Examples 6 in GCD and LCM

Click this to start.

Examples 6 in GCD and LCM

0. Assuming $\boldsymbol{A} = \boldsymbol{x^3 - 2x^2 - 5x + 6}$, and $\boldsymbol{B} = \boldsymbol{x^3 + a^2x^2 + 2ax - 16}$, find all the values of $\boldsymbol{a}$ in each of the cases below.

0.0. The degree of the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$ is ≥ 1.

0.1. The degree of the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$ is 2.

1. Find the GCD of 69300 and 382200.

Suggestions or Solutions

To the Problem 0 in the Example 0

Assuming $A = x^3 – 2x^2 – 5x + 6$, and $B = x^3 + a^2x^2 + 2ax – 16$, find all the values of a in the case where the degree of the GCD of A and B is ≥ 1.

First, $x = 1 \Rightarrow A = x^3 – 2x^2 – 5x + 6 = 1^4 + 2\cdot 1^3 – 1 – 2 = 1 + 2 – 1 – 2 = 0$.

So $x – 1$ is a divisor of A. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

So we get $A = x^3 – 2x^2 – 5x + 6 = (x – 1)(x^2 – x – 6) = (x – 1)(x + 2)(x – 3)$.

Thus, either of $x – 1$, $x + 2$, and $x – 3$ has to be a factor common to A and B.

Assuming first, $x – 1$ is a divisor common to A and B, we get

$x = 1 \Rightarrow B = 1 + a^2 + 2a – 16 = a^2 + 2a – 15 = (a + 5)(a – 3) = 0 \Rightarrow a = -5$ or 3.

Assuming next, $x + 2$ is a divisor common to A and B, we get

$x = -2 \Rightarrow B = -8 + 4a^2 – 4a – 16 = 4a^2 – 4a – 24 = 4(a – 3)(a + 2) = 0 \Rightarrow a = 3$ or -2.

And assuming next, $x – 3$ is a divisor common to A and B, we get

$x = 3 \Rightarrow 27 + 9a^2 + 6a – 16 = 9a^2 + 6a + 11 = (3a + 1) + 10 \geq 10$.

So $x – 3$ cannot be a divisor of B.

Therefore, $a = 3$, -2, or -5.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we mean by the degree of a polynomial?

It is the largest exponent applied to the variable the polynomial is subject to.

So $\boldsymbol{A}$ can be called a polynomial of degree 5.
And a monomial can have a degree, too. For instance, $\mathbf{3}\boldsymbol{x}^{\mathbf{2}}$ is a monomial of degree 2.
What then, about the degree of the GCD?

The GCD of polynomials can be an integer (or constant), a monomial, or a polynomial.

And in particular, an integer (or a constant) can be said to be of degree 0.

That's because we have $\mathbf{5}\boldsymbol{x}^{\mathbf{0}} = \mathbf{5}$, since $\boldsymbol{x}^{\mathbf{0}} = \mathbf{1}$.
And assuming $\boldsymbol{a}$ is constant, we get $\boldsymbol{ax}^{\mathbf{0}} = \boldsymbol{a}$, and $(\boldsymbol{a} + \mathbf{1})\boldsymbol{x}^{\mathbf{0}} = \boldsymbol{a} + \mathbf{1}$, which is constant, since $\boldsymbol{a}$ is constant. So if the degree of the GCD is 0, the GCD is an integer (or constant).

And if for instance, the degree of the GCD is 3, the GCD is a monomial of degree 3 as $\mathbf{2}\boldsymbol{x}^{\mathbf{3}}$ or a polynomial of degree 3 as $\mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \boldsymbol{x} - \mathbf{1}$.

Now, this problem is talking about GCD, which is about divisors, that is, factors.

So let's first, factorize the polynomial $\boldsymbol{A} = \boldsymbol{x}^{\mathbf{3}} - \mathbf{2}\boldsymbol{x}^{\mathbf{2}} - \mathbf{5}\boldsymbol{x} + \mathbf{6}$.

