# Examples 3 on Powers



Click this to start.

# Examples 3 on Powers

0. Assuming that $\sqrt{x}+\frac{1}{\sqrt{x}}=3$, find the values of $A$ and $B$ as follows.

0.0. $A=\dfrac{x^{\frac{3}{2}}+x^{-\frac{3}{2}}+2}{x^{2}+x^{-2}+3}$

0.1. $B=x^{\frac{1}{4}}+x^{-\frac{1}{4}}$

1. Assuming that $2^{x+2}=3$, find the value of $\sqrt{(\frac{1}{8})^{x}}$.

2. Assuming that $4^{2x}=3-2\sqrt{2}$, find the value of $\dfrac{2^{5x}+2^{-3x}}{2^{x}+2^{-x}}$.

## Suggestions or Solutions

### To the Problem 0 in the Example 0

**Assuming that $\sqrt{x}+\frac{1}{\sqrt{x}}=3$, find the value of $A=\frac{x^{\frac{3}{2}}+x^{-\frac{3}{2}}+2}{x^{2}+x^{-2}+3}$.**

To begin with, $x^{\frac{1}{2}}+x^{-\frac{1}{2}}=3 \Rightarrow (x^{\frac{1}{2}}+x^{-\frac{1}{2}})^2=x+2+x^{-1}=3^2 \Rightarrow x+x^{-1}=7$

$\Rightarrow (x+x^{-1})^2=x^2+2+x^{-2}=7^2 \Rightarrow x^2+x^{-2}=47$.

Next, $x^{\frac{1}{2}}+x^{-\frac{1}{2}}=3 \Rightarrow (x^{\frac{1}{2}}+x^{-\frac{1}{2}})^3=x^{\frac{3}{2}}+3(x^{\frac{1}{2}}+x^{-\frac{1}{2}})+x^{-\frac{3}{2}}=3^3 \Rightarrow x^{\frac{3}{2}}+x^{-\frac{3}{2}}=18$.

Therefore, $A=\frac{x^{\frac{3}{2}}+x^{-\frac{3}{2}}+2}{x^{2}+x^{-2}+3}=\frac{18+2}{47+3}=\frac{2}{5}$.

*If not quite sure of the idea behind the processes above, follow the steps below.*

Wouldn't it be nice if we could simply solve $\sqrt{x}+\frac{1}{\sqrt{x}}=3$ for $x$ so that we can put the value of $x$ into the expression?

The equation given is not that easy. Even if we can get the value of $x$, we still need to simplify the fraction that $A$ takes. The fraction looks asking a lot of work. So what are we going to do about this example?

Normally, problems in this kind do not expect us to solve such an equation as above. They want us to make use of though, such an equation. How?

We should be able to use in fact, the right hand side of the equation, which is the number 3, to find the value of $A$, of course. How then do we make use of it?

Putting the equation in terms of powers, we get $x^{\frac{1}{2}}+x^{-\frac{1}{2}}=3$.

Then, we can notice that both the numerator and the denominator in $A = \dfrac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} + 2}{x^2 + x^{-2} + 3}$ can be put in terms of the expression $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$, which is 3. So?

We can get the values of $(x^{\frac{3}{2}} + x^{-\frac{3}{2}})$ and $(x^2 + x^{-2})$.

Let's begin with the value of $(x^2 + x^{-2})$, simply because it looks easier than the other.

We should be able to derive the value from the equation, $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$, of course.

Squaring both sides of $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$, we get $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2 = x + 2 + x^{-1} = 3^2 \Rightarrow x + x^{-1} = 7$.

Then, squaring both the sides again, we get this:

$(x + x^{-1})^2 = x^2 + 2 + x^{-2} = 7^2 \Rightarrow x^2 + x^{-2} = 47$, which is one of the two values we want.

Next, for the other one, we will cube both sides of $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$. Why?

That's because the exponents are $\frac{3}{2}$ and $-\frac{3}{2}$.

And we have a factorization identity, $(a + b)^3 = a^3 + 3ab(a + b) + b^3$.

So we get $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3 \Rightarrow (x^{\frac{1}{2}} + x^{-\frac{1}{2}})^3 = x^{\frac{3}{2}} + 3(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + x^{-\frac{3}{2}} = 3^3$.

And we know $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$.

