Examples J in Polynomial Factorizations

Click this to start.

Examples J

0. Factorize each of the polynomials below.

0.0. $\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}$

0.1. $\boldsymbol{a^3 - 3a^2b + 3ab^2 - b^3}$

1. Show that $\boldsymbol{a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

2. Assuming that $\boldsymbol{a + b = 19}$, and that $\boldsymbol{ab = 29}$, find the value of $\boldsymbol{a^3 + b^3}$.

Suggestions or Solutions

To the Problem 0 in the Example 0

We have $\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}$.

Factorizing it, we can get it the way below.

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3}$$
$$\boldsymbol{= a^2(a + b) + 2ab(a + b) + b^2(a + b) = (a + b)(a^2 + 2ab + b^2) = (a + b)(a + b)^2 = (a + b)^3}.$$

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = 3a^2b + 3ab^2 + a^3 + b^3 = 3ab(a + b) + a^3 + b^3}$$
$$\boldsymbol{= 3ab(a + b) + (a + b)(a^2 - ab + b^2) = (a + b)(3ab + a^2 - ab + b^2)}$$
$$\boldsymbol{= (a + b)(a^2 + 2ab + b^2) = (a + b)(a + b)^2 = (a + b)^3}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $\boldsymbol{P = a^3 + 3a^2b + 3ab^2 + b^3}$.

Basically, factorizing a polynomial, we begin with finding a common divisor.
Such a divisor can be common to all or some of the polynomial.
So if not able to find a divisor common to all the terms, we may want to try finding a divisor common to some of the terms.

Now, we can put $\boldsymbol{P}$ this way, too: $\boldsymbol{a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3}$.

Then, we can see that $\boldsymbol{a^2}$ is common to $\boldsymbol{a^3}$ and $\boldsymbol{a^2b}$, $\boldsymbol{2ab}$ is common to $\boldsymbol{2a^2b}$ and $\boldsymbol{2ab^2}$, and $\boldsymbol{b^2}$ is common to $\boldsymbol{ab^2}$ and $\boldsymbol{b^3}$.

So we get

$\boldsymbol{P = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3 = a^2(a + b) + 2ab(a + b) + b^2(a + b)}$, where $\boldsymbol{a + b}$ is common.

Thus, we get $\boldsymbol{P = (a + b)(a^2 + 2ab + b^2) = (a + b)(a + b)^2 = (a + b)^3}$.

We can get the same the way below, too.

We know that $\boldsymbol{a^3 + b^3}$ gets factorized to $\boldsymbol{(a + b)(a^2 - ab + b^2)}$, and that $\boldsymbol{3a^2b + 3ab^2}$ can get factorized to $\boldsymbol{3ab(a + b)}$.

So $\boldsymbol{a + b}$ can be said to be common to $\boldsymbol{a^3 + b^3}$ and $\boldsymbol{3a^2b + 3ab^2}$. Thus, we can get

$$\boldsymbol{P = a^3 + 3a^2b + 3ab^2 + b^3 = 3a^2b + 3ab^2 + a^3 + b^3 = 3ab(a + b) + a^3 + b^3}$$

$$\boldsymbol{= 3ab(a + b) + (a + b)(a^2 - ab + b^2) = (a + b)(3ab + a^2 - ab + b^2)}$$

$$\boldsymbol{= (a + b)(a^2 + 2ab + b^2) = (a + b)(a + b)^2 = (a + b)^3.}$$

In short:

By one method,

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3}$$

$$\boldsymbol{= a^2(a + b) + 2ab(a + b) + b^2(a + b) = (a + b)(a^2 + 2ab + b^2) = (a + b)(a + b)^2 = (a + b)^3.}$$

And by the other,

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = 3a^2b + 3ab^2 + a^3 + b^3 = 3ab(a + b) + a^3 + b^3}$$

$$\boldsymbol{= 3ab(a + b) + (a + b)(a^2 - ab + b^2) = (a + b)(3ab + a^2 - ab + b^2)}$$

$$\boldsymbol{= (a + b)(a^2 + 2ab + b^2) = (a + b)(a + b)^2 = (a + b)^3.}$$

Suggestions or Solutions
To the Problem 1 in the Example 0

We have $a^3 - 3a^2b + 3ab^2 - b^3$.

Factorizing it, we can get it the way below.

$a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - a^2b - 2a^2b + 2ab^2 + ab^2 - b^3$

$= a^2(a - b) - 2ab(a - b) + b^2(a - b) = (a - b)(a^2 - 2ab + b^2) = (a - b)(a - b)^2 = (a - b)^3$.

$a^3 - 3a^2b + 3ab^2 - b^3 = -3a^2b + 3ab^2 + a^3 - b^3 = -3ab(a - b) + a^3 - b^3$

$= -3ab(a - b) + (a - b)(a^2 + ab + b^2) = (a - b)(-3ab + a^2 + ab + b^2)$

$= (a - b)(a^2 - 2ab + b^2) = (a - b)(a - b)^2 = (a - b)^3$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's set first, $P = a^3 - 3a^2b + 3ab^2 - b^3$.

