Examples H in Polynomial Factorizations

Click this to start.

Examples H

Factorize each of the following polynomials.

0. $\boldsymbol{a}^3 + \boldsymbol{b}^3$

1. $\boldsymbol{a}^3 - \boldsymbol{b}^3$

Suggestions or Solutions
To the Problem in the Example 0

We have $a^3 + b^3$.

Factorizing it, we can get it the way below.

$$a^3 + b^3 = a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 = a^2(a + b) - a^2b - ab^2 + b^2(a + b)$$

$$= (a + b)(a^2 + b^2) - a^2b - ab^2 = (a + b)(a^2 + b^2) - ab(a + b) = (a + b)(a^2 - ab + b^2).$$

If not quite sure of the idea behind the processes above, follow the steps below.

Factorizing an integer or a polynomial, we are expecting that it has divisors.
And such divisors are called factors.

So factorizing a polynomial, we are expecting that the polynomial has factors.

And if factors exist, the polynomial has two factors at least, which can be the same, though. How come?

For instance, x is a factor of a polynomial $x^2 + 3x$, so we get $x^2 + 3x = x(x + 3)$.
Thus, the polynomial has two factors, which are x and $x + 3$.

For another instance, x is a factor of $x^3 + 3x^2 + 2x$, so we get

$x^3 + 3x^2 + 2x = x(x^2 + 3x + 2)$, which in turn, gets factorized to $x(x + 1)(x + 2)$.

For one more instance, $x^2 + 2x + 1$ gets factorized to $(x + 1)^2$, and therefore, has as factors two of $(x + 1)$s.

And a factor can have its factors, too.

For instance, $\boldsymbol{x(x^2 + 3x + 2)}$ has $\boldsymbol{x^2 + 3x + 2}$ as a factor, which has two factors, which are $\boldsymbol{x + 1}$ and $\boldsymbol{x + 2}$, because $\boldsymbol{x^2 + 3x + 2 = (x + 1)(x + 2)}$.

And if a factor is prime, that is, if it is a prime factor, it is often said to have no factor.

So just saying that an integer or an expression has no factor, we mean it has as divisors 1 and itself only.

And thus, if a factor has no divisor other than 1 and itself, it is often said to have no factor, and is said to be prime, so it is called a prime factor.

Now, for instance, the polynomial $\boldsymbol{x^2 + 2x + 1 = (x + 1)^2}$ has two same prime factors, because $\boldsymbol{x + 1}$ is prime, and the polynomial $\boldsymbol{x^3 + 3x^2 + 2x = x(x + 1)(x + 2)}$ has three prime factors, which are $\boldsymbol{x}$, $\boldsymbol{x + 1}$, and $\boldsymbol{x + 2}$.

So factorizing a polynomial, we are expecting the polynomial to have two or more prime factors. And such a factor can be a polynomial, a monomial, or a constant as well as an integer, of course.   So what?

Once having found all the prime factors, we put all those factors in a form of a product. Then, the form is called the factorization.

Therefore, for instance, taking a product of polynomials, that is, expanding the product, we get a polynomial factorable, which sounds, of course, quite natural though.

How then, can we factorize the polynomial $\boldsymbol{a^3 + b^3}$ given in this example?

Let's set first, $\boldsymbol{P = a^3 + b^3}$.
The polynomial $\boldsymbol{P}$ has two terms, which however, have nothing in common.

In other words, there is no divisor common to both of the terms. So is P not factorable?

Even having no divisor common to all the terms, it can still be factorized.

We know multiplying two polynomials, we get a polynomial that is factorable.

Multiplying for instance, $x + 1$ by $x + 2$, we get $x^2 + 3x + 2$, which is thus, factorable.

So if P is factorable, there must be two polynomials that can get multiplied together to produce the two terms, a^3 and b^3. Thus, we may want to think of such two polynomials.

For instance, we can start with two polynomials $a + b$ and $a^2 + b^2$.

Then, expanding $(a + b)(a^2 + b^2)$, we get $(a + b)(a^2 + b^2) = a^3 + ab^2 + a^2b + b^3$.

So we now have $a^3 + b^3$, but we don't want $ab^2 + a^2b$. What do we do then?

We can try doing something to the product $(a + b)(a^2 + b^2)$ so that the two unwanted terms get canceled during the expansion.

That is, we want $ab^2 + a^2b$ to vanish. It doesn't vanish itself, though, of course.

So in other words, we want $-(ab^2 + a^2b)$ to be made, too, during the expansion. How?

Looking at $-(ab^2 + a^2b)$ a bit more closely, we can notice that there is a common divisor, and it is ab. That is, $-(ab^2 + a^2b)$ can be factorized to $-ab(a + b)$.

What then, can we do?

