Examples I in Polynomial Factorizations

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Examples I

Factorize each of the following polynomials.

0. $\boldsymbol{a^3 + b^3 + c^3 - 3abc}$

1. $\boldsymbol{a^4 + a^2b^2 + b^4}$

Suggestions or Solutions

To the Problem in the Example 0

We have $a^3 + b^3 + c^3 - 3abc$.

Factorizing it, we can get it the way below.

To begin with, we can get

$(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$
$= a^3 + b^3 + c^3 + ab(a + b) + bc(b + c) + ca(c + a)$

And we can have

$ab(a + b + c) = ab^2 + a^2b + abc$, $bc(a + b + c) = b^2c + bc^2 + abc$,
and $ca(a + b + c) = c^2a + ca^2 + abc$.

So taking the sum of the three equalities above, we get

$ab(a + b + c) + bc(a + b + c) + ca(a + b + c)$
$= ab^2 + a^2b + abc + b^2c + bc^2 + abc + c^2a + ca^2 + abc$
$= ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc$.

And we know $ab(a + b + c) + bc(a + b + c) + ca(a + b + c) = (a + b + c)(ab + bc + ca)$.

So we get $(a + b + c)(ab + bc + ca) = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc$.

Now, we have

$(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2$, and

$(a + b + c)(ab + bc + ca) = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc$.

That is, assuming $D = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2$, we get

$(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + D$, and $(a + b + c)(ab + bc + ca) = D + 3abc$.

So we get

$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\{(\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2) - (\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca})\}$

$= (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2) - (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca})$

$= \boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3 + \boldsymbol{D} - (\boldsymbol{D} + 3\boldsymbol{abc}) = \boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3 - 3\boldsymbol{abc}$.

Thus, we get $\boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3 - 3\boldsymbol{abc} = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2 - \boldsymbol{ab} - \boldsymbol{bc} - \boldsymbol{ca})$.

If not quite sure of the idea behind the processes above, follow the steps below.

Factorizing a polynomial, we are expecting that the polynomial has factors.
Once having found all the factors, we put them in a form of a product.
Multiplying out those factors or expanding the form, we get the polynomial back.

And such a factor can be a polynomial, and also, can be a monomial, constant, or integer.
So multiplying two polynomials, what do we get?

We get a polynomial that can be factorized.

Let's now set $\boldsymbol{P} = \boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3 - 3\boldsymbol{abc}$.
The polynomial $\boldsymbol{P}$ has four terms, which however, have nothing in common.
In other words, there is no divisor common to all the terms.

So the polynomial $\boldsymbol{P}$ looks quite far from being factorable, doesn't it?

Even having no divisor common to all the terms though, it can still be factorized.

We know multiplying two polynomials, we get a polynomial that is factorable.

Multiplying for instance, $\boldsymbol{x} + \mathbf{1}$ by $\boldsymbol{x} + \mathbf{2}$, we get $\boldsymbol{x}^2 + 3\boldsymbol{x} + \mathbf{2}$, which is thus, factorable.

So if $\boldsymbol{P}$ is factorable, there must be two polynomials that can get multiplied together to produce all the four terms, $\boldsymbol{a}^3$, $\boldsymbol{b}^3$, $\boldsymbol{c}^3$, and $-3\boldsymbol{abc}$. And thus, we may want to think of such two polynomials.

When choosing or coming up with such polynomials though, we want to be reasonable, too, of course. That is, the product of the two should produce a^3, b^3, c^3, and $-3abc$.

For instance, we can begin with two polynomials $a + b + c$ and $a^2 + b^2 + c^2$.

Then, expanding the product of the two, we get

$$(a + b + c)(a^2 + b^2 + c^2) = a^3 + ab^2 + ac^2 + a^2b + b^3 + bc^2 + a^2c + b^2c + c^3.$$

So we now have $a^3 + b^3 + c^3$, but don't have $-3abc$, and we don't want the expression as follows. $ab^2 + ac^2 + a^2b + bc^2 + a^2c + b^2c$.   What do we do then?

We can try doing something to the product so that the terms unwanted get canceled, and the term needed gets created during the expansion.   How?

We should be able to get $-(ab^2 + ac^2 + a^2b + bc^2 + a^2c + b^2c)$ and $-3abc$, both of which need to be produced during the expansion.   How?

