# Examples 1 in GCD and LCM



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# Examples 1 in GCD and LCM

Find the GCD (Greatest Common Divisor) and LCM (Least Common Multiple) in each of the cases below.

0. $\mathbf{72}\boldsymbol{x^2y^3z^5}$ and $\mathbf{16}\boldsymbol{x^2y^4z^7}$

1. $\boldsymbol{x^2(x+y)^4(x+z)}$ and $\boldsymbol{x^5(x+y)^2(x+z)^8}$

2. $\mathbf{72}$ and $\mathbf{16}\boldsymbol{x^2y^4z^7}$

## Suggestions or Solutions

### To the Problem in the Example 0

**Find the GCD and LCM of $72x^2y^3z^5$ and $16x^2y^4z^7$.**

To begin with, what is GCD?

It is the acronym of Greatest Common Divisor. So it is a divisor common and the largest. Looking for GCD though, we don't want to just begin with finding all divisors common. What then, do we begin with?

Suppose for instance, we want to find the GCD of a set of integers.

Then usually, we take a product of all factors, each of which is a power where the base is a prime factor common to all the integers, and the exponent is the smallest of all used for the prime factor. So the product has all the prime factors common to all the integers.

So the product is the divisor the greatest and common to all the integers, and thus, is the GCD. Thus, we want to begin with factorizing all the integers.

And the same is true for monomials and polynomials, too.

Let's find for instance, the GCD of 216, 144, and 360.

• First, factorizing all the integers, we get $216 = 2^3 3^3$, $144 = 2^4 3^2$, and $360 = 2^3 3^2 5$.

• Next, finding all the prime factors common to all the integers, we get 2 and 3.

• Next, we find the smallest of all exponents used for each of the prime factors above.

Then, 3 is the smallest exponent used for the prime factor 2, and for the prime factor 3, 2 is the smallest exponent used.

So we get two powers, in one of which, the base is 2, and the exponent is 3, and in the other, the base is 3, and the exponent is 2. That is, we get $2^3$ and $3^2$.

Thus, taking the product of two powers, we get the GCD, which is $2^2 3^2$.

So in short, finding the GCD of a set of integers, we get all the prime factors common to all the integers, together with the exponents the smallest of all used for the prime factors, and then, take the product of the powers.

And the same is true for a GCD of monomials and polynomials, too.

Now in this problem, we want to find the GCD of two monomials $\mathbf{72}\boldsymbol{x^2y^3z^5}$ and $\mathbf{16}\boldsymbol{x^2y^4z^7}$.

• To begin with, factorizing the monomials, we get $\mathbf{2^3 3^2}\boldsymbol{x^2y^3z^5}$ and $\mathbf{2^4}\boldsymbol{x^2y^4z^7}$.

• Next, finding all the prime factors common to both monomials, we get **2**, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

• Next, apply to each factor above the smallest of all exponents used for the factor.

3 is the smallest exponent used for the prime factor **2**.
2 is the smallest exponent used for the prime factor $\boldsymbol{x}$.
3 is the smallest exponent used for the prime factor $\boldsymbol{y}$.
5 is the smallest exponent used for the prime factor $\boldsymbol{z}$.

• So we get four powers, which are $\mathbf{2^3}$, $\boldsymbol{x^2}$, $\boldsymbol{y^3}$, and $\boldsymbol{z^5}$.

For instance, in the power $\boldsymbol{x^2}$, the base is $\boldsymbol{x}$, which is one of the prime factors common to both monomials, and the exponent is 2, which is the smallest of all the exponents used for the prime factor $\boldsymbol{x}$.

Thus, taking the product of all the powers above, we get $\mathbf{2^3}\boldsymbol{x^2y^3z^5}$, which is the GCD of the two monomials $\mathbf{72}\boldsymbol{x^2y^3z^5}$ and $\mathbf{16}\boldsymbol{x^2y^4z^7}$.

•• Let's next, move on to the LCM. Then, to begin with, what do we mean by LCM?

It is the acronym of Lest Common Multiple. So it is a multiple common and the smallest.