Looking at $\boldsymbol{-ab(a + b)}$ a little more closely, we can notice that it is the product of $\boldsymbol{-ab}$ and $\boldsymbol{(a + b)}$. So what?

We can add $\boldsymbol{-ab}$ to the polynomial $\boldsymbol{a^2 + b^2}$. Then, what?

We know $\boldsymbol{(a + b)(a^2 + b^2)}$ produces $\boldsymbol{a^3 + ab^2 + a^2b + b^3}$, which is $\boldsymbol{a^3 + ab(a + b) + b^3}$.

So $\boldsymbol{(a + b)\{a^2 + b^2 + (-ab)\}}$ produces not only $\boldsymbol{a^3 + ab(a + b) + b^3}$ but $\boldsymbol{-ab(a + b)}$, too. So?

So adding $\boldsymbol{-ab}$ to the polynomial $\boldsymbol{a^2 + b^2}$ in the product $\boldsymbol{(a + b)(a^2 + b^2)}$, we get

$\boldsymbol{(a + b)(a^2 + b^2 - ab) = a^3 + ab(a + b) + b^3 - ab(a + b) = a^3 + b^3}$.

Now, each of ($\boldsymbol{a + b}$) and ($\boldsymbol{a^2 + b^2 - ab}$) is prime.

Therefore, $\boldsymbol{a^3 + b^3}$ gets factorized to $\boldsymbol{(a + b)(a^2 + b^2 - ab)}$.

In short:

$\boldsymbol{a^3 + b^3 = a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 = a^2(a + b) - a^2b - ab^2 + b^2(a + b)}$

$\boldsymbol{= (a + b)(a^2 + b^2) - a^2b - ab^2 = (a + b)(a^2 + b^2) - ab(a + b) = (a + b)(a^2 - ab + b^2)}$.

Doing arithmetic, we sometimes subtract values from other values.
Running math though, we always compensate the values, that is, add the negatives.

Suggestions or Solutions
To the Problem in the Example 1

We have $a^3 - b^3$.

Let's set first, $P = a^3 - b^3$.
In the Example 0 above, we have $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Replacing b with $-b$ in $a^3 + b^3$, we get $a^3 + (-b)^3 = a^3 - b^3$, which is the polynomial P.

So we can simply get $P = a^3 - b^3 = \{a + (-b)\}\{a^2 - a(-b) + (-b)^2\} = (a - b)(a^2 + ab + b^2)$.

Thus, we get $P = a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

So putting threads together, we get $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

In short:

$a^3 - b^3 = a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3 = a^2(a - b) + a^2b - ab^2 + b^2(a - b)$

$= (a - b)(a^2 + b^2) + a^2b - ab^2 = (a - b)(a^2 + b^2) + ab(a - b) = (a - b)(a^2 + ab + b^2)$.

How about the factorizations of polynomials as $a^4 \pm b^4$ and $a^5 \pm b^5$?

(The material starts below and ends before the next example is for those students who want to go beyond the basics.)

We have a factorization identity where $(x + y)^2 = x^2 + 2xy + y^2$.
So replacing x with a^2, and y with b^2, we get $(a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$.
Thus, we get $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$.
Also, we have another factorization identity where $x^2 - y^2 = (x + y)(x - y)$.

So we get $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = \{(a^2 + b^2) + \sqrt{2}ab\}\{(a^2 + b^2) - \sqrt{2}ab\}$

$= (a^2 + b^2 + \sqrt{2}ab)(a^2 + b^2 - \sqrt{2}ab)$.

Thus, we get $a^4 + b^4 = (a^2 + b^2 + \sqrt{2}ab)(a^2 + b^2 - \sqrt{2}ab)$.

Next, using the identity $\boldsymbol{x^2 - y^2 = (x + y)(x - y)}$ twice, we get

$\boldsymbol{a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)}$.

So we get $\boldsymbol{a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)}$.

Next, taking the product of $\boldsymbol{a + b}$ and $\boldsymbol{a^4 + b^4}$, we can get $\boldsymbol{a^5 + b^5}$, along with other terms.

So taking care of the other terms, we can get the factorization of $\boldsymbol{a^5 + b^5}$.

Now, taking the product first, we get

$\boldsymbol{(a + b)(a^4 + b^4) = a^5 + ab^4 + a^4b + b^5 = a^5 + b^5 + ab(a^3 + b^3)}$.

