Examples 5 in GCD and LCM

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Suppose $\boldsymbol{A} = \boldsymbol{x}^4 + 2\boldsymbol{x}^3 - \boldsymbol{x} - 2$, $\boldsymbol{B} = 2\boldsymbol{x}^4 + 3\boldsymbol{x}^3 - \boldsymbol{x}^2 - 4$, and $\boldsymbol{C} = \boldsymbol{x}^3 + 7\boldsymbol{x}^2 + 11\boldsymbol{x} + 2$.

Suppose also, $\boldsymbol{A}$ & $\boldsymbol{B} = \boldsymbol{G}$, where $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$. Then, find these:

0. $(\frac{A}{A\&B})\,\&\,(\frac{B}{A\&B})$, and $(A\,\&\,B)\,\&\,\{(\frac{A}{A\&B})\,\&\,(\frac{B}{A\&B})\}$.

1. $(\boldsymbol{A}$ & $\boldsymbol{B})$ & $\boldsymbol{A}$, $(\boldsymbol{A}$ & $\boldsymbol{C})$ & $\boldsymbol{B}$, $(\boldsymbol{B}$ & $\boldsymbol{C})$ & $\boldsymbol{A}$, and $(\boldsymbol{B}$ & $\boldsymbol{C})$ & $\boldsymbol{B}$.

Suggestions or Solutions
To the Problem in the Example 0

Suppose $A = x^4 + 2x^3 - x - 2$, $B = 2x^4 + 3x^3 - x^2 - 4$, and $C = x^3 + 7x^2 + 11x + 2$.

Suppose also, $A \& B = G$, where G is the GCD of A and B. Then, find these:

$\mathbf{(\frac{A}{A\&B}) \& (\frac{B}{A\&B})}$**, and** $\mathbf{(A \& B) \& \{(\frac{A}{A\&B}) \& (\frac{B}{A\&B})\}}$**.**

> Suppose $\boldsymbol{A = aG}$, and $\boldsymbol{B = bG}$.
>
> Then, $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other, and we get $\boldsymbol{\frac{A}{A\&B} = \frac{aG}{G} = a}$, and $\boldsymbol{\frac{B}{A\&B} = \frac{bG}{G} = b}$.
>
> Thus, $\boldsymbol{(\frac{A}{A\&B}) \& (\frac{B}{A\&B}) = a \& b = 1}$.
>
> So next, we get $\boldsymbol{(A \& B) \& \{(\frac{A}{A\&B}) \& (\frac{B}{A\&B})\} = (A \& B) \& 1 = G \& 1 = 1}$.

If not quite sure of the idea behind the processes above, follow the steps below.

What in the world is &?

'&' in this problem is nothing but an operator, which takes two operands, and produces a result, which is the GCD of the operands.

Thus, in $\boldsymbol{(\frac{A}{A\&B}) \& (\frac{B}{A\&B})}$, $\boldsymbol{\frac{A}{A\&B}}$ and $\boldsymbol{\frac{B}{A\&B}}$ are operands, so $\boldsymbol{(\frac{A}{A\&B}) \& (\frac{B}{A\&B})}$ is no more than the GCD of $\boldsymbol{\frac{A}{A\&B}}$ and $\boldsymbol{\frac{B}{A\&B}}$.

Next, we have $\boldsymbol{A \& B = G}$, where $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$.

So suppose now, $\boldsymbol{A = aG}$, and $\boldsymbol{B = bG}$.

Then, $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$, so we get

$\boldsymbol{\frac{A}{A\&B} = \frac{aG}{G} = a}$, and $\boldsymbol{\frac{B}{A\&B} = \frac{bG}{G} = b}$, since $\boldsymbol{A \& B = G}$, where $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Thus, we get $\boldsymbol{(\frac{A}{A \& B}) \& (\frac{B}{A \& B}) = a \& b}$, which is just 1 since ***a*** and ***b*** are prime to each other.

How come do we get 1 as the GCD?

First of all, 1 can divide any number, constant, monomial, polynomial, etc.

Next, since ***a*** and ***b*** are prime to each other, there is no factor common to both.

That is, other than 1, there is no divisor common to both, so the GCD of both is 1.

- Let's next, move on to $\boldsymbol{(A \& B) \& \{(\frac{A}{A \& B}) \& (\frac{B}{A \& B})\}}$.

First, assuming that $\boldsymbol{A = aG}$, and that $\boldsymbol{B = bG}$, we get $\boldsymbol{(\frac{A}{A \& B}) \& (\frac{B}{A \& B}) = a \& b = 1}$.