Suppose for instance, we want to find the LCM of a set of integers.
Suppose also, all the integers are fully factorized already.

Then, a multiple common to all the integers is a product of powers, has at least all the prime factors all the integers have, and the exponent to be used for each prime factor is at least the largest of all the exponents used for the prime factor.

Thus, the LCM, the least of all the multiples common to a set of integers has *all the prime factors* all the integers have, and the *exponent* to be used for each prime factor is *the largest* of all the exponents used for the prime factor.

●●● So in short, finding the LCM of a set of integers, we get all the prime factors all the integers have, together with the exponents the largest of all used for the prime factors, and then, take the product of the powers.

And the same is true for monomials and polynomials, too.

Now, in the case of $\mathbf{72}\boldsymbol{x^2y^3z^5}$ and $\mathbf{16}\boldsymbol{x^2y^4z^7}$, we can factorize them to $\mathbf{2^3 3^2}\boldsymbol{x^2y^3z^5}$ and $\mathbf{2^4}\boldsymbol{x^2y^4z^7}$.

So a multiple common to both monomials has to have as prime factors at least **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, and the exponents to be used for the prime factors are at least 4, 2, 2, 4, and 7 respectively in the order of **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

●●● So in short, finding the LCM of a set of monomials, we get all the prime factors all the monomials have, together with the exponents the largest of all used for the prime factors, and then, take the product of the powers.

So first, finding all the prime factors the two monomials have, we get **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

Next, 4 is the largest exponent used for the factor 2, 2 is the one used for the factor 3, 2 is the one used for $\boldsymbol{x}$, 4 is the one used for $\boldsymbol{y}$, and 7 is the one for $\boldsymbol{z}$.

Therefore, the LCM of $\mathbf{2^3 3^2}\boldsymbol{x^2y^3z^5}$ and $\mathbf{2^4}\boldsymbol{x^2y^4z^7}$ is $\mathbf{2^4 3^2}\boldsymbol{x^2y^4z^7}$.

## Suggestions or Solutions

### To the Problem in the Example 1

**Find the GCD and LCM of $x^2(x+y)^4(x+z)$ and $x^5(x+y)^2(x+z)^8$.**

A GCD is a divisor common and the largest.

Thus, the GCD of a set of polynomials is the greatest of all the divisors common to all the polynomials. More specifically, it is the product of all factors common to all the polynomials. What are those factors, though?

The factors are powers, in each of which the base is a prime factor common to all the polynomials, and the exponent is the smallest of all used for the prime factor.

So the product has all the prime factors common to all the polynomial, and thus, is the divisor the greatest and common to all the polynomials, that is, the GCD.

Let's now, begin with the GCD of $\mathbf{12x^2(x+y)^4(x+z)}$ and $\mathbf{8x^5(x+y)^2(x+z)^8}$.

• First, factorizing the two polynomials, we get
$\mathbf{2^2 3x^2(x+y)^4(x+z)}$ and $\mathbf{2^3x^5(x+y)^2(x+z)^8}$.

• So next, all the prime factors common to both polynomials are $\mathbf{2}$, $\mathbf{x}$, $\mathbf{x+y}$, and $\mathbf{x+z}$.

• Next, find the smallest of all exponents used for each of the factors above.

Then, 2 is the smallest exponent used for each of the three factors $\mathbf{2}$, $\mathbf{x}$, and $\mathbf{x+y}$, and for the factor $\mathbf{x+z}$, 1 is the smallest exponent used.

• So four powers $\mathbf{2^2}$, $\mathbf{x^2}$, $\mathbf{(x+y)^2}$, and $\mathbf{(x+z)^1}$ are factors common to $\mathbf{2^2 3x^2(x+y)^4(x+z)}$ and $\mathbf{2^3x^5(x+y)^2(x+z)^8}$.

Note that $\boldsymbol{x + z = (x + z)^1}$, so taking ($\boldsymbol{x + z}$) as a power, we take as the base the binomial ($\boldsymbol{x + y}$), which is one of the prime factors common to both polynomials, and we take 1 as the exponent. And 1 is the smallest of all the exponents used for the prime factor $\boldsymbol{x + y}$.