Basically, factorizing a polynomial, we find a common divisor.
Such a divisor can be common to all or some of the polynomial.

So if not able to find a divisor common to all the terms, we may want to try finding a divisor common to some of the terms.

Now, we can put P this way, too: $P = a^3 - a^2b - 2a^2b + 2ab^2 + ab^2 - b^3$.

Then, we can see that a^2 is common to a^3 and $-a^2b$, $2ab$ is common to $-2a^2b$ and $2ab^2$, and b^2 is common to ab^2 and $-b^3$. So we get

$P = a^3 - a^2b - 2a^2b + 2ab^2 + ab^2 - b^3 = a^2(a - b) - 2ab(a - b) + b^2(a - b)$, where $a - b$ is common. Thus, we get $P = (a - b)(a^2 - 2ab + b^2) = (a - b)(a - b)^2 = (a - b)^3$.

• And we can get the same the way below, too.

We know that $\boldsymbol{a^3 - b^3}$ gets factorized to $\boldsymbol{(a - b)(a^2 + ab + b^2)}$, and that $\boldsymbol{3a^2b - 3ab^2}$ can get factorized to $\boldsymbol{3ab(a - b)}$.

So $\boldsymbol{a - b}$ can be said to be common to $\boldsymbol{a^3 - b^3}$ and $\boldsymbol{3a^2b - 3ab^2}$.

Thus, we can get

$$P = a^3 - 3a^2b + 3ab^2 - b^3 = -3a^2b + 3ab^2 + a^3 - b^3 = -3ab(a - b) + a^3 - b^3$$

$$= -3ab(a - b) + (a - b)(a^2 + ab + b^2) = (a - b)(-3ab + a^2 + ab + b^2)$$

$$= (a - b)(a^2 - 2ab + b^2) = (a - b)(a - b)^2 = (a - b)^3.$$

And of course, replacing $\boldsymbol{b}$ with $\boldsymbol{-b}$, too, we get $\boldsymbol{a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3}$.

In short:

By one method,

$$a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - a^2b - 2a^2b + 2ab^2 + ab^2 - b^3$$

$$= a^2(a - b) - 2ab(a - b) + b^2(a - b) = (a - b)(a^2 - 2ab + b^2) = (a - b)(a - b)^2 = (a - b)^3.$$

And by the other,

$$a^3 - 3a^2b + 3ab^2 - b^3 = -3a^2b + 3ab^2 + a^3 - b^3 = -3ab(a - b) + a^3 - b^3$$

$$= -3ab(a - b) + (a - b)(a^2 + ab + b^2) = (a - b)(-3ab + a^2 + ab + b^2)$$

$$= (a - b)(a^2 - 2ab + b^2) = (a - b)(a - b)^2 = (a - b)^3.$$

Suggestions or Solutions
To the Problem in the Example 1

Show that $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$.

Quite frequently, we need to show proofs doing problems in math.
In particular, showing that an equality holds, that is, proving an equality, we need to show two facts. We want to show not only that one side of the equality implies the other, but that the other implies the one side, too.

So for instance, showing that $A = B$, we have to show B implies A as well as A implies B. In short, we must show $A \Rightarrow B$ and $B \Rightarrow A$ both. So both directions have to be OK.

Why both, though?

Sometimes, the other way does not work. That is to say that in some cases, the other way is not always true. For instance, we can say that bread is food, but cannot say that food is bread because some food is not bread. For instance, an apple is food, but is not bread. So in math, it is not true that **bread = food**. That is, **bread ≠ food**, in math. What if it were true that bread = food, though?

Then, all food would have to be bread.

So proving an equality as a factorization identity, we want to show both ways, one way and the other. In this case though, it is not hard to show one way, which is this:

$\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$ implies $a^3 + b^3 + c^3 - 3abc$.

Just expanding (simplifying) the right hand side of the equality, we get the left hand side.
Let's expand it, and see if it is the case. To begin with, we can get this:

$$(a - b)^2 + (b - c)^2 + (c - a)^2\} = a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2$$

$$= 2(a^2 + b^2 + c^2 - ab - bc - ca).$$

So next, we get

$\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + ab^2 + ac^2 - a^2b - abc - a^2c + ba^2 + b^3 + bc^2 - ab^2 - b^2c - abc + ca^2 + cb^2 + c^3 - abc - bc^2 - c^2a = a^3 + b^3 + c^3 - 3abc$.

So such an expansion and simplification show us this:

$\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \Rightarrow a^3 + b^3 + c^3 - 3abc$.

However, it may not be quite straightforward to show that the other way is true.
That is to say that it is not quite simple to show this:

$a^3 + b^3 + c^3 - 3abc \Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$.

Memorizing the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ can be of significant help, but we still need to show this:

$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$.

To make it a bit simpler, we can just show this:

$a^2 + b^2 + c^2 - ab - bc - ca \Rightarrow \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$, since $(a + b + c)$ is common to both of the sides.

