2$ in the case below.

$a - b - c = 12$, and $ab - bc + ca = 21$.

Again, in identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we can notice that replacing b with $-b$, and c with $-c$, we get

$$(a - b - c)^2 = a^2 + b^2 + c^2 + 2\{-ab + (-b)(-c) - ca\} = a^2 + b^2 + c^2 + 2(bc - ab - ca).$$

Thus, we get $(a - b - c)^2 = a^2 + b^2 + c^2 + 2(bc - ab - ca)$.

However, we are given the value of $ab - bc + ca$ and not $bc - ab - ca$.
So what can we do?

We can notice that $bc - ab - ca = -(ab - bc + ca) = -21$.

So setting $A = a^2 + b^2 + c^2$, since we have $(a - b - c)^2 = a^2 + b^2 + c^2 + 2(bc - ab - ca)$, together with $a - b - c = 12$, and $bc - ab - ca = -21$, we get

$$12^2 = A + 2\cdot(-21)$$

$$\Rightarrow A = 12^2 + 2\cdot 21 = (10 + 2)12 - 42 = 120 + 24 - 42 = 144 - 42 = 102$$

$$\Rightarrow A = 102.$$