First, evaluating $\boldsymbol{A} = \boldsymbol{x}^{\mathbf{3}} - \mathbf{2}\boldsymbol{x}^{\mathbf{2}} - \mathbf{5}\boldsymbol{x} + \mathbf{6}$ for $\boldsymbol{x} = \mathbf{1}$, we get
$1^4 + 2 \cdot 1^3 - 1 - 2 = 1 + 2 - 1 - 2 = 0$. How to get $\boldsymbol{x} = \mathbf{1}$ is covered in the previous example.

So $\boldsymbol{x} - \mathbf{1}$ is a divisor of $\boldsymbol{A}$. Thus, doing the synthetic division, we get

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

So we get $\boldsymbol{A} = \boldsymbol{x}^{\mathbf{3}} - \mathbf{2}\boldsymbol{x}^{\mathbf{2}} - \mathbf{5}\boldsymbol{x} + \mathbf{6} = (\boldsymbol{x} - \mathbf{1})(\boldsymbol{x}^{\mathbf{2}} - \boldsymbol{x} - \mathbf{6})$.

Next, we want to check to see if the quotient $x^2 - x - 6$ can still be factorized.

In fact, $x^2 - x - 6$ can be factorized to $(x - 3)(x + 2)$.

Therefore, $A = x^3 - 2x^2 - 5x + 6$ gets factorized to $(x - 1)(x + 2)(x - 3)$.

Besides, knowing the fact that the divisors of A can be $x - 1$, $x + 2$, and $x - 3$ ahead of time, we can do synthetic divisions sequentially in such a way as follows.

1	1	-2	-5	6
		1	-1	-6
-2	**1**	**-1**	**-6**	0
		-2	6	
	1	-3	0	

So having only to look at the arrays of numbers above, we can see these:

- $(x - 1)$ is a divisor, and the quotient is $x^2 - x - 6$ from **1**, **-1**, and **-6**.
- $\{x - (-2)\}$, that is, $(x + 2)$ is a divisor, too, and the quotient is $x - 3$ from **1** and **-3**.

• Now, in this problem, we have $A = (x - 1)(x + 2)(x - 3)$, and $B = x^3 + a^2x^2 + 2ax - 16$.

And we have the fact that the degree of the GCD of A and B is ≥ 1.

That is, the degree of the GCD is at least 1.   What then, can the GCD be?

The GCD is an expression monomial or polynomial, and the degree of the expression is at least 1. So we may want to see first, what the expression is.

To begin with, the GCD of A and B is the greatest of all divisors common to A and B.

That is, the greatest divisor common to A and B is the GCD of A and B.

And we know that a factor is a divisor.
So if the GCD exists, a factor common to $\boldsymbol{A}$ and $\boldsymbol{B}$ has to exist anyway.
Thus, at least one of the factors of $\boldsymbol{A}$ has to be the factor of the polynomial $\boldsymbol{B}$. So?

So we want to look at the factors of $\boldsymbol{A}$.

And we have $\boldsymbol{A = (x - 1)(x + 2)(x - 3)}$.

So displaying all the possible factors of $\boldsymbol{A}$, we get

$\boldsymbol{(x - 1)(x + 2)(x - 3)}$, $\boldsymbol{(x - 1)(x + 2)}$, $\boldsymbol{(x - 1)(x - 3)}$, $\boldsymbol{(x + 2)(x - 3)}$, $\boldsymbol{x - 1}$, $\boldsymbol{x + 2}$, and $\boldsymbol{x - 3}$.

Then, the GCD is a polynomial, which is one of the seven factors above.
And we know that the GCD is an expression of degree at least 1.

So of the seven factors above, either of the last three has to be common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

And the three are $\boldsymbol{x - 1}$, $\boldsymbol{x + 2}$, and $\boldsymbol{x - 3}$.
Thus, at least one of the three above is a common factor of $\boldsymbol{A}$ and $\boldsymbol{B}$.

