Examples 7 in GCD and LCM

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0. Find the GCD of $\boldsymbol{x^4 + 3x^3 - 2x^2 - 3x - 5}$ and $\boldsymbol{x^3 + 7x^2 + 7x + 6}$.

1. Suppose $\boldsymbol{x^2 + 3x + 2}$ is the GCD of two polynomials, $\boldsymbol{x^4 + 5x^3 - 7x^2 - 41x - 30}$ is the LCM of the two, and 1 is the coefficient of the highest term in each of the two.

Then, find the two polynomials.

Suggestions or Solutions
To the Problem in the Example 0

Find the GCD of $x^4 + 3x^3 - 2x^2 - 3x - 5$ and $x^3 + 7x^2 + 7x + 6$.

Suppose $\boldsymbol{P} = x^4 + 3x^3 - 2x^2 - 3x - 5$, and $\boldsymbol{B} = x^3 + 7x^2 + 7x + 6$.
Then first, we can set $\boldsymbol{P} = \boldsymbol{BQ} + \boldsymbol{R}$, where $\boldsymbol{Q}$ is a quotient, and $\boldsymbol{R}$ is the remainder.
So we can do a division as below.

$$
\begin{array}{r|l}
 & x - 4 \\ \hline
x^3 + 7x^2 + 7x + 6 & x^4 + 3x^3 - 2x^2 - 3x - 5 \\
 & x^4 + 7x^3 + 7x^2 + 6x \\ \hline
 & \quad -4x^3 - 9x^2 - 9x - 5 \\
 & \quad -4x^3 - 28x^2 - 28x - 24 \\ \hline
 & \qquad\qquad 19x^2 + 19x + 19
\end{array}
$$

Thus, $\boldsymbol{P} = \boldsymbol{B}(x - 4) + 19x^2 + 19x + 19$.
So setting $\boldsymbol{r} = x^2 + x + 1$, we get $\boldsymbol{P} = \boldsymbol{B}(x - 4) + 19\boldsymbol{r} \Rightarrow \boldsymbol{P} = \boldsymbol{B}(x - 4) + 19\boldsymbol{r}$.

Next, we can set $\boldsymbol{B} = \boldsymbol{rq} + \boldsymbol{R}$, where $\boldsymbol{q}$ is a quotient, and $\boldsymbol{R}$ is the remainder.
So we can do a division as below.

$$
\begin{array}{r|l}
 & x + 6 \\ \hline
x^2 + x + 1 & x^3 + 7x^2 + 7x + 6 \\
 & x^3 + x^2 + x \\ \hline
 & \quad 6x^2 + 6x + 6 \\
 & \quad 6x^2 + 6x + 6 \\ \hline
 & \qquad\qquad 0
\end{array}
$$

Thus, $\boldsymbol{B} = x^3 + 7x^2 + 7x + 6 = (x^2 + x + 1)(x + 6) = \boldsymbol{r}(x + 6) \Rightarrow \boldsymbol{B} = \boldsymbol{r}(x + 6)$.

So we get $\boldsymbol{P} = \boldsymbol{B}(x - 4) + 19\boldsymbol{r} = \boldsymbol{r}(x + 6)(x - 4) + 19\boldsymbol{r} = \boldsymbol{r}\{(x + 6)(x - 4) + 19\}$
$= \boldsymbol{r}(x^2 + 2x - 24 + 19) = \boldsymbol{r}(x^2 + 2x - 5) \Rightarrow \boldsymbol{P} = \boldsymbol{r}(x^2 + 2x - 5)$.

Therefore, the GCD is $\boldsymbol{r}$, which is $x^2 + x + 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

Usually, finding the GCD of a set of polynomials, we take a product of all powers, in each of which, the base is a prime factor common to all the polynomials, and the exponent is the smallest of all used for the prime factor.

So the product has all the prime factors common to all the polynomials, and thus, is the divisor the greatest and common to all the polynomials, that is, the GCD.