And we have a factorization identity where $\boldsymbol{x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)}$. So we get

$\boldsymbol{(a + b)(a^4 + b^4) = a^5 + b^5 + ab(a + b)(a^2 - ab + b^2) = a^5 + b^5 + (a + b)ab(a^2 - ab + b^2)}$

$\boldsymbol{= a^5 + b^5 + (a + b)(a^3b - a^2b^2 + ab^3)}$. Thus, we get

$\boldsymbol{a^5 + b^5 = (a + b)(a^4 + b^4) - (a + b)(a^3b - a^2b^2 + ab^3)}$

$\boldsymbol{= (a + b)\{(a^4 + b^4) - (a^3b - a^2b^2 + ab^3)\} = (a + b)(a^4 + b^4 - a^3b + a^2b^2 - ab^3)}$.

Therefore, we get $\boldsymbol{a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)}$.

Also, replacing $\boldsymbol{b}$ with $\boldsymbol{-b}$ in $\boldsymbol{a^5 + b^5}$, we get $\boldsymbol{a^5 - b^5}$, so we get

$\boldsymbol{a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)}$.

Thus, putting threads together, we get $\boldsymbol{a^5 \pm b^5 = (a \pm b)(a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4)}$.

How about the factorizations of polynomials as $\boldsymbol{a^6 \pm b^6}$ and $\boldsymbol{a^7 \pm b^7}$?

We can use again the factorization identity where $\boldsymbol{(x + y)^2 = x^2 + 2xy + y^2}$.

So replacing x with a^3, and y with b^3, we get $(a^3 + b^3)^2 = a^6 + 2a^3b^3 + b^6$.

Thus, we get $a^6 + b^6 = (a^3 + b^3)^2 - 2a^3b^3$.

Also, we have another factorization identity where $x^2 - y^2 = (x + y)(x - y)$.

So we get $a^6 + b^6 = (a^3 + b^3)^2 - 2a^3b^3 = \{(a^3 + b^3) + \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}}\}\{(a^2 + b^2) - \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}}\}$

$= (a^3 + b^3 + \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}})(a^2 + b^2 - \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}})$.

Thus, we get $a^6 + b^6 = (a^3 + b^3 + \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}})(a^2 + b^2 - \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}})$.

Next, using the identities $x^2 - y^2 = (x + y)(x - y)$ and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, we get $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3) = (a^3 + b^3)(a - b)(a^2 + ab + b^2)$.

So we get $a^6 - b^6 = (a^3 + b^3)(a - b)(a^2 + ab + b^2)$.

Next, taking the product of $a + b$ and $a^6 + b^6$, we can get $a^7 + b^7$, along with other terms.

So removing the other terms, we can get the factorization of $a^7 + b^7$.

Now, taking the product first, we get
$(a + b)(a^6 + b^6) = a^7 + ab^6 + a^6b + b^7 = a^7 + b^7 + ab(a^5 + b^5)$.

We have a factorization identity where $a^5 \pm b^5 = (a \pm b)(a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4)$.

So we get $(a + b)(a^6 + b^6) = a^7 + b^7 + ab(a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

$= a^7 + b^7 + (a + b)(a^5b - a^4b^2 + a^3b^3 - a^2b^4 + ab^5)$. Thus, we get

$a^7 + b^7 = (a + b)(a^6 + b^6) - (a + b)(a^5b - a^4b^2 + a^3b^3 - a^2b^4 + ab^5)$

$= (a + b)\{(a^6 + b^6) - (a^5b - a^4b^2 + a^3b^3 - a^2b^4 + ab^5)\}$

$= (a + b)(a^6 + b^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5)$.

Therefore, we get $a^7 + b^7 = (a + b)(a^4 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6)$.

Also, replacing $\boldsymbol{b}$ with $\boldsymbol{-b}$ in $\boldsymbol{a^7 + b^7}$, we get $\boldsymbol{a^7 - b^7}$, so we get

$$a^7 - b^7 = (a - b)(a^4 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6).$$

Thus, putting threads together, we get

$$a^7 \pm b^7 = (a - b)(a^4 \mp a^5b + a^4b^2 \mp a^3b^3 + a^2b^4 \mp ab^5 + b^6).$$

How about the factorizations of $\boldsymbol{a^n \pm b^n}$ and $\boldsymbol{a^m \pm b^m}$ where $\boldsymbol{n}$ is odd and $\boldsymbol{m}$ is even? And of course, $\boldsymbol{m}$ and $\boldsymbol{n}$ both are integers positive.

Let's first, set $\boldsymbol{n = 2k + 1}$ and $\boldsymbol{m = 2k}$ where $\boldsymbol{k}$ is a positive integer.

