Examples 4 in GCD and LCM

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0. Find the GCD and LCM of 300 and 24255

1. Find the GCD and the LCM of the polynomials as follows.

$ab^2xy^2(x + y)(x + 2y)(x + 3y)$, $a^2bx^2y(x + y)(x + 2y)$, $abx^3y^2(x + 1)(x + y)^2(x + 3y)$,

$xy(x + y)^2$, and $ax^2y^3(x + y)(x + 2y)^2$.

2. Assuming that G is the GCD of A and B, and that L is the LCM of A and B, show that $LG = AB$.

3. Suppose dividing A by B, we get Q as the quotient and R as the remainder.

Then, we get $A = BQ + R$. Show that the GCD of A and B = the GCD of B and R.

Suggestions or Solutions
To the Problem in the Example 0

Find the GCD and LCM of 300 and 24255.

Finding the GCD of a set of integers, we usually take a product. What product is it?

It is a product of all powers, in each of which the base is a prime factor common to all the integers, and the exponent is the smallest of all used for the prime factor.

So the product has all the prime factors common to all the integers, and thus, is the divisor the greatest and common to all the integers, that is, the GCD.

So in short, finding the GCD of a set of integers, we get all the prime factors common to all the integers, together with the exponents the smallest of all used for the prime factors, and then, take the product of the powers.

• Thus, we want to begin with factorizing 300 and 24255. Then, we get

$300 = 3\cdot100 = 3\cdot4\cdot25 = 2^2 3^1 5^2 = 2^2 3\cdot5^2$.

$24255 = 24000 + 240 + 15 = 3\cdot8000 + 3\cdot80 + 3\cdot5 = 3\cdot(8\cdot1000 + 8\cdot10 + 5)$

$= 3(8\cdot5\cdot200 + 8\cdot2\cdot5 + 5) = 3\cdot5(1600 + 16 + 1) = 3\cdot5(1500 + 90 + 27)$

$= 3\cdot5(3\cdot500 + 3^2 10 + 3^3) = 3^2 5(500 + 30 + 9) = 3^2 5(490 + 49) = 3^2 5\cdot7(70 + 7)$

$= 3^2 5\cdot7^2(10 + 1) = 3^2 5^1 7^2 11^1 = 3^2 5\cdot7^2 11$.

• Next, finding all prime factors common to all the integers, we get 3 and 5.

• Next, we find the smallest of all exponents used for each of the prime factors above.

Then, 1 is the smallest exponent used for each of 3 and 5.

So we get two powers, in one of which, the base is 3, and the exponent is 1, and in the other, the base is 5, and the exponent is 1, too. That is, we get 3^1 and 5^1.

Thus, taking the product of two powers, we get the GCD, which is $3^1 5^1$, which is 15.

Next, what is the LCM of a set of integers?

To begin with, a multiple common to a set of integers has at least all prime factors all the integers have, and the exponent to be used for each of the prime factors is at least the largest of all used for the prime factor.

So the LCM, the least of all multiples common to all the integers has all prime factors all the integers have, and the exponent to be used for each of the prime factors is the largest of all used for the prime factor.

• Thus, finding the LCM of a set of integers, too, we want to begin with factorizing the integers. We have already done so, and they are

$300 = 2^2 3^1 5^2$, and $24255 = 3^2 5 \cdot 7^2 11$.

• Next, finding all the prime factors both integers have, we get **2**, **3**, **5**, **7**, and **11**.

• Next, find the largest exponent used for each of the prime factors above.

Then, 2 is the largest used for each of **2**, **3**, **5**, and **7**, and 1 is the one used for **11**.

Therefore, a multiple common to both integers has as prime factors at least **2**, **3**, **5**, **7**, and **11**, and the exponents to be used for the prime factors are at least 2, 2, 2, 2, and 1 respectively.

Thus, the LCM, that is, the least common multiple is $2^2 3^2 5^2 7^2 11$, which is 485100.

Suggestions or Solutions
To the Problem in the Example 1

Find the GCD and the LCM of the polynomials as follows.
$ab^2xy^2(x + y)(x + 2y)(x + 3y)$, $a^2bx^2y(x + y)(x + 2y)$, $abx^3y^2(x + 1)(x + y)^2(x + 3y)$, $xy(x + y)^2$, and $ax^2y^3(x + y)(x + 2y)^2$.

Finding the GCD of a set of polynomials, we usually take a product. What product is it?

It is a product of all powers, in each of which the base is a prime factor common to all the polynomials, and the exponent is the smallest of all used for the prime factor.

So the product has all the prime factors common to all the polynomials, and thus, is the divisor the greatest and common to all the polynomials, that is, the GCD.

Therefore in short, finding the GCD of a set of polynomials, we get all the prime factors common to all the polynomials, together with the exponents the smallest of all used for the prime factors, and then, take the product of the powers.

• Thus, we want to begin with factorizing the polynomials given.
They all have already been fully factorized.