So we get $\boldsymbol{(A \& B) \& \{(\frac{A}{A \& B}) \& (\frac{B}{A \& B})\} = (A \& B) \& 1}$.

Next, the GCD of 1 and any object nonzero as 2, $\boldsymbol{x}$, or $\boldsymbol{2x + y}$ is just 1 no matter what the nonzero object may be.

Only 1 can divide 1, and divisors of an object nonzero include 1, too.

So we can see that $\boldsymbol{(A \& B) \& \{(\frac{A}{A \& B}) \& (\frac{B}{A \& B})\} = 1}$.

Suggestions or Solutions
To the Problem in the Example 1

Suppose $A = x^4 + 2x^3 - x - 2$, $B = 2x^4 + 3x^3 - x^2 - 4$, and $C = x^3 + 7x^2 + 11x + 2$.

Suppose also, A & $B = G$, where G is the GCD of A and B. Then, find these:

(A & B) & A, (A & C) & B, (B & C) & A, and (B & C) & B.

Let's find **(A & B) & A** first.

To begin with, suppose $A = aG$, $B = bG$, and a and b are prime to each other.

Then, we get **(A & B) & $A = G$ & $A = G$** since $A = aG$, and A & $B = G$, so G & $aG = G$.

In fact, we get **(A & B) & $B = G$**, too, since $B = bG$, and A & $B = G$, so G & $bG = G$.

So this time, we want to actually find G, which is the GCD of A and B.

Then first, we want to factorize all the three polynomials, which are as follows.

$A = x^4 + 2x^3 - x - 2$, $B = 2x^4 + 3x^3 - x^2 - 4$, and $C = x^3 + 7x^2 + 11x + 2$.

So let's now get going with A first.

To begin with, suppose $x - e$ is a factor of A.   Then, we can set

$x^4 + 2x^3 - x - 2 = (x - e)Q$, where Q is a polynomials of degree 3 as $ax^3 + bx^2 + cx + d$.

So we get $e^4 + 2e^3 - e - 2 = 0$.   How come?

That's because $x^4 + 2x^3 - x - 2 = (x - e)Q$, and $(e - e)Q = 0$.

How do we get such $\boldsymbol{e}$, though?

Expanding (simplifying) $\mathbf{(x - e)(ax^3 + bx^2 + cx + d)}$, in the result, we can get $\mathbf{-ed = -2}$, that is, we get $\mathbf{ed = 2}$. How?

Setting $\mathbf{Q = ax^3 + bx^2 + cx + d}$, we get

$$\mathbf{x^4 + 2x^3 - x - 2 = (x - e)(ax^3 + bx^2 + cx + d)}$$
$$\mathbf{= ax^4 + bx^3 + cx^2 + dx - aex^3 - bex^2 - cex - ed}$$
$$\mathbf{= ax^4 + (b - ae)x^3 + (c - be)x^2 + (d - ce)x - ed}.$$

Thus, we get $\mathbf{a = 1}$, $\mathbf{b - ad = 2}$, $\mathbf{c - be = 0}$, $\mathbf{d - ce = -1}$, and $\mathbf{-ed = -2}$.

Suppose now, $\boldsymbol{d}$ and $\boldsymbol{e}$ are integers. Then, $\boldsymbol{d}$ and $\boldsymbol{e}$ are divisors of 2.
Thus, $\boldsymbol{e}$ can be one of all the divisors ± 1 and ± 2.

So putting one of the divisors into $\boldsymbol{x}$ in $\mathbf{x^4 + 2x^3 - x - 2}$, and getting 0, the one is the value of $\boldsymbol{e}$. We can keep trying with each of all the divisors until we get 0.

So we can find $\boldsymbol{e}$ by trial and error with those divisors. What if no divisor can make it 0?

Then, $\boldsymbol{e}$ is not an integer, so $\mathbf{x - e}$ where $\boldsymbol{e}$ is an integer is not a factor of $\mathbf{x^4 + 2x^3 - x - 2}$.

We cannot just conclude though, that $\mathbf{x^4 + 2x^3 - x - 2}$ has no factor, and thus, is prime.

That is to say that $\mathbf{x^4 + 2x^3 - x - 2}$ could still be factorized other way.

Now, evaluating $\mathbf{x^4 + 2x^3 - x - 2}$ for $\mathbf{x = 1}$, we get $1^4 + 2\cdot 1^3 - 1 - 2 = 1 + 2 - 1 - 2 = 0$.