Now, compensating, we can get

$a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2} \cdot 2(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = \frac{1}{2}(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + \mathrm{a}^2)$

$= \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$.

So we get $a^2 + b^2 + c^2 - ab - bc - ca \Rightarrow \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$, and in turn, we get $a^3 + b^3 + c^3 - 3abc \Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$.

Therefore, we can now say this:

$\boldsymbol{a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

In short:

To begin with, showing $\boldsymbol{\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \Rightarrow a^3 + b^3 + c^3 - 3abc}$, we get first, $\boldsymbol{(a - b)^2 + (b - c)^2 + (c - a)^2\} = a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2}$ $\boldsymbol{= 2(a^2 + b^2 + c^2 - ab - bc - ca)}$.

So $\boldsymbol{\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}$ $\boldsymbol{= a^3 + ab^2 + ac^2 - a^2b - abc - a^2c + ba^2 + b^3 + bc^2 - ab^2 - b^2c - abc + ca^2 + cb^2 + c^3 - abc - bc^2 - c^2a = a^3 + b^3 + c^3 - 3abc}$.

So we get $\boldsymbol{\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \Rightarrow a^3 + b^3 + c^3 - 3abc}$.

Next, we want to show: $\boldsymbol{a^3 + b^3 + c^3 - 3abc \Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

To begin with, we have an identity as follows.
$\boldsymbol{a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}$.

So we want to show this:
$\boldsymbol{(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

Thus, we want to show $\boldsymbol{(a^2 + b^2 + c^2 - ab - bc - ca) \Rightarrow \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

Now, compensating, we get $\boldsymbol{a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\cdot 2(a^2 + b^2 + c^2 - ab - bc - ca)}$

$\boldsymbol{= \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = \frac{1}{2}(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2)}$

$\boldsymbol{= \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

So we get $\boldsymbol{a^3 + b^3 + c^3 - 3abc \Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

Therefore, $\boldsymbol{a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming that $a + b = 19$, and that $ab = 29$, find the value of $a^3 + b^3$.

$$(a + b)^3 = a^3 + 3ab(a + b) + b^3 \Rightarrow a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 19^3 - 3\cdot 29\cdot 19$$

$$= 6859 - 1653 = 5206.$$

If not quite sure of the idea behind the processes above, follow the steps below.

Solving the system of equations where $a + b = 19$ and $ab = 29$, we can get the values of a and b. And then, putting the values into $a^3 + b^3$, we can get the value of $a^3 + b^3$.

Solving the system is though, nothing but solving the quadratic equation below.

$x^2 - 19x + 29 = 0$.   Why?

That's because $x^2 - (a + b)x + ab = (x - a)(x - b) = 0 \Rightarrow x = a$ or b.

And we have $a + b = 19$, and $ab = 29$.

So if we can factorize $x^2 - 19x + 29$, we can get the solution.

And thus, the values of a and b are the roots of the equation $x^2 - 19x + 29 = 0$.

We have however, $19^2 - 4\cdot 1\cdot 29 = 245$, which is called the discriminant, and is not an integer squared. So the roots are not simple numbers as integers or rational numbers but irrational numbers.

That is, $x^2 - 19x + 29$ is not factorized to $(x - m)(x - n)$ where m and n are integers.

So the roots are numbers not simple enough. Thus, if we put them directly into $\boldsymbol{a^3 + b^3}$, it can hardly be the case where we can get the value of $\boldsymbol{a^3 + b^3}$ fast enough.

Using in fact, the quadratic formula, we get

$x = \frac{19 \pm \sqrt{19^2 - 4 \cdot 1 \cdot 29}}{2} = \frac{19 \pm \sqrt{245}}{2} = \frac{19 \pm \sqrt{5 \cdot 49}}{2} = \frac{19 \pm 7\sqrt{5}}{2}$, which are the values of $\boldsymbol{a}$ and $\boldsymbol{b}$.

And taking the cubes of those two, we get numbers quite complicated.
We can get around with such a mess though, using a factorization identity.
What factorization identity though?

We have a couple of identities as follows.

$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$ and $\boldsymbol{(a + b)^3 = a^3 + 3ab(a + b) + b^3}$.

We can use either of the two above, but the second one is better.
So using the second, we get

$\boldsymbol{(a + b)^3 = a^3 + 3ab(a + b) + b^3 \Rightarrow a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 19^3 - 3 \cdot 29 \cdot 19}$

$\boldsymbol{= 6859 - 1653 = 5206}$.

Let's next, use the first identity.

Then, we want to get the value of $\boldsymbol{a^2 + b^2}$, first, which can be obtained from another identity where $\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$.

Then, we get first,

$\boldsymbol{a^2 + b^2 = (a + b)^2 - 2ab = 19^2 - 2 \cdot 29 = (20 - 1)^2 - 58 = 400 - 40 + 1 - 58}$

$\boldsymbol{= 301 + 100 - 98 = 303}$.

So next, we get $\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2) = 19(303 - 29) = 19 \cdot 274 = 5206}$.