Examples 3 in GCD and LCM

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Examples 3 in GCD and LCM

Find the GCD (Greatest Common Divisor) and LCM (Least Common Multiple) in each of the cases below.

0. $\mathbf{(x + 1)^4(y - 1)}$ and $\mathbf{(x + 2)(y + 1)^2}$

1. $\mathbf{2x^4 + 20x^3 + 62x^2 + 60x}$ and $\mathbf{x^4 + x^3 - 2x^2}$

2. $\mathbf{t^2(t^2 - 9)(t + 5)^{-2}(t + 3)^2(t - 2)}$ and $\mathbf{t^3(t + 3)^4(t - 3)^2(t - 1)(t + 5)^2}$

3. $\mathbf{(9x - 3x^2 - 27)(x + 2)^3(x - 1)^3(x + 1)(x - 3)}$ and $\mathbf{9(x^3 + 27)^2(x^2 + 4x + 4)(x^2 - 1)^2}$

4. $\mathbf{12(x^2 - 1)^2(x^3 - 1)(x + 3)}$ and $\mathbf{54(x + 1)^3(x + 3)^2(x^2 - x - 2)^2(x - 3)}$

Suggestions or Solutions

To the Problem in the Example 0

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Find the GCD and LCM of $(x + 1)^4(y – 1)$ and $(x + 2)(y + 1)^2$.

Getting the GCD of a set of polynomials, we get the product of all powers, in each of which the base is a prime factor common to all the polynomials, and the exponent is the smallest of all used for the prime factor.

Now, we are given two polynomials, fully factorized already, so we are ready to get the GCD. Finding however, all the prime factors common to both polynomials, we get none. What then, is the GCD?

Every polynomial has at least two divisors, which are 1 and itself.
So what are divisors common to $(x + 1)^4(y – 1)$ and $(x + 2)(y + 1)^2$?

It is 1, which is the only divisor, and therefore, is the GCD. So 1 is the only divisor common to every polynomial. And the same is true, too, for integers, monomials, etc.

Let's next, move on to the LCM, which is a multiple common and the smallest.

A multiple common to a set of polynomials has at least all the prime factors all the polynomials have, and the exponent to be used for each prime factor is at least the largest of all used for the prime factor.

Thus, the least multiple common to all the polynomials has *all prime factors* all the polynomials have, and the *exponent* to be used for each prime factor is *the largest* of all used for the prime factor.

Now, in the case of two polynomials $\mathbf{(x + 1)^4(y - 1)}$ and $\mathbf{(x + 2)(y + 1)^2}$, all the prime factors the two have are $\mathbf{x + 1}$, $\mathbf{y - 1}$, $\mathbf{x + 2}$, $\mathbf{x}$, and $\mathbf{y + 1}$,

4 is the largest exponent used for $\mathbf{x + 1}$,

1 is the largest exponent used for each of $\mathbf{y - 1}$ and $\mathbf{x + 2}$,

and 2 is largest exponent used for $\mathbf{y + 1}$.

So a multiple common to the two polynomials has to have as prime factors <u>at least</u> $\mathbf{x + 1}$, $\mathbf{y - 1}$, $\mathbf{x + 2}$, and $\mathbf{y + 1}$, and the exponents to be used for the prime factors are <u>at least</u> 4, 1, 1, and 2 respectively in the order of $\mathbf{x + 1}$, $\mathbf{y - 1}$, $\mathbf{x + 2}$, and $\mathbf{y + 1}$.

Therefore, the LCM of the three polynomials is $\mathbf{(x + 1)^4(y - 1)(x + 2)(y + 1)^2}$.

Suggestions or Solutions
To the Problem in the Example 1

Find the GCD and LCM of $2x^4 + 20x^3 + 62x^2 + 60x$ and $x^4 + x^3 - 2x^2$.

> The GCD of a set of polynomials is the product of all powers, in each of which the base is a prime factor common to all the polynomials, and the exponent is the smallest of all used for the prime factor. And the same is true for integers and monomials, too.

Now, in this problem, we are given two polynomials, which are not factorized yet.

• So first, we want to factorize the two polynomials.

To begin with, we get $2x^4 + 20x^3 + 62x^2 + 60x = 2x(x^3 + 10x^2 + 31x + 30)$.

Next, if $x - d$ is a factor of $x^3 + 10x^2 + 31x + 30$, we can set

$x^3 + 10x^2 + 31x + 30 = (x - d)Q$, where Q is a polynomials of degree 2 as $ax^2 + bx + c$.