Therefore, we want to begin with factorizing all the polynomials.
Factorizing though, such a polynomial as $x^4 + 3x^3 - 2x^2 - 3x - 5$ is not easy.
In fact, no divisor of 5 can be the value of x that can make the polynomial 0. So in this case, we cannot find an integer d for which $x - d$ can divide the polynomial above.

When we solve this kind of problem, too, though, the theorem used in the example 1 in **Examples 6 in GCD and LCM** can be quite handy, and is as follows.

Suppose dividing A by B, we get Q as the quotient, and R as the remainder, which also, divides B.
Then, we get $A = BQ + R$, and the GCD of A and B is the GCD of B and R.
That is, the GCD of a dividend and a divisor is the GCD of the divisor and the remainder.
So if R divides B, R is the GCD of A and B.

We've seen how it is the case in the Example 1 in **Examples 6 in GCD and LCM**.

Let's however, see now again, how R is the GCD of A and B if R divides B.

Note first, that normally, factorizing polynomials, we assume that factors are integers or expressions with integer coefficients. So for instance, we consider as a factor $2x + y + 1$, $3y^2$, or $2c$ and not $\frac{3}{2}x + y + 1$, $\frac{1}{3}y^2$, or $\frac{2c}{5}$.

Now, suppose, A and B are polynomials, $A = aG$, $B = bG$, and G is the GCD of A and B. Then, a and b are prime to each other, and we can get

$$A = BQ + R \Rightarrow aG = bGQ + R \Rightarrow R = (a - bQ)G.$$

So we now have $\boldsymbol{B = bG}$, and $\boldsymbol{R = (a - bQ)G}$, where $\boldsymbol{G}$ is the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$.

Therefore, $\boldsymbol{b}$ and $(\boldsymbol{a - bQ})$ both can be integers, constants, monomials, or polynomials, and are prime to each other, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{B}$ and $\boldsymbol{R}$.

Suppose now, $\boldsymbol{B = kR}$, where $\boldsymbol{k}$ is an integer, constant, monomial, or polynomial. That is, $\boldsymbol{k}$ and $\boldsymbol{R}$ are factors of $\boldsymbol{B}$. In other words, each of $\boldsymbol{k}$ and $\boldsymbol{R}$ divides $\boldsymbol{B}$.

Then, we get $\boldsymbol{B = kR \Rightarrow k = \frac{B}{R} = \frac{bG}{(a-bQ)G} = \frac{b}{a-bQ}}$, so $(\boldsymbol{a - bQ})$ has to be 1.   How?

If $\boldsymbol{a - bQ \neq 1}$, $\boldsymbol{k}$ cannot be a factor positive, constant, monomial, or polynomial, because $\boldsymbol{b}$ and $(\boldsymbol{a - bQ})$ are integers (or constants), monomials, or polynomials, and are prime to each other.   How?

First, $\boldsymbol{b}$ is an integer (or constant), monomial, or polynomial, and so is $(\boldsymbol{a - bQ})$.

So next, if $\boldsymbol{a - bQ \neq 1}$, since $\boldsymbol{b}$ and $(\boldsymbol{a - bQ})$ are prime to each other, $\boldsymbol{k = \frac{b}{a-bQ}}$ cannot be a factor that is an integer, a constant, a monomial, or a polynomial, but $\boldsymbol{k = \frac{b}{a-bQ}}$ is a fractional number as $\frac{3}{2}$ or expression as $\frac{2x}{x-y}$, $\frac{3}{2x-y}$, or $\frac{2c}{y}$, which is neither a monomial nor a polynomial. Technically of course, $\frac{3c}{2}$, for instance, can be taken as a constant, too, if $\boldsymbol{c}$ is a constant, but is a fractional constant.

Thus, $(\boldsymbol{a - bQ})$ has to be $\mathbf{1}$ if $\boldsymbol{k}$ is a factor that is integer (or constant), monomial, or polynomial.

Also, even though $\boldsymbol{a - bQ = 1}$, $\boldsymbol{b}$ and $(\boldsymbol{a - bQ})$ are still prime to each other, because 1 is the only divisor common to $\boldsymbol{b}$ and $(\boldsymbol{a - bQ})$ which is 1.