So $\mathbf{e = 1}$, and thus, $\mathbf{x - e = x - 1}$ is a divisor, that is, a factor of $\mathbf{x^4 + 2x^3 - x - 2}$.

So next, doing the synthetic division by $\boldsymbol{x - 1}$, we get

1	1	2	0	-1	-2
		1	3	3	2
	1	3	3	2	0

Thus, dividing $\boldsymbol{x^4 + 2x^3 - x - 2}$ by $\boldsymbol{x - 1}$, we get $\boldsymbol{x^3 + 3x^2 + 3x + 2}$ as the quotient.

So we get $\boldsymbol{x^4 + 2x^3 - x - 2 = (x - 1)(x^3 + 3x^2 + 3x + 2)}$.   What then is the next?

We want to check to see if the quotient $\boldsymbol{x^3 + 3x^2 + 3x + 2}$ can still be factorized.
Then, by the same token, we can set

$\boldsymbol{x^3 + 3x^2 + 3x + 2 = (x - d)Q}$, where $\boldsymbol{Q}$ is a polynomials of degree 2 as $\boldsymbol{ax^2 + bx + c}$.

So we can use the fact that if $\boldsymbol{d^3 + 3d^2 + 3d + 2 = 0}$, $\boldsymbol{x - d}$ is a divisor of $\boldsymbol{x^3 + 3x^2 + 3x + 2}$.

And in fact, we can get $\boldsymbol{x^3 + 3x^2 + 3x + 2 = 0}$ for $\boldsymbol{x = -2}$, which is a divisor of the numeric term 2 in the polynomial $\boldsymbol{x^3 + 3x^2 + 3x + 2}$.

And actually, evaluating $\boldsymbol{x^3 + 3x^2 + 3x + 2}$ for $\boldsymbol{x = -2}$, we get

$(-2)^3 + 3\cdot(-2)^2 + 3(-2) + 2 = -8 + 12 - 6 + 2 = 0.$

So $\boldsymbol{x + 2}$ is a divisor, that is, a factor of $\boldsymbol{x^3 + 3x^2 + 3x + 2}$.

Thus next, doing the synthetic division by $\boldsymbol{x + 2}$, we get

-2	1	3	3	2
		-2	-2	-2
	1	1	1	0

Thus, dividing $\boldsymbol{x^3 + 3x^2 + 3x + 2}$ by $\boldsymbol{x + 2}$, we get $\boldsymbol{x^2 + x + 1}$ as the quotient.

So we get $\boldsymbol{x^3 + 3x^2 + 3x + 2 = (x + 2)(x^2 + x + 1)}$.

Next, we want to check to see if the quotient $\boldsymbol{x^2 + x + 1}$ can still be factorized.

In fact, $\boldsymbol{x^2 + x + 1}$ cannot be factorized further.
That is, $\boldsymbol{x^2 + x + 1}$ has no factor, and thus, is prime.

So $\boldsymbol{x^4 + 2x^3 - x - 2 = (x - 1)(x^3 + 3x^2 + 3x + 2) = (x - 1)(x + 2)(x^2 + x + 1)}$.

Therefore, $\boldsymbol{A = x^4 + 2x^3 - x - 2}$ gets factorized to $\boldsymbol{(x - 1)(x + 2)(x^2 + x + 1)}$.

• Let's next, factorize $\boldsymbol{B = 2x^4 + 3x^3 - x^2 - 4}$.

Then, by the same method for the polynomial $\boldsymbol{A}$, we can set

$\boldsymbol{2x^4 + 3x^3 - x^2 - 4 = (x - e)Q}$, where $\boldsymbol{Q}$ is a polynomials of degree 3 as $\boldsymbol{ax^3 + bx^2 + cx + d}$.

So we can use the fact that if $\boldsymbol{2e^4 + 3e^3 - e^2 - 4 = 0}$, $\boldsymbol{x - e}$ is a divisor of $\boldsymbol{B}$.

Putting a divisor of -4 into $\boldsymbol{x}$ in $\boldsymbol{2x^4 + 3x^3 - x^2 - 4}$, and getting 0, we can say that the divisor is $\boldsymbol{e}$.
In fact, evaluating $\boldsymbol{2x^4 + 3x^3 - x^2 - 4}$ for $\boldsymbol{x = 1}$, we get $2 \cdot 1^4 + 3 \cdot 1^3 - 1 - 4 = 5 - 5 = 0$.

So $\boldsymbol{x - 1}$ is a divisor, that is, a factor of $\boldsymbol{2x^4 + 3x^3 - x^2 - 4}$.