So we get $d^3 + 10d^2 + 31d + 30 = 0$. How?

That's because $x^3 + 10x^2 + 31x + 30 = (x - d)Q$, and $(d - d)Q = 0$.

How do we get such d, though?

Expanding (or simplifying) $(x - d)(ax^2 + bx + c)$, we get $-dc = 30$. How?

Setting $\boldsymbol{Q = ax^2 + bx + c}$, we get $\boldsymbol{x^3 + 10x^2 + 31x + 30 = (x - d)(ax^2 + bx + c)}$

$\boldsymbol{= ax^3 + bx^2 + cx - adx^2 - bdx - cd = ax^3 + (b - ad)x^2 + (c - bd)x - cd}$.

Thus, we get $\boldsymbol{a = 1}$, $\boldsymbol{b - ad = 10}$, $\boldsymbol{c - bd = 31}$, and $\boldsymbol{-dc = 30}$.

So $\boldsymbol{d}$ is a divisor of 30, and thus, is one of ±1, ±2, ±3, ±5, ±6, ±10, ±15, and ±30.

Now, we have set $\boldsymbol{x^3 + 10x^2 + 31x + 30 = (x - d)(ax^2 + bx + c)}$.

So putting a divisor of 30 into $\boldsymbol{x}$ in $\boldsymbol{x^3 + 10x^2 + 31x + 30}$, and getting 0, the divisor is the value of $\boldsymbol{d}$. And if any, we can find $\boldsymbol{d}$ by trial and error with divisors of 30.

What if no divisor can make it 0?

Then, $\boldsymbol{d}$ is not an integer, and $\boldsymbol{x - d}$ where $\boldsymbol{d}$ is an integer is not a factor of the polynomial $\boldsymbol{x^3 + 10x^2 + 31x + 30}$, so we can conclude that the polynomial has no factor, and is prime.

That is, $\boldsymbol{x^3 + 10x^2 + 31x + 30}$ is a prime factor of $\boldsymbol{2x^4 + 20x^3 + 62x^2 + 60x}$. So we can say that $\boldsymbol{2x^4 + 20x^3 + 62x^2 + 60x}$ is fully factorized to $\boldsymbol{2x(x^3 + 10x^2 + 31x + 30)}$.

Now, evaluating $\boldsymbol{x^3 + 10x^2 + 31x + 30}$ for $\boldsymbol{x = -2}$, we get

$(-2)^3 + 10(-2)^2 + 31(-2) + 30 = -8 + 40 - 62 + 30 = -70 + 70 = 0$.

So $\boldsymbol{d = -2}$, and thus, $\boldsymbol{x - d = x + 2}$ is a divisor, that is, a factor of $\boldsymbol{x^3 + 10x^2 + 31x + 30}$.

And next, we can get $\boldsymbol{Q}$, which is the quotient we get dividing the polynomial by $\boldsymbol{x + 2}$.

And getting the quotient, we can apply to the polynomial the synthetic division by $\boldsymbol{x + 2}$.

Then, we get

-2	1	10	31	30
		-2	-16	-30
	1	8	15	0

Thus, dividing $\boldsymbol{x^3 + 10x^2 + 31x + 30}$ by $\boldsymbol{x + 2}$, we get $\boldsymbol{x^2 + 8x + 15}$ as the quotient.

So we get $\boldsymbol{x^3 + 10x^2 + 31x + 30 = (x + 2)(x^2 + 8x + 15)}$.

Next, we want to check to see if the quotient $\boldsymbol{x^2 + 8x + 15}$ can still be factorized.

In fact, $\boldsymbol{x^2 + 8x + 15}$ can be factorized to $\boldsymbol{(x + 5)(x + 3)}$.

That is, $\boldsymbol{2x^4 + 20x^3 + 62x^2 + 60x = 2x(x^3 + 10x^2 + 31x + 30) = 2x(x + 2)(x + 5)(x + 3)}$.

Therefore, $\boldsymbol{2x^4 + 20x^3 + 62x^2 + 60x}$ gets factorized to $\boldsymbol{2x(x + 2)(x + 5)(x + 3)}$.

Next, moving on to $\boldsymbol{x^4 + x^3 - 2x^2}$, we get $\boldsymbol{x^4 + x^3 - 2x^2 = x^2(x^2 + x - 2) = x^2(x + 2)(x - 1)}$.

• So factorizing the two polynomials given in the problem, we get
$\boldsymbol{2x(x + 2)(x + 5)(x + 3)}$, and $\boldsymbol{x^2(x + 2)(x - 1)}$.