So in the case where $\boldsymbol{B = bG}$, and $\boldsymbol{R = (a - bQ)G}$, if we get $\boldsymbol{B = kR}$ where $\boldsymbol{k}$ is an integer, a constant, a monomial, or a polynomial, we get $\boldsymbol{a - bQ = 1}$, so we get $\boldsymbol{R = G}$, which is the GCD.

So using the theorem above, we can get the GCD in such a way as follows, too.

Suppose $\boldsymbol{P = x^4 + 3x^3 - 2x^2 - 3x - 5}$, and $\boldsymbol{B = x^3 + 7x^2 + 7x + 6}$.

Then first, we can set $\boldsymbol{P = BQ + R}$, where $\boldsymbol{Q}$ is a quotient, and $\boldsymbol{R}$ is the remainder.

So we can do a division the way as follows.

$$
\begin{array}{r|l}
 & \quad x - 4 \\ \cline{2-2}
x^3 + 7x^2 + 7x + 6 & x^4 + 3x^3 - 2x^2 - 3x - 5 \\
 & x^4 + 7x^3 + 7x^2 + 6x \\ \cline{2-2}
 & \qquad -4x^3 - 9x^2 - 9x - 5 \\
 & \qquad -4x^3 - 28x^2 - 28x - 24 \\ \cline{2-2}
 & \qquad\qquad 19x^2 + 19x + 19
\end{array}
$$

Thus, $\boldsymbol{P = (x^3 + 7x^2 + 7x + 6)(x - 4) + 19x^2 + 19x + 19}$, where $\boldsymbol{x^3 + 7x^2 + 7x + 6}$ is $\boldsymbol{B}$ in the theorem, and $\boldsymbol{19x^2 + 19x + 19}$ is $\boldsymbol{R}$.

So next, $\boldsymbol{x^3 + 7x^2 + 7x + 6 = (19x^2 + 19x + 19)Q + R}$, where $\boldsymbol{19x^2 + 19x + 19}$ is now $\boldsymbol{B}$ in the theorem, and $\boldsymbol{R}$ is of course, $\boldsymbol{R}$ in the theorem.

Meanwhile though, we have $\boldsymbol{19x^2 + 19x + 19 = 19(x^2 + x + 1)}$.

So setting $\boldsymbol{r = x^2 + x + 1}$, we get $\boldsymbol{x^3 + 7x^2 + 7x + 6 = 19rQ + R = r19Q + R}$.

Now, anyway, $\boldsymbol{Q}$ above simply indicates just a quotient.

Thus, setting $\boldsymbol{q = 19Q}$, we get $\boldsymbol{x^3 + 7x^2 + 7x + 6 = (x^2 + x + 1)q + R}$, where $\boldsymbol{q}$ merely indicates a quotient, too.

So $\boldsymbol{x^2 + x + 1}$ is now $\boldsymbol{B}$ in the theorem, and $\boldsymbol{R}$ is of course, $\boldsymbol{R}$ in the theorem.

And thus, doing the division, we get

$$\begin{array}{r|l} & \quad \mathbf{x + 6} \\ \cline{2-2} \mathbf{x^2 + x + 1} & \mathbf{x^3 + 7x^2 + 7x + 6} \\ & \underline{\mathbf{x^3 + x^2 + x}} \\ & \qquad \mathbf{6x^2 + 6x + 6} \\ & \qquad \underline{\mathbf{6x^2 + 6x + 6}} \\ & \qquad\qquad\qquad 0 \end{array}$$

Thus, we get $\mathbf{x^3 + 7x^2 + 7x + 6 = (x^2 + x + 1)(x + 6) + 0}$, so the quotient is $\mathbf{x + 6}$, and the remainder is $\mathbf{0}$. So $\mathbf{x^2 + x + 1}$ which is $\boldsymbol{R}$ in the theorem, divides $\mathbf{x^3 + 7x^2 + 7x + 6}$ which is $\boldsymbol{B}$ in the theorem. Thus, we can see that $\mathbf{x^2 + x + 1}$ is the GCD.