So we get $x^{\frac{3}{2}} + 3(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + x^{-\frac{3}{2}} = 3^3 \Rightarrow x^{\frac{3}{2}} + 3 \cdot 3 + x^{-\frac{3}{2}} = 3^3$.

Thus, we get $x^{\frac{3}{2}} + x^{-\frac{3}{2}} = 3^3 - 3^2 = 18$.

Therefore, $A = \dfrac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} + 2}{x^2 + x^{-2} + 3} = \dfrac{18 + 2}{47 + 3} = \dfrac{20}{50} = \dfrac{2}{5}$.

## Suggestions or Solutions

### To the **Problem 1** in the Example **0**

**Assuming that $\sqrt{x}+\frac{1}{\sqrt{x}}=3$, find the value of $B=x^{\frac{1}{4}}+x^{-\frac{1}{4}}$.**

$$B^2=(x^{\frac{1}{4}}+x^{-\frac{1}{4}})^2=x^{\frac{1}{2}}+2+x^{-\frac{1}{2}}=2+3=5\Rightarrow B=\sqrt{5}, \text{because } x^{\frac{1}{4}}+x^{-\frac{1}{4}}>0.$$

*If not quite sure of the idea behind the processes above, follow the steps below.*

In this case, conversion of the equation given doesn't seem to help. So is it of no use?

If one way doesn't work, try the other way. Running math, we are in fact, finding ways, where we get to the solution, of course.

So this time, getting $B$ squared, we can get the left hand side of $\sqrt{x}+\frac{1}{\sqrt{x}}=3$.

Now, squaring $B$, we get $B^2=(x^{\frac{1}{4}}+x^{-\frac{1}{4}})^2=x^{\frac{1}{2}}+2+x^{-\frac{1}{2}}$, which is $\sqrt{x}+\frac{1}{\sqrt{x}}$.

We know $x^{\frac{1}{2}}+x^{-\frac{1}{2}}=3$. So we get $B^2=x^{\frac{1}{2}}+2+x^{-\frac{1}{2}}=3+2=5\Rightarrow B^2=5\Rightarrow B=\pm\sqrt{5}$.

Then, since $B>0$, we get $B=\sqrt{5}$. What about $-\sqrt{5}$ ? That is, why $B>0$?

That's because both $x^{\frac{1}{4}}$ and $x^{-\frac{1}{4}}>0$, and $B=x^{\frac{1}{4}}+x^{-\frac{1}{4}}$. Why both $x^{\frac{1}{4}}$ and $x^{-\frac{1}{4}}>0$?

That's becuase $x>0$. Why $x>0$?

That's because if $x < 0$, both $x^{\frac{1}{4}}$ and $x^{-\frac{1}{4}}$ cannot be real.

If $b < 0$, $m$ is odd, and $n$ is even, $b^{\frac{m}{n}}$ is not real. For instance, $(-2)^{\frac{1}{2}}$ is not a real number.

Besides, if $x = 0$, $x^{-\frac{1}{4}}$ cannot exist, because $x^{-\frac{1}{4}} = \dfrac{1}{x^{\frac{1}{4}}}$, and no division by 0 is allowed.

**In short:**

$B^2 = (x^{\frac{1}{4}} + x^{-\frac{1}{4}})^2 = x^{\frac{1}{2}} + 2 + x^{-\frac{1}{2}} = 2 + 3 = 5 \Rightarrow B = \sqrt{5}$, because $x^{\frac{1}{4}} + x^{-\frac{1}{4}} > 0$.

## Suggestions or Solutions
## To the Problem in the Example 1

**Assuming that $2^{x+2} = 3$, find the value of $\sqrt{(\frac{1}{8})^x}$.**

In math, we never get anything out of nothing.
Working in math, we can get something out of something else.
Normally, doing a problem in math, we find the solution inside the problem, and not outside. Running math, we break the problem apart, and put the pieces together to form the solution.

The equation given has a power, which is $2^{x+2}$, where the base is 2. So if we put in a power the radical given $\sqrt{(\frac{1}{8})^x}$, the power can probably have the same base, which is 2. So let's try such a modification.

Then, we get $\sqrt{(\frac{1}{8})^x} = (\frac{1}{8})^{\frac{x}{2}} = 8^{-\frac{x}{2}} = (2^3)^{-\frac{x}{2}} = 2^{-\frac{3x}{2}}$.