• So next, we want to find all the prime factors common to all the polynomials.
Then, we get x, y, and $x + y$.

• Next, find the smallest of all the exponents used for each of the prime factors above.
Then, 1 is the smallest exponent used for each of all the prime factors above.

So we get three powers, in one of which, the base is x, and the exponent is 1, in another, the base is y, and the exponent is 1, and in the other, the base is $x + y$, and the exponent is 1, too.
That is, we get x^1, y^1, and $(x + y)^1$, which are just x, y, and $x + y$, of course.
Thus, taking the product of the three powers, we get the GCD, which is $x^1y^1(x + y)^1$, which is $xy(x + y)$.

Next, what is the LCM of a set of polynomials?

To begin with, a multiple common to a set of polynomials has at least all prime factors all the polynomials have, and the exponent to be used for each of the prime factors is at least the largest of all used for the prime factor.

So the LCM, the least of all multiples common to all the polynomials has all prime factors all the polynomials have, and the exponent to be used for each of the prime factors is the largest of all used for the prime factor.

• Thus, finding the LCM of a set of polynomials, too, we want to begin with factorizing the polynomials.

They all have already been fully factorized, and they are

$ab^2xy^2(x + y)(x + 2y)(x + 3y)$, $a^2bx^2y(x + y)(x + 2y)$, $abx^3y^2(x + 1)(x + y)^2(x + 3y)$, $xy(x + y)^2$, and $ax^2y^3(x + y)(x + 2y)^2$.

• So next, finding all the prime factors all the polynomials have, we get

a, b, x, y, $x + 1$, $x + y$, $x + 2y$, and $x + 3y$.

• Next, we want to find the largest exponent used for each of the prime factors above.

Then, 2 is the largest used for each of a, b, $x + y$, and $x + 2y$, and 1 is the one used for each of $x + 1$ and $x + 3y$, and 3 is the one used for each of x and y.

Therefore, a multiple common to all the polynomials has as prime factors at least a, b, x, y, $x + 1$, $x + y$, $x + 2y$, and $x + 3y$, and the exponents to be used for the prime factors are at least 2, 2, 3, 3, 1, 2, 2, and 1 respectively.

Thus, the LCM is $a^2b^2x^3y^3(x + 1)(x + y)^2(x + 2y)^2(x + 3y)$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming that *G* is the GCD of *A* and *B*, and that *L* is the LCM of *A* and *B*, show that *LG* = *AB*.

Suppose ***a*** and ***b*** are prime to each other.

Then, ***A* = *aG***, and ***B* = *bG***, since ***G*** is the GCD of ***A*** and ***B***.

So ***L* = *abG***, and therefore, ***LG* = *abGG* = *aGbG* = *AB***.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what is GCD?

It is the divisor common and the greatest. So for instance, the GCD of a set of integers is the greatest of all divisors common to all the integers.

Suppose ***A* = *aG***, ***B* = *bG***, and ***a*** and ***b*** are prime to each other, that is, there is no factor common to ***a*** and ***b***. Then, ***G*** is the GCD of ***A*** and ***B***.

Next, what is LCM?

It is the multiple common and the least. So for instance, the LCM of a set of integers is the least of all multiples common to all the integers in the set.

Suppose now, ***L*** is the LCM of ***A*** and ***B***. Then, ***L* = *abG***. How come?

To begin with, we have a fact below.

• For instance, a multiple common to a set of integers is a product of powers, and has at least all the prime factors all the integers have, and the exponent to be used for each prime factor is at least the largest of all the exponents used for the prime factor.

So the LCM, the least of all multiples common to all the integers has all prime factors all the integers have, and the exponent to be used for each prime factor is the largest of all used for the prime factor.

And thus, getting the LCM of $\boldsymbol{A}$ and $\boldsymbol{B}$, we get the product of all prime factors that $\boldsymbol{A}$ and $\boldsymbol{B}$ have, and then, apply to each factor the largest of all the exponents used for the factor.

Now, suppose first, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{G}$ are prime to each other. For instance, $\boldsymbol{G = cd}$.

Then, we get $\boldsymbol{A = aG = acd}$, and $\boldsymbol{B = bG = bcd}$.

We know getting the LCM of $\boldsymbol{A}$ and $\boldsymbol{B}$, we get the product of all prime factors that $\boldsymbol{A}$ and $\boldsymbol{B}$ have, and then, apply to each factor the largest of all the exponents used for the factor.

So the LCM of $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\boldsymbol{abcd}$, since all the largest exponents are 1.

Thus, we get $\boldsymbol{L = abG}$, since $\boldsymbol{G = cd}$.

Next, suppose $\boldsymbol{a}$ and $\boldsymbol{G}$ are not prime to each other, for instance, $\boldsymbol{G = ac}$.

Then, we get $\boldsymbol{A = aG = aac = a^2c}$, and $\boldsymbol{B = bG = bac}$.