And let's now, put the processes above the way below.

To begin with, we set $\boldsymbol{P} = \mathbf{x^4 + 3x^3 - 2x^2 - 3x - 5}$, and $\boldsymbol{B} = \mathbf{x^3 + 7x^2 + 7x + 6}$.

In the first step, dividing $\boldsymbol{P}$ by $\boldsymbol{B}$, we get $\boldsymbol{P = B}\mathbf{(x - 4) + 19x^2 + 19x + 19}$.

Next, we set $\boldsymbol{B} = \mathbf{x^3 + 7x^2 + 7x + 6 = (19x^2 + 19x + 19)}\boldsymbol{Q} + \boldsymbol{R}$.

Meanwhile, we set $\mathbf{19x^2 + 19x + 19 = 19}\boldsymbol{r}$, and $\boldsymbol{q} = \mathbf{19}\boldsymbol{Q}$. So we get $\boldsymbol{B = rq + R}$.

In the second step, dividing $\boldsymbol{B}$ by $\boldsymbol{r}$, we get $\boldsymbol{B} = \mathbf{x^3 + 7x^2 + 7x + 6 = (x^2 + x + 1)(x + 6)}$.

So we can notice that $\mathbf{x^2 + x + 1}$ is the GCD of $\boldsymbol{P}$ and $\boldsymbol{B}$ since the remainder is 0, that is, $\mathbf{x^2 + x + 1}$ divides $\mathbf{x^3 + 7x^2 + 7x + 6}$.

Now, we have

$$\boldsymbol{P = B}\mathbf{(x - 4) + 19x^2 + 19x + 19} = \boldsymbol{B}\mathbf{(x - 4) + 19}\boldsymbol{r} \Rightarrow \boldsymbol{P = B}\mathbf{(x - 4) + 19}\boldsymbol{r}.$$

$$\boldsymbol{B} = \mathbf{x^3 + 7x^2 + 7x + 6 = (x^2 + x + 1)(x + 6)} = \boldsymbol{r}\mathbf{(x + 6)} \Rightarrow \boldsymbol{B = r}\mathbf{(x + 6)}.$$

So putting threads together, we get

$$\boldsymbol{P = B}\mathbf{(x - 4) + 19}\boldsymbol{r} = \boldsymbol{r}\mathbf{(x + 6)(x - 4) + 19}\boldsymbol{r} = \boldsymbol{r}\mathbf{\{(x + 6)(x - 4) + 19\}} = \boldsymbol{r}\mathbf{(x^2 + 2x - 24 + 19)}$$
$$\mathbf{= (x^2 + x + 1)(x^2 + 2x - 5)}.$$

Thus, we can see that $\mathbf{x^4 + 3x^3 - 2x^2 - 3x - 5}$ gets factorized to $\mathbf{(x^2 + x + 1)(x^2 + 2x - 5)}$.

And of course, $\boldsymbol{B} = \boldsymbol{x^3 + 7x^2 + 7x + 6}$ gets factorized to $\boldsymbol{(x^2 + x + 1)(x + 6)}$.
Therefore, the GCD is $\boldsymbol{x^2 + x + 1}$.

In short:

Suppose $\boldsymbol{P = x^4 + 3x^3 - 2x^2 - 3x - 5}$, and $\boldsymbol{B = x^3 + 7x^2 + 7x + 6}$.
Then first, we can set $\boldsymbol{P = BQ + R}$, where $\boldsymbol{Q}$ is a quotient, and $\boldsymbol{R}$ is the remainder.
So we can do a division the way as follows.

$$
\begin{array}{r|l}
 & x - 4 \\ \cline{2-2}
x^3 + 7x^2 + 7x + 6 & x^4 + 3x^3 - 2x^2 - 3x - 5 \\
 & x^4 + 7x^3 + 7x^2 + 6x \\ \cline{2-2}
 & \quad -4x^3 - 9x^2 - 9x - 5 \\
 & \quad -4x^3 - 28x^2 - 28x - 24 \\ \cline{2-2}
 & \qquad\qquad 19x^2 + 19x + 19
\end{array}
$$

Thus, $\boldsymbol{P = B(x - 4) + 19x^2 + 19x + 19}$.
So setting $\boldsymbol{r = x^2 + x + 1}$, we get $\boldsymbol{P = B(x - 4) + 19r \Rightarrow P = B(x - 4) + 19r}$.