So next, doing the synthetic division by $\boldsymbol{x - 1}$, we get

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -1 & 0 & -4 \\ & & 2 & 5 & 4 & 4 \\ \hline & 2 & 5 & 4 & 4 & 0 \end{array}$$

Thus, dividing $\boldsymbol{2x^4 + 3x^3 - x^2 - 4}$ by $\boldsymbol{x - 1}$, we get $\boldsymbol{2x^3 + 5x^2 + 4x + 4}$ as the quotient.

So we get $\boldsymbol{2x^4 + 3x^3 - x^2 - 4 = (x - 1)(2x^3 + 5x^2 + 4x + 4)}$.

Next, we want to check to see if the quotient $\boldsymbol{2x^3 + 5x^2 + 4x + 4}$ can still be factorized.
Then, by the same token, we can set

$\boldsymbol{2x^3 + 5x^2 + 4x + 4 = (x - d)Q}$, where $\boldsymbol{Q}$ is a polynomials of degree 2 as $\boldsymbol{ax^2 + bx + c}$. So we can use the fact that if $\boldsymbol{2d^3 + 5d^2 + 4d + 4 = 0}$, $\boldsymbol{x - d}$ is a divisor of $\boldsymbol{2x^3 + 5x^2 + 4x + 4}$.

Putting a divisor of 4 into $\boldsymbol{x}$ in $\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{5}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{4}$, and getting 0, we can say that the divisor is $\boldsymbol{d}$.

In fact, evaluating $\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{5}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{4}$ for $\boldsymbol{x} = \mathbf{-2}$, we get

$2(-2)^3 + 5\cdot(-2)^2 + 4(-2) + 4 = -16 + 20 - 8 + 4 = 0.$

So $\boldsymbol{x} + \mathbf{2}$ is a divisor, that is, a factor of $\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{5}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{4}$.

So next, doing the synthetic division $\boldsymbol{x} + \mathbf{2}$, we get

$$
\begin{array}{r|rrrr}
-2 & 2 & 5 & 4 & 4 \\
 & & -4 & -2 & -4 \\
\hline
 & 2 & 1 & 2 & 0
\end{array}
$$

Thus, dividing $\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{5}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{4}$ by $\boldsymbol{x} + \mathbf{2}$, we get $\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x} + \mathbf{2}$ as the quotient.

So we get $\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{5}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{4} = (\boldsymbol{x} + \mathbf{2})(\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x} + \mathbf{2})$.   What then, is the next?

We want to check to see if the quotient $\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x} + \mathbf{2}$ can still be factorized.

In fact, $\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x} + \mathbf{2}$ cannot be factorized further, so it has no factor, and thus, is prime.

So $\mathbf{2}\boldsymbol{x}^{\mathbf{4}} + \mathbf{3}\boldsymbol{x}^{\mathbf{3}} - \boldsymbol{x}^{\mathbf{2}} - \mathbf{4} = (\boldsymbol{x} - \mathbf{1})(\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{5}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{4}) = (\boldsymbol{x} - \mathbf{1})(\boldsymbol{x} + \mathbf{2})(\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x} + \mathbf{2})$.

Therefore, $\boldsymbol{B} = \mathbf{2}\boldsymbol{x}^{\mathbf{4}} + \mathbf{3}\boldsymbol{x}^{\mathbf{3}} - \boldsymbol{x}^{\mathbf{2}} - \mathbf{4}$ gets factorized to $(\boldsymbol{x} - \mathbf{1})(\boldsymbol{x} + \mathbf{2})(\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x} + \mathbf{2})$.

- Let's next, factorize $\boldsymbol{C} = \boldsymbol{x}^{\mathbf{3}} + \mathbf{7}\boldsymbol{x}^{\mathbf{2}} + \mathbf{11}\boldsymbol{x} + \mathbf{2}$.

Then, by the same method for the polynomial $\boldsymbol{A}$ or $\boldsymbol{B}$, we can set

$\boldsymbol{x}^{\mathbf{3}} + \mathbf{7}\boldsymbol{x}^{\mathbf{2}} + \mathbf{11}\boldsymbol{x} + \mathbf{2} = (\boldsymbol{x} - \boldsymbol{d})\boldsymbol{Q}$, where $\boldsymbol{Q}$ is a polynomials of degree 2 as $\boldsymbol{a}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{b}\boldsymbol{x} + \boldsymbol{c}$.

So we can use the fact that if $\boldsymbol{d}^{\mathbf{3}} + \mathbf{7}\boldsymbol{d}^{\mathbf{2}} + \mathbf{11}\boldsymbol{d} + \mathbf{2} = \mathbf{0}$, $\boldsymbol{x} - \boldsymbol{d}$ is a divisor of $\boldsymbol{C}$.