So let's begin with $\boldsymbol{x - 1}$.
Suppose now, $\boldsymbol{x - 1}$ is a common factor, that is, a divisor common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

What then, is $\boldsymbol{B}$ when $\boldsymbol{x = 1}$?

It has to be 0, since $\boldsymbol{x - 1}$ is a divisor of $\boldsymbol{B}$.
That is, $\boldsymbol{B}$ has to be in a form of $\boldsymbol{(x - 1)Q}$, where $\boldsymbol{Q}$ is of degree 2, for $\boldsymbol{B}$ is of degree 3.
So?

We have $\boldsymbol{B = x^3 + a^2x^2 + 2ax - 16}$.
So we get $\boldsymbol{x = 1 \Rightarrow B = 1 + a^2 + 2a - 16 = a^2 + 2a - 15 = (a + 5)(a - 3) = 0}$.

Therefore, we get $\boldsymbol{a} = \mathbf{-5}$ or $\mathbf{3}$.

Suppose next, $\boldsymbol{x} + \mathbf{2}$ is a common factor, that is, a divisor common to $\boldsymbol{A}$ and $\boldsymbol{B}$.
Then, what is $\boldsymbol{B}$ when $\boldsymbol{x} = \mathbf{-2}$?

It has to be 0, too, since $\boldsymbol{x} - \mathbf{2}$ is a divisor of $\boldsymbol{B}$.

That is, $\boldsymbol{B}$ has to be put in $(\boldsymbol{x} + \mathbf{2})\boldsymbol{Q}$, where $\boldsymbol{Q}$ is of degree 2, since $\boldsymbol{B}$ is of degree 3.

And we have $\boldsymbol{B} = \boldsymbol{x}^3 + \boldsymbol{a}^2\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{ax} - \mathbf{16}$.

So we get $\boldsymbol{x} = \mathbf{-2} \Rightarrow \boldsymbol{B} = \mathbf{-8} + \mathbf{4}\boldsymbol{a}^2 - \mathbf{4}\boldsymbol{a} - \mathbf{16} = \mathbf{4}\boldsymbol{a}^2 - \mathbf{4}\boldsymbol{a} - \mathbf{24} = \mathbf{4}(\boldsymbol{a} - \mathbf{3})(\boldsymbol{a} + \mathbf{2}) = \mathbf{0}$.

Therefore, we get $\boldsymbol{a} = \mathbf{3}$ or $\mathbf{-2}$.

And suppose next, $\boldsymbol{x} - \mathbf{3}$ is a factor common to $\boldsymbol{A}$ and $\boldsymbol{B}$. What then, is $\boldsymbol{B}$ when $\boldsymbol{x} = \mathbf{3}$?

It has to be 0, too, of course, since $\boldsymbol{x} - \mathbf{3}$ is a divisor of $\boldsymbol{B}$.

That is, $\boldsymbol{B}$ needs to be put in $(\boldsymbol{x} - \mathbf{3})\boldsymbol{Q}$, where $\boldsymbol{Q}$ is of degree 2, since $\boldsymbol{B}$ is of degree 3.

And since $\boldsymbol{B} = \boldsymbol{x}^3 + \boldsymbol{a}^2\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{ax} - \mathbf{16}$, we get

$\boldsymbol{x} = \mathbf{3} \Rightarrow \boldsymbol{B} = \mathbf{27} + \mathbf{9}\boldsymbol{a}^2 + \mathbf{6}\boldsymbol{a} - \mathbf{16} = \mathbf{9}\boldsymbol{a}^2 + \mathbf{6}\boldsymbol{a} + \mathbf{11} = (\mathbf{3}\boldsymbol{a} + \mathbf{1})^2 + \mathbf{10} \geq \mathbf{10}$. What then?

This time, $\boldsymbol{B}$ cannot be 0 for any value of $\boldsymbol{a}$, and thus, $\boldsymbol{x} - \mathbf{3}$ cannot be a divisor of $\boldsymbol{B}$.