• Next, finding all the prime factors common to both polynomials, we get $\boldsymbol{x}$ and $\boldsymbol{x + 2}$.

• Next, we find the smallest of all exponents used for each of the prime factors above.
Then, 1 is the smallest exponent used for each of the prime factors $\boldsymbol{x}$ and $\boldsymbol{x + 1}$.

• Next, we take the product of all the prime factors above, and apply the smallest of all exponents used for each of the prime factors.

Then, we get $\boldsymbol{x(x + 1)}$, which is the GCD of $\boldsymbol{2x(x + 2)(x + 5)(x + 3)}$, and $\boldsymbol{x^2(x + 2)(x - 1)}$.

●●● Let's next, move on to the LCM, which is the multiple common and the smallest.

A multiple common to a set of polynomials has at least all prime factors all the polynomials have, and the exponent to be used for each prime factor is at least the largest of all used for the prime factor.

So the multiple the least and common to a set of polynomials has *all prime factors* all the polynomials have, and the *exponent* to be used for each prime factor is *the largest* of all used for the prime factor.
And the same is true for integers and monomials, too.

• So first, finding all the prime factors both polynomials have, we get

$\boldsymbol{x}$, $\boldsymbol{x+2}$, $\boldsymbol{x+5}$, $\boldsymbol{x+3}$, and $\boldsymbol{x-1}$.

• Next, we take the product of all the prime factors above since the LCM has all the prime factors both polynomials have.

Then, we can simply get $\boldsymbol{x(x+2)(x+5)(x+3)(x-1)}$.

• Next, we find the largest exponent of all used for each of the prime factors above.

Then, 2 is the largest used for $\boldsymbol{x}$, and 1 is the one used for each of $\boldsymbol{x+2}$, $\boldsymbol{x+5}$, $\boldsymbol{x+3}$, and $\boldsymbol{x-1}$.

Therefore, the LCM is $\boldsymbol{x^2(x+2)(x+5)(x+3)(x-1)}$.

Suggestions or Solutions
To the Problem in the Example 2

Find the GCD and LCM of two polynomials below.
$\boldsymbol{t^2(t^2 - 9)(t + 5)^{-2}(t + 3)^2(t - 2)}$ **and** $\boldsymbol{t^3(t + 3)^4(t - 3)^2(t - 1)(t + 5)^2}$**.**

The two polynomials given look already factorized. The first of the two is however, not fully factorized. It has $\boldsymbol{t^2 - 9}$, which can get factorized to $\boldsymbol{(t + 3)(t - 3)}$. So we get

$\boldsymbol{t^2(t^2 - 9)(t + 5)^{-2}(t + 3)^2(t - 2) = t^2(t + 3)(t - 3)(t + 5)^{-2}(t + 3)^2(t - 2)}$
$\boldsymbol{= t^2(t + 3)^3(t - 3)(t + 5)^{-2}(t - 2)}$.

Thus, we now have $\boldsymbol{t^2(t + 3)^3(t - 3)(t + 5)^{-2}(t - 2)}$ and $\boldsymbol{t^3(t + 3)^4(t - 3)^2(t - 1)(t + 5)^2}$.

• So first, finding the GCD, we want to find first, all the prime factors common to both polynomials. Then, we get $\boldsymbol{t}$, $\boldsymbol{t + 3}$, $\boldsymbol{t - 3}$, and $\boldsymbol{t + 5}$.

• Next, we find the smallest of all exponents used for each of the prime factors above.

Then, 2 is the smallest exponent used for each of the prime factors $\boldsymbol{t}$, 3 is the smallest used for $\boldsymbol{t + 3}$, 1 is the one for $\boldsymbol{t - 3}$, and -2 is used for $\boldsymbol{t + 5}$.

• Next, we take the product of all the prime factors above, and then, apply to each of the factors the smallest of all exponents used for the factor.

Then, we get $\boldsymbol{t^2(t + 3)^3(t - 3)(t + 5)^{-2}}$, which is the GCD.

Let's next, move on to the LCM, which is a multiple common and the smallest.

• First, we take the product of <u>all the prime factors</u> all the polynomials have.

Then, we can simply get $\boldsymbol{t(t + 3)(t - 3)(t + 5)(t - 2)(t - 1)}$.

• Next, we apply to each factor above <u>the largest exponent</u> of all used for the factor.