Putting a divisor of 2 into $\boldsymbol{x}$ in $\boldsymbol{x^3 + 7x^2 + 11x + 2}$, and getting 0, we can say that the divisor is $\boldsymbol{d}$.

In fact, evaluating $\boldsymbol{x^3 + 7x^2 + 11x + 2}$ for $\boldsymbol{x = -2}$, we get

$(-2)^3 + 7(-2)^2 + 11(-2) + 2 = -8 + 28 - 22 + 2 = 0.$

So $\boldsymbol{x + 2}$ is a divisor, that is, a factor of $\boldsymbol{x^3 + 7x^2 + 11x + 2}$.

So next, doing the synthetic division by $\boldsymbol{x + 2}$, we get

$$\begin{array}{r|rrrr} -2 & 1 & 7 & 11 & 2 \\ & & -2 & -10 & -2 \\ \hline & 1 & 5 & 1 & 0 \end{array}$$

Thus, dividing $\boldsymbol{x^3 + 7x^2 + 11x + 2}$ by $\boldsymbol{x + 2}$, we get $\boldsymbol{x^2 + 5x + 1}$ as the quotient.

So we get $\boldsymbol{x^3 + 7x^2 + 11x + 2 = (x + 2)(x^2 + 5x + 1)}$.

Next, we want to check to see if the quotient $\boldsymbol{x^2 + 5x + 1}$ can still be factorized.

In fact, $\boldsymbol{x^2 + 5x + 1}$ cannot be factorized further, so it has no factor, and thus, is prime.

Therefore, $\boldsymbol{C = x^3 + 7x^2 + 11x + 2}$ gets factorized to $\boldsymbol{(x + 2)(x^2 + 5x + 1)}$.

So we now have

$\boldsymbol{A = x^4 + 2x^3 - x - 2}$ factorized to $\boldsymbol{(x - 1)(x + 2)(x^2 + x + 1)}$.

$\boldsymbol{B = 2x^4 + 3x^3 - x^2 - 4}$ factorized to $\boldsymbol{(x - 1)(x + 2)(2x^2 + x + 2)}$.

$\boldsymbol{C = x^3 + 7x^2 + 11x + 2}$ factorized to $\boldsymbol{(x + 2)(x^2 + 5x + 1)}$.

Thus, the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\boldsymbol{(x - 1)(x + 2)}$.

So **(*A* & *B*) & *A*** $\boldsymbol{= (x - 1)(x + 2)}$. In fact, **(*A* & *B*) & *B*** $\boldsymbol{= (x - 1)(x + 2)}$, too. How come?

We know $\boldsymbol{A}$ **&** $\boldsymbol{B} = \boldsymbol{G}$, and $\boldsymbol{B} = \boldsymbol{bG}$.

So $\mathbf{(}\boldsymbol{A}$ **&** $\boldsymbol{B}\mathbf{)}$ **&** $\boldsymbol{B} = \boldsymbol{G}$ **&** $\boldsymbol{bG} = \boldsymbol{G} = \mathbf{(}\boldsymbol{x} \mathbf{- 1)(}\boldsymbol{x} \mathbf{+ 2)}$.

- Let's next, move on to $\mathbf{(}\boldsymbol{A}$ **&** $\boldsymbol{C}\mathbf{)}$ **&** $\boldsymbol{B}$.

We know $\boldsymbol{A}$ **&** $\boldsymbol{C}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{C}$. So do we actually want to get that GCD, first?

Not necessarily.

Suppose first, $\boldsymbol{A} = \boldsymbol{aG}$, $\boldsymbol{C} = \boldsymbol{cG}$, and $\boldsymbol{a}$ and $\boldsymbol{c}$ are prime to each other.

Then, $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{C}$.

Suppose next, $\boldsymbol{G} = \boldsymbol{gD}$, $\boldsymbol{B} = \boldsymbol{bD}$, and $\boldsymbol{g}$ and $\boldsymbol{b}$ are prime to each other.

Then, $\boldsymbol{D}$ is the GCD of $\boldsymbol{G}$ and $\boldsymbol{B}$, so we get $\mathbf{(}\boldsymbol{A}$ **&** $\boldsymbol{C}\mathbf{)}$ **&** $\boldsymbol{B} = \boldsymbol{D}$.