To begin with, rearranging the first of the two expressions above, we get

$$-(ab^2 + ac^2 + a^2b + bc^2 + a^2c + b^2c) = -(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2).$$

Next, looking at $a^2b + ab^2$ a bit more closely, we can notice that there is a common divisor, which is ab.

That is to say that $a^2b + ab^2$ can be factorized to $ab(a + b)$.

And the same is true for $b^2c + bc^2$ and $c^2a + ca^2$, too.

So we get $ab^2 + a^2b = ab(a + b)$, $b^2c + bc^2 = bc(b + c)$, and $c^2a + ca^2 = ca(c + a)$.

And let's next, move on to $\boldsymbol{-3abc}$.

Looking at first, the three expressions above, we can notice that adding each of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ to each of the three expressions, we can get three of ($\boldsymbol{abc}$)s in such a way as follows.

$$\boldsymbol{ab^2 + a^2b = ab(a + b) \Rightarrow ab(a + b + c) = ab^2 + a^2b + abc},$$

$$\boldsymbol{b^2c + bc^2 = bc(b + c) \Rightarrow bc(a + b + c) = b^2c + bc^2 + abc},$$

and $$\boldsymbol{c^2a + ca^2 = ca(c + a) \Rightarrow ca(a + b + c) = c^2a + ca^2 + abc}.$$

So taking the sum of the three equalities above, we get

$$\boldsymbol{ab(a + b + c) + bc(a + b + c) + ca(a + b + c)}$$

$$\boldsymbol{= ab^2 + a^2b + abc + b^2c + bc^2 + abc + c^2a + ca^2 + abc}$$

$$\boldsymbol{= ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc}.$$

And factorizing $\boldsymbol{ab(a + b + c) + bc(a + b + c) + ca(a + b + c)}$, which is the top of the equalities above, we get $\boldsymbol{(a + b + c)(ab + bc + ca)}$.

So we get $\boldsymbol{(a + b + c)(ab + bc + ca) = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc}$.

Now, we can see what we need. What then, is it?

It is $\boldsymbol{-(a + b + c)(ab + bc + ca)}$.

And since $\boldsymbol{(a + b + c)(ab + bc + ca) = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc}$, we get

$$\begin{aligned}&\boldsymbol{-(a + b + c)(ab + bc + ca)}\\ &\boldsymbol{= -(ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc)}\\ &\boldsymbol{= -(ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2) - 3abc}.\end{aligned}$$

So we can add $\boldsymbol{-(ab + bc + ca)}$ to the polynomial $\boldsymbol{a^2 + b^2 + c^2}$. What then?

We know $(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$.

So on the left hand side of the equality above, adding $-(ab + bc + ca)$ to $(a^2 + b^2 + c^2)$, we get $(a + b + c)\{(a^2 + b^2 + c^2) - (ab + bc + ca)\}$, which produces

not only $(a + b + c)(a^2 + b^2 + c^2)$ but $-(a + b + c)(ab + bc + ca)$, too.

That is to say that we get $-(ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2) - 3abc$ as well as

$a^3 + b^3 + c^3 + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$.

And thus, putting threads together, we have

$(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2$, and

$(a + b + c)(ab + bc + ca) = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2 + 3abc$.

That is, assuming $D = ab^2 + a^2b + b^2c + bc^2 + c^2a + ca^2$, we get

$(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + D$, and $(a + b + c)(ab + bc + ca) = D + 3abc$.

So we get

$$\begin{aligned}
&(a + b + c)\{(a^2 + b^2 + c^2) - (ab + bc + ca)\}\\
&= (a + b + c)(a^2 + b^2 + c^2) - (a + b + c)(ab + bc + ca)\\
&= a^3 + b^3 + c^3 + D - (D + 3abc) = a^3 + b^3 + c^3 - 3abc.
\end{aligned}$$

And we know

$(a + b + c)\{(a^2 + b^2 + c^2) - (ab + bc + ca)\} = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

Thus, we get $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

In short:

To begin with, we can get

$$(a+b+c)(a^2+b^2+c^2) = a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$$
$$= a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)$$

And we can have

$ab(a+b+c) = ab^2+a^2b+abc$, $bc(a+b+c) = b^2c+bc^2+abc$,
and $ca(a+b+c) = c^2a+ca^2+abc$.

So taking the sum of the three equalities above, we get

$$ab(a+b+c)+bc(a+b+c)+ca(a+b+c)$$
$$= ab^2+a^2b+abc+b^2c+bc^2+abc+c^2a+ca^2+abc$$
$$= ab^2+a^2b+b^2c+bc^2+c^2a+ca^2+3abc.$$

And we know $ab(a+b+c)+bc(a+b+c)+ca(a+b+c) = (a+b+c)(ab+bc+ca)$.