Then, 3 is the largest for the factor $\boldsymbol{t}$, 4 is used for the factor $\boldsymbol{t + 3}$, 2 is used for each of the factors $\boldsymbol{t - 3}$ and $\boldsymbol{t - 5}$, and 1 is used for each of $\boldsymbol{t - 2}$ and $\boldsymbol{t - 1}$.

Therefore, the LCM is $\boldsymbol{t^3(t + 3)^4(t - 3)^2(t + 5)^2(t - 2)(t - 1)}$.

Suggestions or Solutions
To the Problem in the Example 3

Find the GCD and LCM of two polynomials below.
$\mathbf{(9x - 3x^2 - 27)(x + 2)^3(x - 1)^3(x + 1)(x - 3)}$ **and** $\mathbf{9(x^3 + 27)^2(x^2 + 4x + 4)(x^2 - 1)^2}$**.**

• To begin with, we want to factorize the two polynomials given.

Though they both look already factorized, they are not fully factorized.

The first one has $\mathbf{9x - 3x^2 - 27}$, which can still be factorized.
So factorizing it, we get $\mathbf{9x - 3x^2 - 27 = -3(x^2 - 3x + 9)}$.

Next, the second one has $\mathbf{x^2 + 4x + 4}$ and $\mathbf{x^2 - 1}$, both of which can still be factorized.

Factorizing them both, we get $\mathbf{x^2 + 4x + 4 = (x + 2)^2}$, and $\mathbf{x^2 - 1 = (x + 1)(x - 1)}$.

And also, $\mathbf{9(x^3 + 27)^2(x^2 + 4x + 4)(x^2 - 1)^2}$ has $\mathbf{x^3 + 27}$, which can still be factorized, too.
How then, can we get it factorized?

We have a factorization identity, $\mathbf{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$.

So using the identity, we can get $\mathbf{x^3 + 27 = x^3 + 3^3 = (x + 3)(x^2 - 3x + 9)}$. Thus, we get

$\mathbf{9(x^3 + 27)^2(x^2 + 4x + 4)(x^2 - 1)^2 = 9(x + 3)^2(x^2 - 3x + 9)^2(x + 2)^2(x + 1)^2(x - 1)^2}$.
In addition, we have $9 = 3^2$, too.

Therefore, factorizing fully the two polynomials given, we get

$\mathbf{-3(x^2 - 3x + 9)(x + 2)^3(x - 1)^3(x + 1)(x - 3)}$ and
$\mathbf{3^2(x + 3)^2(x^2 - 3x + 9)^2(x + 2)^2(x + 1)^2(x - 1)^2}$.

• So next, finding all the prime factors common to the two polynomials, we get
$\mathbf{3}$, $\mathbf{x^2 - 3x + 9}$, $\mathbf{x + 2}$, $\mathbf{x - 1}$, and $\mathbf{x + 1}$.

• Next, we find the smallest of all exponents used for each of the prime factors above.

Then, 1 is the smallest exponent used for each of **3** and $x^2 - 3x + 9$, 2 is the one for each of $x + 2$ and $x - 1$, and again, 1 is the one used for $x + 1$.

• Next, we take the product of all the prime factors above, and apply the smallest of all exponents used for each of the prime factors.

Then, we get $3(x^2 - 3x + 9)(x + 2)^2(x - 1)^2(x + 1)$, which is the GCD.

Let's next, move on to the LCM, which is a multiple common and the smallest.

• First, we want to collect <u>all the prime factors</u> all the polynomials have, and take the product of the factors.

Then, we can simply get $3(x^2 - 3x + 9)(x + 2)(x - 1)(x + 1)(x - 3)(x + 3)$.

• Next, we want to get <u>the largest exponents</u> of all used for all the prime factors above.

Then, 2 is the largest used for each of 3, $x + 1$, and $x^2 - 3x + 9$, 3 is the one used for each of $x + 2$ and $x - 1$, 1 is the one used for $x - 3$, and 2 is the one used for $x + 3$.

Therefore, the LCM is $3^2(x^2 - 3x + 9)^2(x + 2)^3(x - 1)^3(x + 1)^2(x - 3)(x + 3)^2$.

We can put it this way, too: $9(x^2 - 3x + 9)^2(x + 2)^3(x^2 - 1)^2(x - 1)(x^2 - 9)(x + 3)$.
That's because
$(x - 1)^3(x + 1)^2 = (x - 1)(x - 1)^2(x + 1)^2 = (x - 1)\{(x - 1)(x + 1)\}^2 =(x - 1)(x^2 - 1)^2$.
$(x - 3)(x + 3)^2 = (x - 3)(x + 3)(x + 3) = (x^2 - 9)(x + 3)$.