Next, we can set $\boldsymbol{B = rq + R}$, where $\boldsymbol{q}$ is a quotient, and $\boldsymbol{R}$ is the remainder.
So we can do a division the way as follows.

$$
\begin{array}{r|l}
 & x + 6 \\ \cline{2-2}
x^2 + x + 1 & x^3 + 7x^2 + 7x + 6 \\
 & x^3 + x^2 + x \\ \cline{2-2}
 & \quad 6x^2 + 6x + 6 \\
 & \quad 6x^2 + 6x + 6 \\ \cline{2-2}
 & \qquad\qquad 0
\end{array}
$$

Thus, $\boldsymbol{B = x^3 + 7x^2 + 7x + 6 = (x^2 + x + 1)(x + 6) = r(x + 6) \Rightarrow B = r(x + 6)}$.

So we get $\boldsymbol{P = B(x - 4) + 19r = r(x + 6)(x - 4) + 19r = r\{(x + 6)(x - 4) + 19\}}$
$\boldsymbol{= r(x^2 + 2x - 24 + 19) = r(x^2 + 2x - 5) \Rightarrow P = r(x^2 + 2x - 5)}$.

Therefore, the GCD is $\boldsymbol{r}$, which is $\boldsymbol{x^2 + x + 1}$.

Suggestions or Solutions
To the Problem in the Example 1

Suppose $x^2 + 3x + 2$ is the GCD of two polynomials, $x^4 + 5x^3 - 7x^2 - 41x - 30$ is the LCM of the two, and 1 is the coefficient of the highest term in each of the two. Then, find the two polynomials.

Suppose $\boldsymbol{G = x^2 + 3x + 2}$, $\boldsymbol{L = x^4 + 5x^3 - 7x^2 - 41x - 30}$, the two polynomials are $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

Then, we get $\boldsymbol{A = aG}$, $\boldsymbol{B = bG}$, and $\boldsymbol{L = abG}$.

$$
\begin{array}{rrrrrr}
 & x^2 & -2x & -15 & & \\
\cline{2-6}
x^2+3x+2 \,\big) & x^4 & +5x^3 & -7x^2 & -41x & -30 \\
 & x^4 & +3x^3 & +2x^2 & & \\
\cline{2-6}
 & & 2x^3 & -9x^2 & -41x & -30 \\
 & & 2x^3 & -6x^2 & +4x & \\
\cline{3-6}
 & & & -15x^2 & -45x & +30 \\
 & & & -15x^2 & -45x & +30 \\
\cline{4-6}
 & & & & & 0
\end{array}
$$

Therefore, $\boldsymbol{ab = x^2 - 2x - 15}$, which is factorized to $\boldsymbol{(x - 3)(x + 5)}$.

Next, $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other, and thus, are $\boldsymbol{x - 3}$ and $\boldsymbol{x + 5}$, or are $\boldsymbol{(x - 3)(x + 5)}$, and 1.

Thus, since $\boldsymbol{A = aG}$, and $\boldsymbol{B = bG}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{x^2 + 3x + 2}$ and $\boldsymbol{(x - 3)(x + 5)(x^2 + 3x + 2)}$, or are $\boldsymbol{(x - 3)(x^2 + 3x + 2)}$ and $\boldsymbol{(x + 5)(x^2 + 3x + 2)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we mean by the highest term?

Normally, it is a term using the largest of all exponents used for all the terms.