And thus, putting threads together, we get $\boldsymbol{a} = \mathbf{3}$, $\mathbf{-2}$, or $\mathbf{-5}$.

Suggestions or Solutions

To the Problem 1 in the Example 0

Assuming $A = x^3 - 2x^2 - 5x + 6$, and $B = x^3 + a^2x^2 + 2ax - 16$, find all the values of a in the case where the degree of the GCD of A and B is 2.

If $\boldsymbol{a = 3}$ or **-5**, $\boldsymbol{x - 1}$ is a factor common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

If $\boldsymbol{a = 3}$ or **-2**, $\boldsymbol{x + 2}$ is a common factor.

So $\boldsymbol{(x - 1)(x + 2)}$ is the GCD, and satisfying both conditions above, $\boldsymbol{a = 3}$ only.

If not quite sure of the idea behind the processes above, follow the steps below.

In this problem, we have $\boldsymbol{A = (x - 1)(x + 2)(x - 3)}$, and $\boldsymbol{B = x^3 + a^2x^2 + 2ax - 16}$, along with the fact that the degree of the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$ is 2.

So the GCD is a polynomial of degree 2, which is a factor common to $\boldsymbol{A}$ and $\boldsymbol{B}$.
Thus, we may want to get back to the factors of $\boldsymbol{A}$, and they are as follows.

$\boldsymbol{(x - 1)(x + 2)(x - 3)}$, $\boldsymbol{(x - 1)(x + 2)}$, $\boldsymbol{(x - 1)(x - 3)}$, $\boldsymbol{(x + 2)(x - 3)}$, $\boldsymbol{x - 1}$, $\boldsymbol{x + 2}$, and $\boldsymbol{x - 3}$.

Of the seven factors above, either of the three below has to be common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

$\boldsymbol{(x - 1)(x + 2)}$, $\boldsymbol{(x - 1)(x - 3)}$, and $\boldsymbol{(x + 2)(x - 3)}$.

That's because the degree of the GCD is 2.
Thus, at least one of the three above is a common factor of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Of the three however, the last two cannot be applicable.
How come?

We have facts as follows.

If $\boldsymbol{a} = \mathbf{3}$ or $\mathbf{-5}$, then $\boldsymbol{x} - \mathbf{1}$ can be a factor common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

If $\boldsymbol{a} = \mathbf{3}$ or $\mathbf{-2}$, then $\boldsymbol{x} + \mathbf{2}$ can be a common factor.

However, $\boldsymbol{x} - \mathbf{3}$ cannot be a common factor for any value of $\boldsymbol{a}$.   So?

If $\mathbf{(}\boldsymbol{x} - \mathbf{1)(}\boldsymbol{x} - \mathbf{3)}$ is a common factor, $\mathbf{(}\boldsymbol{x} - \mathbf{1)}$ and $\mathbf{(}\boldsymbol{x} - \mathbf{3)}$ both have to be common factors.

If $\mathbf{(}\boldsymbol{x} + \mathbf{2)(}\boldsymbol{x} - \mathbf{3)}$ is a common factor, $\mathbf{(}\boldsymbol{x} + \mathbf{1)}$ and $\mathbf{(}\boldsymbol{x} - \mathbf{3)}$ both have to be common factors.

So neither of $\mathbf{(}\boldsymbol{x} - \mathbf{1)(}\boldsymbol{x} - \mathbf{3)}$ and $\mathbf{(}\boldsymbol{x} + \mathbf{2)(}\boldsymbol{x} - \mathbf{3)}$ can be applicable.

What then, about $\mathbf{(}\boldsymbol{x} - \mathbf{1)(}\boldsymbol{x} + \mathbf{2)}$?

It is in fact, the GCD, because both of $\mathbf{(}\boldsymbol{x} - \mathbf{1)}$ and $\mathbf{(}\boldsymbol{x} + \mathbf{2)}$ can be common factors.