We can put it this way, too: $9(x^2 - 3x + 9)^2(x + 2)^3(x^2 - 1)^2(x^2 - 9)(x^2 + 2x - 3)$.
That's because $(x - 1)(x + 3) = x^2 + 2x - 3$.

We can put it this way, too: $9(x^2 - 3x + 9)^2(x + 2)^3(x^4 - 10x^2 + 9)(x^2 - 1)(x^2 + 2x - 3)$.

That's because $(x^2 - 1)^2(x^2 - 9) = (x^2 - 1)(x^2 - 1)(x^2 - 9) = (x^2 - 1)(x^4 - 10x^2 + 9)$.

Suggestions or Solutions
To the Problem in the Example 4

Find the GCD and LCM of two polynomials below.
$12(x^2-1)^2(x^3-1)(x+3)$ and $54(x+1)^3(x+3)^2(x^2-x-2)^2(x-3)$.

• First, we want to factorize the two polynomials given.

Though they both look already factorized, they are not fully factorized.

The first one has $\boldsymbol{x^2-1}$ and $\boldsymbol{x^3-1}$, both of which can still be factorized.
Factorizing them both, we get

$\boldsymbol{x^2-1=(x+1)(x-1)}$, and $\boldsymbol{x^3-1=(x-1)(x^2+x+1)}$.

Besides, we have $12=2^2 3$, too.

Thus, we get $\boldsymbol{12\underline{(x^2-1)^2}\underline{(x^3-1)}(x+3)=2^2 3\underline{(x+1)(x-1)}\underline{(x-1)(x^2+x+1)}(x+3)}$

$\boldsymbol{=2^2 3(x+1)(x-1)^2(x^2+x+1)(x+3)}$.

Next, the second one has $\boldsymbol{x^2-x-2}$, which can still be factorized.

Factorizing it, we get $\boldsymbol{x^2-x-2=(x-2)(x+1)}$.

In addition, we have $54=2\cdot 3^3$, too.

Thus, $\boldsymbol{54(x+1)^3(x+3)^2(x^2-x-2)^2(x-3)=2\cdot 3^3(x+1)^3(x+3)^2(x-2)(x+1)(x-3)}$

$\boldsymbol{=2\cdot 3^3(x+1)^4(x+3)^2(x-2)(x-3)}$.

And thus, fully factorizing the two polynomials given, we get

$\boldsymbol{2^2 3(x+1)(x-1)^2(x^2+x+1)(x+3)}$ and $\boldsymbol{2\cdot 3^3(x+1)^4(x+3)^2(x-2)(x-3)}$.

So let's now begin with the GCD of the two above.

• First, finding all the prime factors common to the two polynomials, we get

2, **3**, $\boldsymbol{x+1}$, and $\boldsymbol{x+3}$.

• Next, we take the product of all the prime factors above, and apply to each prime factor the smallest of all exponents used for the prime factor.

Then, we get $\mathbf{2^1 \cdot 3^1 (\boldsymbol{x} + 1)^1 (\boldsymbol{x} + 3)^1}$, which equals $\mathbf{6(\boldsymbol{x} + 1)(\boldsymbol{x} + 3)}$, which is the GCD.

Let's next, move on to the LCM, which is a multiple common and the smallest.

• First, we want to collect <u>all the prime factors</u> all the polynomials have.

Then, we get **2**, **3**, $\boldsymbol{x+1}$, $\boldsymbol{x-1}$, $\boldsymbol{x^2+x+1}$, $\boldsymbol{x+3}$, $\boldsymbol{x-2}$, and $\boldsymbol{x-3}$.

• Next, we get <u>the largest exponents</u> of all used for all the prime factors above.

Then, 2 is the largest used for each of **2**, $\boldsymbol{x-1}$, and $\boldsymbol{x+3}$, 3 is the one used for 3, 4 is the one used for $\boldsymbol{x+1}$, and 1 is the one used for each of $\boldsymbol{x^2+x+1}$, $\boldsymbol{x-2}$, and $\boldsymbol{x-3}$.

• Next, we want to take the product of all the prime factors above, and apply the exponents above to those prime factors respectively.

Then, we get $\mathbf{2^2 3^3 (\boldsymbol{x}^2 + \boldsymbol{x} + 1)(\boldsymbol{x} - 1)^2 (\boldsymbol{x} + 3)^2 (\boldsymbol{x} + 1)^4 (\boldsymbol{x} - 2)(\boldsymbol{x} - 3)}$, which is the LCM.