Now, we have $2^{x+2} = 3$, and have $2^{-\frac{3x}{2}}$, of which we want to find the value. So?

We can notice that the two powers $2^{x+2}$ and $2^{-\frac{3x}{2}}$ can be put in terms of $2^x$.
So let's put them that way.

Then, we get $2^{x+2} = 2^2 2^x = 4 \cdot 2^x = 3 \Rightarrow 2^x = \frac{3}{4}$, and also, $2^{-\frac{3x}{2}} = 2^{x \cdot (-\frac{3}{2})} = (2^x)^{-\frac{3}{2}}$.

Now, we can see the solution.

$2^x = \frac{3}{4} \Rightarrow (2^x)^{-\frac{3}{2}} = (\frac{3}{4})^{-\frac{3}{2}} = (\frac{4}{3})^{\frac{3}{2}} = \sqrt{(\frac{4}{3})^3}$. Therefore, we get $\sqrt{(\frac{1}{8})^x} = \sqrt{(\frac{4}{3})^3}$.

**In short:**

$2^{x+2} = 2^2 2^x = 4 \cdot 2^x = 3 \Rightarrow 2^x = \frac{3}{4}$. $\sqrt{(\frac{1}{8})^x} = (\frac{1}{8})^{\frac{x}{2}} = 8^{-\frac{x}{2}} = (2^3)^{-\frac{x}{2}} = 2^{-\frac{3x}{2}} = (2^x)^{-\frac{3}{2}}$.

Thus, $\sqrt{(\frac{1}{8})^x} = (2^x)^{-\frac{3}{2}} = (\frac{3}{4})^{-\frac{3}{2}} = (\frac{4}{3})^{\frac{3}{2}} = \sqrt{(\frac{4}{3})^3}$.

## Suggestions or Solutions

### To the Problem in the Example 2

**Assuming that $4^{2x} = 3 - 2\sqrt{2}$, find the value of $\dfrac{2^{5x}+2^{-3x}}{2^{x}+2^{-x}}$.**

$4^{2x} = 3-2\sqrt{2} = 3-2\sqrt{2}\cdot 1 = 2-2\sqrt{2}\cdot 1+1 = (\sqrt{2}-1)^2 \Rightarrow 4^{2x} = (\sqrt{2}-1)^2 = (2^{\frac{1}{2}}-1)^2$.

Also, $4^{2x} = (4^x)^2 = \{(2^2)^x\}^2 = (2^{2x})^2$.

So $4^{2x} = (2^{2x})^2 = (2^{\frac{1}{2}}-1)^2 \Rightarrow 2^{2x} = 2^{\frac{1}{2}}-1$, since $2^{2x} > 0$ and $2^{\frac{1}{2}} > 1$.

Next, $\dfrac{2^{5x}+2^{-3x}}{2^{x}+2^{-x}} = \dfrac{2^x(2^{5x}+2^{-3x})}{2^x(2^x+2^{-x})} = \dfrac{2^{6x}+2^{-2x}}{2^{2x}+2^0} = \dfrac{(2^{2x})^3+2^{-2x}}{2^{2x}+1}$.

So $\dfrac{2^{5x}+2^{-3x}}{2^{x}+2^{-x}} = \dfrac{(2^{2x})^3+2^{-2x}}{2^{2x}+1}$. Besides, we have $2^{2x} = 2^{\frac{1}{2}}-1$.

So $\dfrac{(2^{2x})^3+2^{-2x}}{2^{2x}+1} = \dfrac{(2^{\frac{1}{2}}-1)^3+(2^{\frac{1}{2}}-1)^{-1}}{2^{\frac{1}{2}}-1+1} = \dfrac{2^{\frac{3}{2}}-3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1)-1+(2^{\frac{1}{2}}-1)^{-1}}{2^{\frac{1}{2}}}$.

Meanwhile, $(2^{\frac{1}{2}}-1)^{-1} = \dfrac{1}{2^{\frac{1}{2}}-1} = \dfrac{2^{\frac{1}{2}}+1}{(2^{\frac{1}{2}}+1)(2^{\frac{1}{2}}-1)} = \dfrac{2^{\frac{1}{2}}+1}{2-1} = 2^{\frac{1}{2}}+1$, and

$2^{\frac{3}{2}}-3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1)-1 = 2\cdot 2^{\frac{1}{2}}-3\cdot 2+3\cdot 2^{\frac{1}{2}}-1 = 5\cdot 2^{\frac{1}{2}}-7$.