So we can say that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are all the prime factors that $\boldsymbol{A}$ and $\boldsymbol{B}$ have.

Next, the largest exponent used for $\boldsymbol{a}$ is 2, the one used for each of $\boldsymbol{b}$ and $\boldsymbol{c}$ is 1.

Thus, we get $\boldsymbol{L = a^2bc}$. And we know $\boldsymbol{G = ac}$.

So we get $\boldsymbol{L = a^2bc = abac = abG \Rightarrow L = abG}$.

We will get the same result, too, in the case where $\boldsymbol{b}$ and $\boldsymbol{G}$ are not prime to each other, for instance, $\boldsymbol{G = bc}$.

So putting threads together, we get $\boldsymbol{L = abG}$ if $\boldsymbol{A = aG}$ and $\boldsymbol{B = bG}$.

Thus, we get $\boldsymbol{LG = abGG = aGbG = AB}$.

In short:

Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

Then, $\boldsymbol{A = aG}$, and $\boldsymbol{B = bG}$, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$.

So $\boldsymbol{L = abG}$, and therefore, $\boldsymbol{LG = abGG = aGbG = AB}$.

What if $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, and $\boldsymbol{L}$ is the LCM of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$?

Then, we get $\boldsymbol{LG^2 = ABC}$. How come?

Suppose $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are prime to each other.

Then, $\boldsymbol{A = aG}$, $\boldsymbol{B = bG}$, and $\boldsymbol{C = cG}$, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

So $\boldsymbol{L = abcG}$, and therefore, $\boldsymbol{LG^2 = abcGG^2 = aGbGcG = ABC}$.

Thus by the same token,

suppose $\boldsymbol{G}$ is the GCD of $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, … and $\boldsymbol{A_n}$, and $\boldsymbol{L}$ is the LCM of $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, … and $\boldsymbol{A_n}$.

Then, we get $\boldsymbol{LG^{n-1} = A_1A_2 \ldots A_n}$. How?

Suppose $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, … and $\boldsymbol{a_n}$ are prime to each other.

Then, $\boldsymbol{A_1 = a_1G}$, $\boldsymbol{A_2 = a_2G}$, …, and $\boldsymbol{A_n = a_nG}$, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, …, and $\boldsymbol{A_n}$.

Thus, $\boldsymbol{L = a_1a_2 \ldots a_nG}$, so $\boldsymbol{LG^{n-1} = a_1a_2 \ldots a_nGG^{n-1} = a_1Ga_2 \ldots a_nG = A_1A_2 \ldots A_n}$.

Suggestions or Solutions
To the Problem in the Example 3

Suppose dividing A by B, we get Q as the quotient and R as the remainder. Then, $A = BQ + R$. Show that the GCD of A and B = the GCD of B and R.

Suppose $\boldsymbol{A = aG}$, $\boldsymbol{B = bG}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

Then, $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$, and we get

$\boldsymbol{A = BQ + R \Rightarrow aG = bGQ + R \Rightarrow R = (a - bQ)G}$.

So we now have $\boldsymbol{B = bG}$, and $\boldsymbol{R = (a - bQ)G}$.

Thus, $\boldsymbol{G}$ is a divisor common to $\boldsymbol{B}$ and $\boldsymbol{R}$.

Therefore, if showing that $\boldsymbol{b}$ and $\boldsymbol{(a - bQ)}$ are prime to each other, we show that $\boldsymbol{G}$ is not just a divisor common to $\boldsymbol{B}$ and $\boldsymbol{R}$ but the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$, too.

Now, suppose that $\boldsymbol{b}$ and $\boldsymbol{(a - bQ)}$ are **not** prime to each other.

In other words, we assume that $\boldsymbol{G}$ is **not** the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$.

Then, there is a factor common to $\boldsymbol{b}$ and $\boldsymbol{(a - bQ)}$.

Suppose now, the common factor is $\boldsymbol{m}$.
Then, we can set $\boldsymbol{b = mk}$, and $\boldsymbol{a - bQ = mp}$.

Thus, we get $\boldsymbol{a - bQ = mp \Rightarrow a = bQ + mp = mkQ + mp = m(kQ + p)}$, since $\boldsymbol{b = mk}$.

So we now have $\boldsymbol{a = m(kQ + p)}$, and $\boldsymbol{b = mk}$.

Thus, $\boldsymbol{m}$ is a common divisor of $\boldsymbol{a}$ and $\boldsymbol{b}$, which however, contradicts the fact that $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

So it is not true that ***b*** and **(*a* – *bQ*)** are **not** prime to each other.

That is, ***b*** and **(*a* – *bQ*)** are prime to each other.

So ***G*** is the GCD of ***B*** and ***R***, since ***B* = *bG*** and ***R* = (*a* – *bQ*)*G***.

And we know that ***G*** is the GCD of ***A*** and ***B***.

Therefore, the GCD of ***A*** and ***B*** is the GCD of ***B*** and ***R***.