• And next, take the product of all the four powers above.

Then, we get $\mathbf{2^2}\boldsymbol{x^2(x + y)^2(x + z)}$, which is the GCD.    How come though?

First, each of the two polynomials $\mathbf{2^2 3^1}\boldsymbol{x^2(x + y)^4(x + z)}$ and $\mathbf{2^3}\boldsymbol{x^5(x + y)^2(x + z)^8}$ is a product of powers, in each of which the base is 2, 3, $\boldsymbol{x}$, $\boldsymbol{x + y}$, or $\boldsymbol{x + z}$.

And next,

Of all powers of 2, the one where the exponent is 2 can divide both polynomials, and is the greatest.

Of all powers of $\boldsymbol{x}$, the one where the exponent is 2 can divide both polynomials, and is the greatest.

Of all powers where the base is $\boldsymbol{x + y}$, the one where the exponent is 2 can divide both polynomials, and is the greatest.

Of all powers of ($\boldsymbol{x + z}$), the one where the exponent is 1 can divide both polynomials, and is the greatest.

Therefore, the product of all the four powers, that is, $\mathbf{2^2}\boldsymbol{x^2(x + y)^2(x + z)}$ is the greatest divisor common to both polynomials, and is called the GCD.

Usually though, we find the GCD the way below, which is in fact, no other than the way above though.

• First, fully factorize all in the polynomials, $\mathbf{12}\boldsymbol{x^2(x + y)^4(x + z)}$ and $\mathbf{8}\boldsymbol{x^5(x + y)^2(x + z)^8}$.

Then, we get $\mathbf{2^2 3}\boldsymbol{x^2(x + y)^4(x + z)}$ and $\mathbf{2^3}\boldsymbol{x^5(x + y)^2(x + z)^8}$.

• Next, find all the prime factors common to all the polynomials.

Then, we get **2**, $\boldsymbol{x}$, $\boldsymbol{x + y}$, and $\boldsymbol{x + z}$.

• Next, take the product of all the prime factors above, and then, apply the smallest of all exponents used for each of the prime factors.

Then, we get $\mathbf{2^2}\boldsymbol{x^2(x + y)^2(x + z)}$, which is the GCD.

••• Therefore in short, finding the GCD of a set of polynomials, we get all the prime factors common to all the polynomials, together with the exponents the smallest of all used for the prime factors, and then, take the product of the powers.

And the same is true for integers and monomials, too.

• Let's next, move on to the LCM.

The LCM is the least common multiple.
So the LCM of a set of polynomial is the smallest of all multiples common to all the polynomials in the set. How then, can we get the LCM?

The LCM has to be a common multiple, first, and next, has to be the least.
So beginning with a common multiple, we can put it the way below.

A common multiple of a set of polynomials has to have at least all the prime factors all the polynomials have, because each of all the polynomials has to divide the common multiple.
And the exponent to be used for each prime factor is at least the largest of all the exponents used for the prime factor.

So next, the least of all the multiples common to all the polynomials has to have *all the prime factors* all the polynomials have, and the *exponent* to be used for each prime factor is *the largest* of all the exponents used for the prime factor.

And the same is true for integers and monomials, too.

Now, in this example, we want to get the LCM of two polynomials, which are below.

$\mathbf{12}\boldsymbol{x}^{2}\mathbf{(}\boldsymbol{x}+\boldsymbol{y}\mathbf{)}^{4}\mathbf{(}\boldsymbol{x}+\boldsymbol{z}\mathbf{)}$ and $\mathbf{8}\boldsymbol{x}^{5}\mathbf{(}\boldsymbol{x}+\boldsymbol{y}\mathbf{)}^{2}\mathbf{(}\boldsymbol{x}+\boldsymbol{z}\mathbf{)}^{8}$, which are factorized to $\mathbf{2^{2}3}\boldsymbol{x}^{2}\mathbf{(}\boldsymbol{x}+\boldsymbol{y}\mathbf{)}^{4}\mathbf{(}\boldsymbol{x}+\boldsymbol{z}\mathbf{)}$ and $\mathbf{2^{3}}\boldsymbol{x}^{5}\mathbf{(}\boldsymbol{x}+\boldsymbol{y}\mathbf{)}^{2}\mathbf{(}\boldsymbol{x}+\boldsymbol{z}\mathbf{)}^{8}$.