Meanwhile, $\boldsymbol{A} = \boldsymbol{aG}$, and $\boldsymbol{C} = \boldsymbol{cG}$, so we get $\boldsymbol{A} = \boldsymbol{aG} = \boldsymbol{agD}$, and $\boldsymbol{C} = \boldsymbol{cgD}$ since $\boldsymbol{G} = \boldsymbol{gD}$.

Thus, we get $\boldsymbol{A} = \boldsymbol{agD}$, $\boldsymbol{C} = \boldsymbol{cgD}$, and $\boldsymbol{B} = \boldsymbol{bD}$.

So we get $\boldsymbol{A}$ **&** $\boldsymbol{C} = \boldsymbol{gD}$, which is $\boldsymbol{G}$, of course, since $\boldsymbol{A}$ **&** $\boldsymbol{C}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{C}$.

Thus, we get $\mathbf{(}\boldsymbol{A}$ **&** $\boldsymbol{C}\mathbf{)}$ **&** $\boldsymbol{B} = \boldsymbol{gD}$ **&** $\boldsymbol{bD} = \boldsymbol{D}$ since $\boldsymbol{D}$ is the GCD of $\boldsymbol{gD}$ **&** $\boldsymbol{bD}$.

Now, what is the GCD of $\boldsymbol{A} = \boldsymbol{agD}$, $\boldsymbol{C} = \boldsymbol{cgD}$, and $\boldsymbol{B} = \boldsymbol{bD}$?

It is $\boldsymbol{D}$, of course. So $\mathbf{(}\boldsymbol{A}$ **&** $\boldsymbol{C}\mathbf{)}$ **&** $\boldsymbol{B}$ is nothing but the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

Now, we have

$\boldsymbol{A = x^4 + 2x^3 - x - 2}$ factorized to $\boldsymbol{(x - 1)(x + 2)(x^2 + x + 1)}$.

$\boldsymbol{B = 2x^4 + 3x^3 - x^2 - 4}$ factorized to $\boldsymbol{(x - 1)(x + 2)(2x^2 + x + 2)}$.

$\boldsymbol{C = x^3 + 7x^2 + 11x + 2}$ factorized to $\boldsymbol{(x + 2)(x^2 + 5x + 1)}$.

So the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ is $\boldsymbol{x + 2}$.

In fact, the same is true for $\boldsymbol{(A \,\&\, B) \,\&\, C}$ and $\boldsymbol{(B \,\&\, C) \,\&\, A}$, too. How come?

Suppose first, $\boldsymbol{A = aG}$, $\boldsymbol{B = bG}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

Then, $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$. That is, $\boldsymbol{A \,\&\, B = G}$.

Suppose next, $\boldsymbol{G = gD}$, $\boldsymbol{C = cD}$, and $\boldsymbol{g}$ and $\boldsymbol{c}$ are prime to each other.

Then, $\boldsymbol{D}$ is the GCD of $\boldsymbol{G}$ and $\boldsymbol{C}$, so we get $\boldsymbol{(A \,\&\, B) \,\&\, C = D}$.

Meanwhile, $\boldsymbol{A = aG}$, and $\boldsymbol{B = bG}$, so we get $\boldsymbol{A = aG = agD}$, and $\boldsymbol{B = bgD}$ since $\boldsymbol{G = gD}$.

Thus, we get $\boldsymbol{A = agD}$, $\boldsymbol{B = bgD}$, and $\boldsymbol{C = cD}$. So $\boldsymbol{D}$ is the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

Thus, $\boldsymbol{(A \,\&\, B) \,\&\, C}$ is the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, too.

Suppose next, $\boldsymbol{B = bG}$, $\boldsymbol{C = cG}$, and $\boldsymbol{b}$ and $\boldsymbol{c}$ are prime to each other.

Then, $\boldsymbol{G}$ is the GCD of $\boldsymbol{B}$ and $\boldsymbol{C}$. That is, $\boldsymbol{B \,\&\, C = G}$.

Suppose next, $\boldsymbol{G = gD}$, $\boldsymbol{A = aD}$, and $\boldsymbol{g}$ and $\boldsymbol{a}$ are prime to each other.

Then, $\boldsymbol{D}$ is the GCD of $\boldsymbol{G}$ and $\boldsymbol{A}$, so we get $\boldsymbol{(B \,\&\, C) \,\&\, A = D}$.

Meanwhile, $\boldsymbol{B = bG}$, and $\boldsymbol{C = cG}$, so we get $\boldsymbol{B = bG = bgD}$, and $\boldsymbol{C = cgD}$ since $\boldsymbol{G = gD}$.