What then, about the value of $\boldsymbol{a}$?

We know $\mathbf{(}\boldsymbol{x} - \mathbf{1)}$ and $\mathbf{(}\boldsymbol{x} + \mathbf{2)}$ can be common factors, and both have to be common factors at the same time if $\mathbf{(}\boldsymbol{x} - \mathbf{1)(}\boldsymbol{x} + \mathbf{2)}$ is a common factor.

So the value of $\boldsymbol{a}$ has to satisfy both conditions that $\boldsymbol{x} - \mathbf{1}$ and $\boldsymbol{x} + \mathbf{2}$ both are factors common to $\boldsymbol{A}$ and $\boldsymbol{B}$ at the same time. And we have the facts as follows.

If $\boldsymbol{x} - \mathbf{1}$ is a factor common to $\boldsymbol{A}$ and $\boldsymbol{B}$, we get $\boldsymbol{a} = \mathbf{3}$ or $\mathbf{-5}$.

If $\boldsymbol{x} + \mathbf{2}$ is a common factor, we get $\boldsymbol{a} = \mathbf{3}$ or $\mathbf{-2}$.   So?

So satisfying both cases above, we need to have $\boldsymbol{a} = \mathbf{3}$ only.

That is, only if $\boldsymbol{a} = \mathbf{3}$, $\mathbf{(}\boldsymbol{x} - \mathbf{1)(}\boldsymbol{x} + \mathbf{2)}$ is the GCD.

Suggestions or Solutions
To the Problem in the Example 1

Find the GCD of 69300 and 382200.

Usually, finding the GCD of a set of integers, we take a product of all powers, in each of which the base is a prime factor common to all the integers, and the exponent is the smallest of all used for the prime factor.

So the product has all prime factors common to all the integers, and thus, is the divisor the greatest and common to all the integers, that is, the GCD. And the same is true for monomials and polynomials, too.

So what should we begin with finding the GCD of the two integers given?

We want to begin with factorizing the two integers.

So factorizing both integers given, we get $69300 = 2^2 3^2 5^2 7 \cdot 11$, and $382200 = 2^3 3 \cdot 5^2 7^2 13$.

• Next, finding all the prime factors common to both integers, we get 2, 3, 5, and 7.

• Next, finding the smallest of all exponents used for each of the prime factors above, we get 2 for the prime factor 2, 1 for the factor 3, 2 for 5, and 1 for 7.

• Next, taking the product of all the powers, we get $2^2 3 \cdot 5^2 7$, which is the GCD, which is 2100 if expanded.

Besides, we can use a theorem regarding the GCD, and the theorem is as follows.

• Suppose dividing $\boldsymbol{A}$ by $\boldsymbol{B}$, we get $\boldsymbol{Q}$ as the quotient, and $\boldsymbol{R}$ as the remainder, which also, divides $\boldsymbol{B}$.
Then, we get $\boldsymbol{A = BQ + R}$, and the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$ = the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$.

That is, the GCD of a dividend and a divisor is the GCD of the divisor and the remainder.

So if $\boldsymbol{R}$ divides $\boldsymbol{B}$, $\boldsymbol{R}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$.   How?

The theorem has been proved in the Example 3 earlier.

Let's however, see now again, how the theorem works.

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are integers, $\boldsymbol{A = aG}$, $\boldsymbol{B = bG}$, and $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Then, $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other, and we can get
$\boldsymbol{A = BQ + R \Rightarrow aG = bGQ + R \Rightarrow R = (a - bQ)G}$.

Thus, we now have $\boldsymbol{B = bG}$, and $\boldsymbol{R = (a - bQ)G}$, where $\boldsymbol{G}$ is the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$.

So $\boldsymbol{b}$ and ($\boldsymbol{a - bQ}$) are prime to each other, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$.

Suppose now, $\boldsymbol{B = kR}$, where $\boldsymbol{k}$ is an integer.
That is, $\boldsymbol{R}$ is a factor of $\boldsymbol{B}$. In other words, $\boldsymbol{R}$ divides $\boldsymbol{B}$.