So all the prime factors both polynomials have are **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{x+y}$, and $\boldsymbol{x+z}$.

And next,

3 is the largest exponent used for the factor 2,

1 is the largest exponent used for the factor 3,

5 is the largest exponent used for $\boldsymbol{x}$,

4 is the largest exponent used for $\boldsymbol{x+y}$,

and 8 is the largest exponent used for $\boldsymbol{x+z}$.

So a multiple common to both polynomials has as prime factors <u>at least</u> **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{x+y}$, and $\boldsymbol{x+z}$. And the exponents to be used for the prime factors are <u>at least</u> 3, 1, 5, 4, and 8 respectively in the order of **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{x+y}$, and $\boldsymbol{x+z}$.

And thus, the LCM, the <u>least</u> of all the multiples common to both polynomials is

$\mathbf{2^{3}3}\boldsymbol{x}^{5}\mathbf{(}\boldsymbol{x}+\boldsymbol{y}\mathbf{)}^{4}\mathbf{(}\boldsymbol{x}+\boldsymbol{z}\mathbf{)}^{8}$.

## Suggestions or Solutions
### To the Problem in the Example 2

**Find the GCD and LCM of 72 and $16x^2y^4z^7$.**

To begin with, a GCD is a divisor common and the largest.

So the GCD of **72** and $\mathbf{16x^2y^4z^7}$ is the greatest of all divisors common to **72** and $\mathbf{16x^2y^4z^7}$.

And getting the GCD, we get the product of all powers, in each of which the base is a prime factor common to both **72** and $\mathbf{16x^2y^4z^7}$, and the exponent is the smallest of all used for the prime factor.

• So we want to begin with factorizing **72** and $\mathbf{16x^2y^4z^7}$. Then, we get $\mathbf{2^3 3^2}$ and $\mathbf{2^4x^2y^4z^7}$.

• Next, finding all the prime factors common to $\mathbf{2^3 3^2}$ and $\mathbf{2^4x^2y^4z^7}$, we get **2** only.

So in this case, we get one power only where the base is a prime factor common to $\mathbf{2^3 3^2}$ and $\mathbf{2^4x^2y^4z^7}$, and therefore, we don't need to take such a product as described above.

Thus, the GCD is just a power, where the base is **2**, which is the prime factor, and the exponent is the smallest of all the exponents used for the prime factor **2**.

The smallest exponent is 3. So the GCD is $\mathbf{2^3}$.

Let's next, move on to the LCM, which is the smallest and common multiple.

To begin with, for instance, a multiple common to a set of integers has at least all the prime factors all the integers have, and the exponent to be used for each prime factor is at least the largest of all used for the prime factor.

So the LCM, the multiple the least and common to all the integers has to have *all the prime factors* all the integers have, and the *exponent* to be used for each prime factor is *the largest* of all used for the prime factor.

Now, in the case of **72** and $\boldsymbol{16x^2y^4z^7}$, which are fully factorized to $\mathbf{2^3 3^2}$ and $\boldsymbol{2^4x^2y^4z^7}$, all the prime factors used in both are **2**, **3**, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

4 is the largest exponent used for the factor 2,

2 is the largest exponent used for 3,

2 is the largest exponent used for $\boldsymbol{x}$,

4 is the largest exponent used for $\boldsymbol{y}$,

and 7 is the largest exponent used for $\boldsymbol{z}$.

So a multiple common to $\mathbf{2^3 3^2}$ and $\boldsymbol{2^4x^2y^4z^7}$ has as prime factors <u>at least</u> 2, 3, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, and the exponents to be used for the prime factors are <u>at least</u> 4, 2, 2, 4, and 7 respectively.

Therefore, the LCM of $\mathbf{2^3 3^2}$ and $\boldsymbol{2^4x^2y^4z^7}$ is $\boldsymbol{2^4 3^2 x^2y^4z^7}$.