Thus, $2^{\frac{3}{2}}-3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1)-1+(2^{\frac{1}{2}}-1)^{-1} = 5\cdot 2^{\frac{1}{2}}-7+2^{\frac{1}{2}}+1 = 6\cdot 2^{\frac{1}{2}}-6$.

So we get $\dfrac{(2^{2x})^3+2^{-2x}}{2^{2x}+1} = \dfrac{2^{\frac{3}{2}}-3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1)-1+(2^{\frac{1}{2}}-1)^{-1}}{2^{\frac{1}{2}}} = \dfrac{6\cdot 2^{\frac{1}{2}}-6}{2^{\frac{1}{2}}}$.

Thus, we get $\frac{2^{5x}+2^{-3x}}{2^x+2^{-x}}=\frac{6\cdot 2^{\frac{1}{2}}-6}{2^{\frac{1}{2}}}=\frac{2^{\frac{1}{2}}\cdot(6\cdot 2^{\frac{1}{2}}-6)}{2^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}}=\frac{6\cdot 2-6\cdot 2^{\frac{1}{2}}}{2}=3(2-2^{\frac{1}{2}})$.

*If not quite sure of the idea behind the processes above, follow the steps below.*

This example is not much more than the previous one. It's a bit more involved though. It just requires some more exponential algebra with powers.

Now, there is something common to both the power in the equation given and the one in the expression we want to find the value of. What then is common?

It is a power of 2, which is $\mathbf{2^x}$. So it looks like we want to put the equation and expression in terms of $\mathbf{2^x}$, get the value of $\mathbf{2^x}$ from the equation, and then, put the value into the expression. However, we may not want to do so this time.

That's because putting into the expression the value of $\mathbf{2^x}$, we will get to see the algebra seriously involved, that is, the exponential algebra will be a real big mess.
To those of you though who really love calculations much, it will be a lot of fun.

Let's see though, how it looks if we do the substitution.

First, we get $4^{2x}=(2^2)^{2x}=2^{4x}=3-2\sqrt{2}\Rightarrow 2^x=(3-2\sqrt{2})^{\frac{1}{4}}$.

So next, putting it directly into the expression given, we get this:

$\frac{(3-2\sqrt{2})^{\frac{5}{4}}+(3-2\sqrt{2})^{-\frac{3}{4}}}{(3-2\sqrt{2})^{\frac{1}{4}}+(3-2\sqrt{2})^{-\frac{1}{4}}}$, simplification of which can be an awful pain to most of us.

Of course, we can just leave it at that in the solution sheet, since it is a number, too. What if however, this problem were multiple-choice, and the list of the choices did not show the number? Besides, your teacher probably wouldn't allow much credit if you left the number as is. Whatelse then can be done?

Running math is not for nothing. We don't like to do a lot of calculation, do we? One of good reasons for running math is that we can get around with too much of calculation.

Normally, the solution is inside the problem, so we may want to give a little closer look to the problem, particularly, the equation given, which is $\mathbf{4^{2x} = 3 - 2\sqrt{2}}$.

Then, we can notice that $\mathbf{3 - 2\sqrt{2}}$ can be put in a form of a complete square.
So let's break it apart, and then, put the pieces together.

$$\mathbf{4^{2x} = 3 - 2\sqrt{2} = 3 - 2\sqrt{2} \cdot 1 = 2 - 2\sqrt{2} \cdot 1 + 1 = (\sqrt{2} - 1)^2 \Rightarrow 4^{2x} = (\sqrt{2} - 1)^2 = (2^{\frac{1}{2}} - 1)^2}.$$

Next, we want to take advantage of the result above, of course.

To begin with, let's do a little modification on $\mathbf{4^{2x}}$.

Then, we get $\mathbf{4^{2x} = (4^x)^2 = \{(2^2)^x\}^2 = (2^{2x})^2}$.

So we get $\mathbf{4^{2x} = (2^{2x})^2 = (2^{\frac{1}{2}} - 1)^2 \Rightarrow 2^{2x} = 2^{\frac{1}{2}} - 1}$. Why not $\mathbf{2^{2x} = \pm(2^{\frac{1}{2}} - 1)}$, though?