Then, $\boldsymbol{B = kR \Rightarrow k = \frac{B}{R} = \frac{bG}{(a-bQ)G} = \frac{b}{a-bQ}}$, so ($\boldsymbol{a - bQ}$) has to be 1.   How come?

That's because if $\boldsymbol{a - bQ \neq 1}$, $\boldsymbol{k}$ cannot be an integer, because $\boldsymbol{b}$ and ($\boldsymbol{a - bQ}$) are prime to each other. So we need to have $\boldsymbol{a - bQ = 1}$ if $\boldsymbol{k}$ is an integer.

Also, even though $\boldsymbol{a - bQ = 1}$, $\boldsymbol{b}$ and ($\boldsymbol{a - bQ}$) are still prime to each other, because 1 is the only divisor common to $\boldsymbol{b}$ and ($\boldsymbol{a - bQ}$) which is 1.

So in the case where $\boldsymbol{B = bG}$, and $\boldsymbol{R = (a - bQ)G}$, if we get $\boldsymbol{B = kR}$ where $\boldsymbol{k}$ is an integer, we get $\boldsymbol{a - bQ = 1}$, so we get $\boldsymbol{R = G}$, which is the GCD.

And the same is true, too, for cases where $\boldsymbol{A}$ and $\boldsymbol{B}$ are monomials or polynomials.

So using the theorem above, we can get the GCD the way below, too.

First, 382200 = 69300·5 + 35700, where 69300 is ***B*** in the theorem, and 35700 is ***R***.

So next, 69300 = 35700·1 + 33600, where 35700 is ***B***, and 33600 is ***R***.

So next, 35700 = 33600·1 + 2100, where 33600 is ***B***, and 2100 is ***R***.

So next, 33600 = 2100·16 + 0, and thus, ***R*** above divides ***B*** above.

Thus, ***R***, which is 2100 is the GCD. And let's see now, if it really is the case.

We get $33600 = 21 \cdot 100 \cdot 16 = 3 \cdot 7 \cdot 2^2 \cdot 5^2 2^4 = 2^4 \underline{2^2 3 \cdot 5^2 7}$, and $2100 = 3 \cdot 7 \cdot 2^2 \cdot 5^2 = \underline{2^2 3 \cdot 5^2 7}$.

So the GCD of 33600 and 2100 is $2^2 3 \cdot 5^2 7$, which is the GCD of the original two integers 382200 and 69300.

And let's for reference, put the processes above the way below.

First, 382200 = 69300·5 + 35700, where 5 is the quotient, and 35700 is the remainder.

That is, $2^3 3 \cdot 5^2 7^2 13 = (2^2 3^2 5^2 7 \cdot 11) \cdot 5 + 2^2 5^2 3 \cdot 7 \cdot 17$.

So next, 69300 = 35700·1 + 33600, where 1 is the quotient, and 33600 is the remainder.

That is, $2^2 3^2 5^2 7 \cdot 11 = (2^2 5^2 3 \cdot 7 \cdot 17) \cdot 1 + 2^6 5^2 3 \cdot 7$.

So next, 35700 = 33600·1 + 2100, where 1 is the quotient, and 2100 is the remainder.

That is, $2^2 5^2 3 \cdot 7 \cdot 17 = (2^6 5^2 3 \cdot 7) \cdot 1 + 2^2 5^2 3 \cdot 7$.

So next, 33600 = 2100·16 + 0, where 16 is the quotient, and 0 is the remainder.

That is, $2^6 5^2 3 \cdot 7 = (2^2 5^2 3 \cdot 7) \cdot 2^4 + 0$.

So $2^2 5^2 3 \cdot 7$ is the GCD, which is the GCD of $382200 = 2^3 3 \cdot 5^2 7^2 13$, and $2100 = 2^2 3 \cdot 5^2 7$.