Thus, we get $\boldsymbol{A = aD}$, $\boldsymbol{B = bgD}$, and $\boldsymbol{C = cgD}$. So $\boldsymbol{D}$ is the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

Thus, $(B \mathbin{\&} C) \mathbin{\&} A$ is the GCD of A, B, and C, too.

Therefore, $(A \mathbin{\&} B) \mathbin{\&} C = (B \mathbin{\&} C) \mathbin{\&} A = (C \mathbin{\&} A) \mathbin{\&} B$ is the GCD of A, B, and C.

• Let's next, move on to $(B \mathbin{\&} C) \mathbin{\&} B$.

Suppose first, $B = bG$, $C = cG$, and b and c are prime to each other.

Then, G is the GCD of B and C. That is, $B \mathbin{\&} C = G$.

Thus, we get $(B \mathbin{\&} C) \mathbin{\&} B = G \mathbin{\&} B = G$ since $B = bG$, and $B \mathbin{\&} C = G$, so $G \mathbin{\&} bG = G$.

In fact, we get $(B \mathbin{\&} C) \mathbin{\&} C = G$, too, since $C = cG$, and $B \mathbin{\&} C = G$, so $G \mathbin{\&} cG = G$.

So this time, too, we want to actually find G, which is the GCD of B and C.

Now, we have $A = (x - 1)(x + 2)(x^2 + x + 1)$, $B = (x - 1)(x + 2)(2x^2 + x + 2)$, and
$C = (x + 2)(x^2 + 5x + 1)$. Thus, the GCD of B and C is $x + 2$. So $(B \mathbin{\&} C) \mathbin{\&} B = x + 2$.

In short:

Let's first, factorize all the three polynomials, which are as follows.

$A = x^4 + 2x^3 - x - 2$, $B = 2x^4 + 3x^3 - x^2 - 4$, and $C = x^3 + 7x^2 + 11x + 2$.

First, evaluating $A = x^4 + 2x^3 - x - 2$ for $x = 1$, we get
$1^4 + 2\cdot1^3 - 1 - 2 = 1 + 2 - 1 - 2 = 0$.

So $x - 1$ is a divisor of A. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 0 & -1 & -2 \\ & & 1 & 3 & 3 & 2 \\ \hline & 1 & 3 & 3 & 2 & 0 \end{array}$$

So we get $A = x^4 + 2x^3 - x - 2 = (x - 1)(x^3 + 3x^2 + 3x + 2)$.

Next, evaluating $\boldsymbol{x^3 + 3x^2 + 3x + 2}$ for $\boldsymbol{x = -2}$, we get

$(-2)^3 + 3\cdot(-2)^2 + 3(-2) + 2 = -8 + 12 - 6 + 2 = 0.$

So $\boldsymbol{x + 2}$ is a divisor of $\boldsymbol{x^3 + 3x^2 + 3x + 2}$. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 1 & 3 & 3 & 2 \\ & & -2 & -2 & -2 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

So we get $\boldsymbol{x^3 + 3x^2 + 3x + 2 = (x + 2)(x^2 + x + 1)}$.

Thus, $\boldsymbol{A = x^4 + 2x^3 - x - 2 = (x - 1)(x^3 + 3x^2 + 3x + 2) = (x - 1)(x + 2)(x^2 + x + 1)}$.

Next, evaluating $\boldsymbol{B = 2x^4 + 3x^3 - x^2 - 4}$ for $\boldsymbol{x = 1}$, we get

$2\cdot1^4 + 3\cdot1^3 - 1 - 4 = 5 - 5 = 0.$

So $\boldsymbol{x - 1}$ is a divisor of $\boldsymbol{2x^4 + 3x^3 - x^2 - 4}$. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -1 & 0 & -4 \\ & & 2 & 5 & 4 & 4 \\ \hline & 2 & 5 & 4 & 4 & 0 \end{array}$$

So we get $\boldsymbol{B = 2x^4 + 3x^3 - x^2 - 4 = (x - 1)(2x^3 + 5x^2 + 4x + 4)}$.

Next, evaluating $\boldsymbol{2x^3 + 5x^2 + 4x + 4}$ for $\boldsymbol{x = -2}$, we get

$2(-2)^3 + 5\cdot(-2)^2 + 4(-2) + 4 = -16 + 20 - 8 + 4 = 0.$

So $\boldsymbol{x + 2}$ is a divisor of $\boldsymbol{2x^3 + 5x^2 + 4x + 4}$. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 2 & 5 & 4 & 4 \\ & & -4 & -2 & -4 \\ \hline & 2 & 1 & 2 & 0 \end{array}$$

So we get $\boldsymbol{2x^3 + 5x^2 + 4x + 4 = (x + 2)(2x^2 + x + 2)}$.