That's because we have $\mathbf{2^{2x} > 0}$, and also, $\mathbf{2^{\frac{1}{2}} > 1}$, that is, $\mathbf{2^{\frac{1}{2}} - 1 > 0}$.

Next, let's take a closer look at the expression in question.

The expression is $\mathbf{\dfrac{2^{5x} + 2^{-3x}}{2^x + 2^{-x}}}$, which can be put in terms of $\mathbf{2^{2x}}$.

Thus, we may want to do some modifications on the expression above.
So let's break it apart, and then, put the pieces together.

Then, we get $\mathbf{\dfrac{2^{5x} + 2^{-3x}}{2^x + 2^{-x}} = \dfrac{2^x(2^{5x} + 2^{-3x})}{2^x(2^x + 2^{-x})} = \dfrac{2^{6x} + 2^{-2x}}{2^{2x} + 2^0} = \dfrac{(2^{2x})^3 + 2^{-2x}}{2^{2x} + 1}}$.

So we get $\mathbf{\dfrac{2^{5x} + 2^{-3x}}{2^x + 2^{-x}} = \dfrac{(2^{2x})^3 + 2^{-2x}}{2^{2x} + 1}}$.

Besides, we have $\mathbf{2^{2x} = 2^{\frac{1}{2}} - 1}$. So let's plug it in now.

$$\frac{(2^{2x})^3 + 2^{-2x}}{2^{2x}+1} = \frac{(2^{\frac{1}{2}}-1)^3 + (2^{\frac{1}{2}}-1)^{-1}}{2^{\frac{1}{2}}-1+1} = \frac{2^{\frac{3}{2}} - 3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1) - 1 + (2^{\frac{1}{2}}-1)^{-1}}{2^{\frac{1}{2}}}.$$ How?

We have a factorization identity as follows.

$\boldsymbol{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - 3ab(a-b) - b^3}$.

Thus, putting $\mathbf{2^{\frac{1}{2}}}$ into $\boldsymbol{a}$, and -1 into $\boldsymbol{b}$, we get $\mathbf{(2^{\frac{1}{2}}-1)^3 = 2^{\frac{3}{2}} - 3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1) - 1}$.

Meanwhile, $\mathbf{(2^{\frac{1}{2}}-1)^{-1} = \dfrac{1}{2^{\frac{1}{2}}-1} = \dfrac{2^{\frac{1}{2}}+1}{(2^{\frac{1}{2}}+1)(2^{\frac{1}{2}}-1)} = \dfrac{2^{\frac{1}{2}}+1}{2-1} = 2^{\frac{1}{2}}+1}$, and

$$2^{\frac{3}{2}} - 3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1) - 1 = 2\cdot 2^{\frac{1}{2}} - 3\cdot 2 + 3\cdot 2^{\frac{1}{2}} - 1 = 5\cdot 2^{\frac{1}{2}} - 7.$$

Thus, we get $\mathbf{2^{\frac{3}{2}} - 3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1) - 1 + (2^{\frac{1}{2}}-1)^{-1} = 5\cdot 2^{\frac{1}{2}} - 7 + 2^{\frac{1}{2}} + 1 = 6\cdot 2^{\frac{1}{2}} - 6}$.

So we get $\mathbf{\dfrac{(2^{2x})^3 + 2^{-2x}}{2^{2x}+1} = \dfrac{2^{\frac{3}{2}} - 3\cdot 2^{\frac{1}{2}}(2^{\frac{1}{2}}-1) - 1 + (2^{\frac{1}{2}}-1)^{-1}}{2^{\frac{1}{2}}} = \dfrac{6\cdot 2^{\frac{1}{2}} - 6}{2^{\frac{1}{2}}}}$.

Therefore, we get $\mathbf{\dfrac{(2^{2x})^3 + 2^{-2x}}{2^{2x}+1} = \dfrac{6\cdot 2^{\frac{1}{2}} - 6}{2^{\frac{1}{2}}} = \dfrac{2^{\frac{1}{2}}\cdot(6\cdot 2^{\frac{1}{2}} - 6)}{2^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}} = \dfrac{6\cdot 2 - 6\cdot 2^{\frac{1}{2}}}{2} = 3(2-2^{\frac{1}{2}})}$.