So for instance, a polynomial $\boldsymbol{x^3 + 3x^2 + x + 2}$ has four terms, which are $\boldsymbol{x^3}$, $\boldsymbol{3x^2}$, $\boldsymbol{x}$, and **2**, and the term $\boldsymbol{x^3}$ is the highest term, for the exponent 3 is the largest of all the exponents.

Thus, the degree of the polynomial indicates the degree of the highest term.

Technically though, a term itself can be a polynomial, too, as $\boldsymbol{x+(2x+1)^2+\frac{2y+1}{3}}$. In such a case, we expand all the terms in polynomial, and then, determine the highest term.

So expanding them all, we get $\boldsymbol{x+4x^2+4x+\frac{2}{3}y+\frac{4}{3}}$, and the highest term is $\boldsymbol{4x^2}$.

And of course, a polynomial can have many highest terms, too.

For instance, expanding $\boldsymbol{x+(2x+y)^2+\frac{2y+1}{3}}$, we get $\boldsymbol{x+4x^2+4xy+y^2+\frac{2}{3}y+\frac{1}{3}}$, so the highest terms are $\boldsymbol{4x^2}$, $\boldsymbol{4xy}$, and $\boldsymbol{y^2}$.

So the highest term in a polynomial can be said to be a term where the number of variable multiplied together is the largest.

Suppose now, $\boldsymbol{G = x^2 + 3x + 2}$, $\boldsymbol{L = x^4 + 5x^3 - 7x^2 - 41x - 30}$, the two polynomials are $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.

Then, since $\boldsymbol{G}$ is the GCD of $\boldsymbol{A}$ and $\boldsymbol{B}$, we can set: $\boldsymbol{A = aG}$, and $\boldsymbol{B = bG}$, and also, we get

$\boldsymbol{L = abG}$. Thus, dividing $\boldsymbol{L}$ by $\boldsymbol{G}$, we can find $\boldsymbol{ab}$.

So getting $\boldsymbol{ab}$, we can do a division the way as follows.

$$
\begin{array}{r|l}
 & x^2 - 2x - 15 \\ \cline{2-2}
x^2 + 3x + 2 & x^4 + 5x^3 - 7x^2 - 41x - 30 \\
 & x^4 + 3x^3 + 2x^2 \\ \cline{2-2}
 & \qquad 2x^3 - 9x^2 - 41x - 30 \\
 & \qquad 2x^3 - 6x^2 + 4x \\ \cline{2-2}
 & \qquad\qquad -15x^2 - 45x + 30 \\
 & \qquad\qquad -15x^2 - 45x + 30 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

Therefore, we can see that $ab = x^2 - 2x - 15$. So we now have

$G = x^2 + 3x + 2$, $L = x^4 + 5x^3 - 7x^2 - 41x - 30$, $A = aG$, $B = bG$, and $ab = x^2 - 2x - 15$.

Next, finding a and b, we may want to factorize $x^2 - 2x - 15$.

So factorizing it, we get $x^2 - 2x - 15 = (x - 3)(x + 5)$.

Next, we know a and b are prime to each other, so we can have two cases as follows.

Either of a and b is $x - 3$, and the other is $x + 5$.

Either of a and b is $(x - 3)(x + 5)$, and the other is 1.

Now, we have $A = aG$, and $B = bG$.

So if first, $a = x - 3$, and $b = x + 5$, we get $A = aG = (x - 3)G$, and $B = bG = (x + 5)G$.

If next, $a = 1$, and $b = (x - 3)(x + 5)$, we get $A = aG = G$, and $B = bG = (x - 3)(x + 5)G$.

So the two polynomials are

$(x - 3)G$ and $(x + 5)G$, or G and $(x - 3)(x + 5)G$.

And we have $G = x^2 + 3x + 2$.

Thus, the two polynomials can be in either of the two cases as below.

$(x - 3)(x^2 + 3x + 2)$ and $(x + 5)(x^2 + 3x + 2)$

$x^2 + 3x + 2$ and $(x - 3)(x + 5)(x^2 + 3x + 2)$