To begin with, putting in a sequence the polynomials of the form $\boldsymbol{a^{2k+1} \pm b^{2k+1}}$, we get

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2).$$
$$a^5 \pm b^5 = (a \pm b)(a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4).$$
$$a^7 \pm b^7 = (a - b)(a^4 \mp a^5b + a^4b^2 \mp a^3b^3 + a^2b^4 \mp ab^5 + b^6).$$

Then, we can notice a pattern in the sequence of the exponents applied to $\boldsymbol{a}$ and $\boldsymbol{b}$, and reflecting the pattern, we get

$$a^{2k+1} \pm b^{2k+1} = (a \pm b)(a^{2k} \mp a^{2k-1}b + a^{2k-2}b^2 \mp a^{2k-3}b^3 + a^{2k-4}b^4 \dots \mp ab^{2k-1} + b^{2k}).$$

Next, putting in a sequence the polynomials of the form $\boldsymbol{a^{2k} + b^{2k}}$, we get

$$a^2 + b^2 = (a + b)^2 - 2ab = (a + b + \sqrt{2ab}\,)(a + b - \sqrt{2ab}\,).$$
$$a^4 + b^4 = (a^2 + b^2 + \sqrt{2}ab\,)(a^2 + b^2 - \sqrt{2}ab\,).$$
$$a^6 + b^6 = (a^3 + b^3 + \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}}\,)(a^2 + b^2 - \sqrt{2}a^{\frac{3}{2}}b^{\frac{3}{2}}\,).$$

Then, we can notice a pattern in the sequence of the exponents applied to a and b, and reflecting the pattern, we get

$$a^{2k} + b^{2k} = (a^k + b^k + \sqrt{2}a^{\frac{k}{2}}b^{\frac{k}{2}})(a^2 + b^2 - \sqrt{2}a^{\frac{k}{2}}b^{\frac{k}{2}}).$$

Next, putting in a sequence the polynomials of the form $a^{2k} - b^{2k}$, we get

$a^2 - b^2 = (a + b)(a - b)$.

$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$.

$a^6 - b^6 = (a^3 + b^3)(a - b)(a^2 + ab + b^2)$.

$a^8 - b^8 = (a^4 + b^4)(a^4 - b^4) = (a^4 + b^4)(a^2 + b^2)(a^2 - b^2) = (a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$.

$a^{10} - b^{10} = (a^5 + b^5)(a^5 - b^5)$, and we have $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

So we get $a^{10} - b^{10} = (a^5 + b^5)(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

Next, we get

$a^{12} - b^{12} = (a^6 + b^6)(a^3 + b^3)(a^3 - b^3) = (a^6 + b^6)(a^3 + b^3)(a - b)(a^2 + ab + b^2)$, and

$a^{14} - b^{14} = (a^7 + b^7)(a^7 - b^7) = (a^7 + b^7)(a^7 - b^7)$

$= (a^7 + b^7)(a - b)(a^4 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6)$.

Next, putting in a sequence the polynomials of the form $a^n - b^n$, where $n = 2^k$, we get a sequence as follows.

$$a^2 - b^2 = (a + b)(a - b).$$

$$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b).$$

$$a^8 - b^8 = (a^4 + b^4)(a^2 + b^2)(a + b)(a - b).$$

$$a^{16} - b^{16} = (a^8 + b^8)(a^4 + b^4)(a^2 + b^2)(a + b)\ (a - b).$$

Next, putting in a sequence the polynomials of the form $a^n - b^n$, where $n = 6k$, we get a sequence as follows.

$$a^6 - b^6 = (a^3 + b^3)(a - b)(a^2 + ab + b^2).$$

$$a^{12} - b^{12} = (a^6 + b^6)(a^3 + b^3)(a - b)(a^2 + ab + b^2).$$

$$a^{18} - b^{18} = (a^9 + b^9)(a^6 + b^6)(a^3 + b^3)(a - b)(a^2 + ab + b^2).$$

Otherwise, we get a sequence as follows.

$$a^{10} - b^{10} = (a^5 + b^5)(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

$$a^{14} - b^{14} = (a^7 + b^7)(a - b)(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6).$$

$$a^{20} - b^{20} = (a^{10} + b^{10})(a^5 + b^5)(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

$$a^{22} - b^{22} = (a^{11} + b^{11})(a^{11} - b^{11}).$$

And we have

$$a^{2k+1} \pm b^{2k+1} = (a \pm b)(a^{2k} \mp a^{2k-1}b + a^{2k-2}b^2 \mp a^{2k-3}b^3 + a^{2k-4}b^4 \dots \mp ab^{2k-1} + b^{2k}).$$

So if $\boldsymbol{k} = \mathbf{5}$, we get

$$\boldsymbol{a^{11} - b^{11} = (a - b)(a^{10} + a^9b + a^8b^2 + a^7b^3 + a^6b^4 + \ldots + ab^9 + b^{10})}.$$

Thus, we get

$$\boldsymbol{a^{22} - b^{22} = (a^{11} + b^{11})(a - b)(a^{10} + a^9b + a^8b^2 + a^7b^3 + a^6b^4 + \ldots + ab^9 + b^{10})}.$$