So we get $(a+b+c)(ab+bc+ca) = ab^2+a^2b+b^2c+bc^2+c^2a+ca^2+3abc$.

Now, we have

$(a+b+c)(a^2+b^2+c^2) = a^3+b^3+c^3+ab^2+a^2b+b^2c+bc^2+c^2a+ca^2$, and

$(a+b+c)(ab+bc+ca) = ab^2+a^2b+b^2c+bc^2+c^2a+ca^2+3abc$.

That is, assuming $D = ab^2+a^2b+b^2c+bc^2+c^2a+ca^2$, we get

$(a+b+c)(a^2+b^2+c^2) = a^3+b^3+c^3+D$, and $(a+b+c)(ab+bc+ca) = D+3abc$.

So we get

$$(a+b+c)\{(a^2+b^2+c^2)-(ab+bc+ca)\}$$
$$= (a+b+c)(a^2+b^2+c^2)-(a+b+c)(ab+bc+ca)$$
$$= a^3+b^3+c^3+D-(D+3abc) = a^3+b^3+c^3-3abc.$$

Thus, we get $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Suggestions or Solutions
To the Problem in the Example 1

We have $\boldsymbol{a^4 + a^2b^2 + b^4}$.

Factorizing it, we can get it the way below.

$\boldsymbol{(a^2 + b^2)^2 - a^2b^2 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + b^4}$, and also,

$\boldsymbol{(a^2 + b^2)^2 - a^2b^2 = \{(a^2 + b^2) + ab\}\{(a^2 + b^2) - ab\} = (a^2 + b^2 + ab)(a^2 + b^2 - ab)}$.

Therefore, $\boldsymbol{a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

First, we know $\boldsymbol{a^2 + 2ab + b^2}$ gets factorized to $\boldsymbol{(a + b)^2}$.

So we may want to begin with the expansion of $\boldsymbol{(a^2 + b^2)^2}$.

Then, we get

$\boldsymbol{(a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4}$, which is $\boldsymbol{a^4 + a^2b^2 + a^2b^2 + b^4}$, in which however, we have too many of ($\boldsymbol{a^2b^2}$)s, since we need only one of those.

So next, we want to compensate $\boldsymbol{a^2b^2}$ so that we can have $\boldsymbol{a^4 + a^2b^2 + b^4}$ only. How?

We have $\boldsymbol{(a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4 = a^4 + a^2b^2 + a^2b^2 + b^4}$, so we want to compensate $\boldsymbol{a^2b^2}$. What then can we do?

We can do this ($\boldsymbol{a^4 + a^2b^2 + a^2b^2 + b^4}$) + ($\boldsymbol{-a^2b^2}$).

Then, we get $a^4 + a^2b^2 + a^2b^2 + b^4 - a^2b^2$, and thus, we can get $a^4 + a^2b^2 + b^4$.

Now, we have $(a^2 + b^2)^2 = a^4 + a^2b^2 + a^2b^2 + b^4$.

(Doing one thing to the right hand side of an equality, we want to do the same to the left hand side, too, if we want to maintain the equality.)

So we may want to try $(a^2 + b^2)^2 - a^2b^2$.
We are not going to just do the subtraction only, but take advantage of a factorization identity as follows, too. $x^2 - y^2 = (x + y)(x - y)$.

Then, we can get

$(a^2 + b^2)^2 - a^2b^2 = \{(a^2 + b^2) + ab\}\{(a^2 + b^2) - ab\} = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$ as well as $a^4 + 2a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + b^4$.

That is to say that we get

$(a^2 + b^2)^2 - a^2b^2 = \{(a^2 + b^2) + ab\}\{(a^2 + b^2) - ab\} = \underline{(a^2 + b^2 + ab)(a^2 + b^2 - ab)}$, and

$(a^2 + b^2)^2 - a^2b^2 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + a^2b^2 + b^4 - a^2b^2 = \underline{a^4 + a^2b^2 + b^4}$.

Therefore, we get $a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$.

In short:

$(a^2 + b^2)^2 - a^2b^2 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + b^4$, and also,

$(a^2 + b^2)^2 - a^2b^2 = \{(a^2 + b^2) + ab\}\{(a^2 + b^2) - ab\} = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$.

Therefore, $a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$.