Thus, $\boldsymbol{B = 2x^4 + 3x^3 - x^2 - 4 = (x - 1)(2x^3 + 5x^2 + 4x + 4) = (x - 1)(x + 2)(2x^2 + x + 2)}$.

Next, evaluating $\boldsymbol{x^3 + 7x^2 + 11x + 2}$ for $\boldsymbol{x = -2}$, we get

$(-2)^3 + 7(-2)^2 + 11(-2) + 2 = -8 + 28 - 22 + 2 = 0$.

So $\boldsymbol{x + 2}$ is a divisor of $\boldsymbol{x^3 + 7x^2 + 11x + 2}$. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 1 & 7 & 11 & 2 \\ & & -2 & -10 & -2 \\ \hline & 1 & 5 & 1 & 0 \end{array}$$

Thus, dividing $\boldsymbol{x^3 + 7x^2 + 11x + 2}$ by $\boldsymbol{x + 2}$, we get $\boldsymbol{x^2 + 5x + 1}$ as the quotient.

Therefore, $\boldsymbol{C = x^3 + 7x^2 + 11x + 2 = (x + 2)(x^2 + 5x + 1)}$.

So we now have

$\boldsymbol{A = (x - 1)(x + 2)(x^2 + x + 1)}$.

$\boldsymbol{B = (x - 1)(x + 2)(2x^2 + x + 2)}$.

$\boldsymbol{C = (x + 2)(x^2 + 5x + 1)}$.

So the GCD of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ is $\boldsymbol{x + 2}$.

Let's now, find $\boldsymbol{(A \mathbin{\&} B) \mathbin{\&} A}$, $\boldsymbol{(A \mathbin{\&} C) \mathbin{\&} B}$, $\boldsymbol{(B \mathbin{\&} C) \mathbin{\&} A}$, and $\boldsymbol{(B \mathbin{\&} C) \mathbin{\&} B}$.

Suppose $\boldsymbol{A = aG}$, $\boldsymbol{B = bG}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

Then, $\boldsymbol{(A \mathbin{\&} B) \mathbin{\&} A = G \mathbin{\&} A = G}$, which is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$, which is $\boldsymbol{(x - 1)(x + 2)}$.

Let's next, move on to $\boldsymbol{(A \mathbin{\&} C) \mathbin{\&} B}$.

Suppose first, $\boldsymbol{A = aG}$, $\boldsymbol{C = cG}$, and $\boldsymbol{a}$ and $\boldsymbol{c}$ are prime to each other.

Then, $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{C}$.

Suppose next, $G = gD$, $B = bD$, and g and b are prime to each other.

Then, D is the GCD of G and B, so we get $(A \ \& \ C) \ \& \ B = D$.

Meanwhile, $A = aG$, and $C = cG$, so we get $A = aG = agD$, and $C = cgD$ since $G = gD$.

Thus, we get $A = agD$, $C = cgD$, and $B = bD$.

So we get $A \ \& \ C = gD$, which is G.

Thus, we get

$(A \ \& \ C) \ \& \ B = gD \ \& \ bD = D$, which is the GCD of $A = agD$, $C = cgD$, and $B = bD$.

The GCD of A, B, and C is $x + 2$.

And in fact, the same is true for $(B \ \& \ C) \ \& \ A$, too. The reason is as follows.

Suppose $B = bG$, $C = cG$, and b and c are prime to each other.

Then, G is the GCD of B and C. That is, $B \ \& \ C = G$.

Suppose next, $G = gD$, $A = aD$, and g and a are prime to each other.

Then, D is the GCD of G and A, so we get $(B \ \& \ C) \ \& \ A = D$.

Meanwhile, $B = bG$, and $C = cG$, so we get $B = bG = bgD$, and $C = cgD$ since $G = gD$.

Thus, we get $A = aD$, $B = bgD$, and $C = cgD$.

So D is the GCD of A, B, and C.

Thus, $(B \ \& \ C) \ \& \ A$ is the GCD of A, B, and C, too.

Let's next, move on to $(B \mathbin{\&} C) \mathbin{\&} B$.

Suppose $B = bG$, $C = cG$, and b and c are prime to each other.

Then, G is the GCD of B and C.

Thus, $(B \mathbin{\&} C) \mathbin{\&} B = G \mathbin{\&} B = G$, which is the GCD of B and C, which is $x + 2$.