Examples 4 on Powers

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0. Suppose $\boldsymbol{f(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ where $\boldsymbol{e}$ is Euler's number, $\boldsymbol{f(a) = \frac{1}{2}}$, and $\boldsymbol{f(b) = \frac{1}{3}}$.

Then, find $\boldsymbol{f(a + b)}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

1. Suppose that $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z \neq 0}$.

Suppose also, $\boldsymbol{a}$ and $\boldsymbol{b}$ are positive integers, $\boldsymbol{a^x = b^y = 51^z}$, and $\boldsymbol{\frac{1}{x} + \frac{1}{y} = \frac{1}{z}}$.

Then, find the value of $\boldsymbol{a + b}$.

Suggestions or Solutions
To the Problem in the Example 0

Suppose $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ where e is Euler's number, $f(a) = \frac{1}{2}$, and $f(b) = \frac{1}{3}$.
Then, find $f(a + b)$ where a and b are constant.

$$f(a+b) = \frac{e^{a+b} - e^{-(a+b)}}{e^{(a+b)} + e^{-(a+b)}},\ f(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}} = \frac{1}{2},\ \text{and } f(b) = \frac{e^b - e^{-b}}{e^b + e^{-b}} = \frac{1}{3}.$$

$$\frac{e^a - e^{-a}}{e^a + e^{-a}} = \frac{e^{2a} - 1}{e^{2a} + 1} = \frac{1}{2} \Rightarrow 2(e^{2a} - 1) = e^{2a} + 1 \Rightarrow e^{2a} = 3.$$

$$\frac{e^b - e^{-b}}{e^b + e^{-b}} = \frac{e^{2b} - 1}{e^{2b} + 1} = \frac{1}{3} \Rightarrow 3(e^{2b} - 1) = e^{2b} + 1 \Rightarrow e^{2b} = 2.$$

$$f(a+b) = \frac{e^{a+b} - e^{-(a+b)}}{e^{(a+b)} + e^{-(a+b)}} = \frac{e^{2(a+b)} - 1}{e^{2(a+b)} + 1} = \frac{e^{2a}e^{2b} - 1}{e^{2a}e^{2b} + 1} = \frac{2 \cdot 3 - 1}{2 \cdot 3 + 1} = \frac{5}{7}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

This example is probably not easy for those of you who are not familiar with functions. If not much familiar with functions yet, you may want to refer to the book **BASIC FUNCTIONS**.

Now, we have $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. What number though, do we use as x?

Since no specification is given to x, x is assumed to be real.
What then is the Euler's number e?

It is an irrational number, and is conceptually $(1 + 0)^{\infty}$ where 0 is called an infinitesimal which is as good as zero but not zero itself, and ∞ is called infinity. Actually,

$e = \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$, which converges to 2.718181828459045…, which is an irrational number.

It is said that about 200 years ago, Euler used his name to name the number when he was working with the number. Who is Euler?

Leonhard Paul Euler was a Swiss mathematician and physicist, but is said to had spent most of his life in Russia and Germany.

Next, what do we mean by $f(a + b)$?

It is the output for $x = a + b$.
Thus, plugging $a + b$ into the expression of the function f, we get $f(a + b)$.
In other words, replacing x in $f(x)$ with $a + b$, we get $f(a + b)$.

Specifically, we get $f(a + b) = \frac{e^{a+b} - e^{-(a+b)}}{e^{(a+b)} + e^{-(a+b)}}$.

And by the same token, we get $f(a) = \frac{e^{a} - e^{-a}}{e^{a} + e^{-a}} = \frac{1}{2}$, and $f(b) = \frac{e^{b} - e^{-b}}{e^{b} + e^{-b}} = \frac{1}{3}$. So?

Closely looking at the exponents in the expression of f, we can notice that we can put it in terms of e^{2x}.

We have $(a^m)^n = a^{mn}$ for $a > 0$. So we get $e^x e^x = e^{x+x} = e^{2x}$, and $e^x e^{-x} = e^{x-x} = e^0 = 1$.

So now, all we have to do is exponential algebra.

To begin with, we have $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Multiplying both the numerator and the denominator by e^x respectively, we get this:

$$\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^x(e^x-e^{-x})}{e^x(e^x+e^{-x})}=\frac{e^{2x}-1}{e^{2x}+1}\Rightarrow f(x)=\frac{e^{2x}-1}{e^{2x}+1}.$$

Then, we get $f(a+b)=\frac{e^{2(a+b)}-1}{e^{2(a+b)}+1}=\frac{e^{2a+2b}-1}{e^{2a+2b}+1}=\frac{e^{2a}e^{2b}-1}{e^{2a}e^{2b}+1}$.

By the same token, we get $\frac{e^a-e^{-a}}{e^a+e^{-a}}=\frac{e^{2a}-1}{e^{2a}+1}=\frac{1}{2}\Rightarrow 2(e^{2a}-1)=e^{2a}+1\Rightarrow e^{2a}=3$, and

$$\frac{e^b-e^{-b}}{e^b+e^{-b}}=\frac{e^{2b}-1}{e^{2b}+1}=\frac{1}{3}\Rightarrow 3(e^{2b}-1)=e^{2b}+1\Rightarrow e^{2b}=2.$$

Thus, we get $f(a+b)=\frac{e^{2a}e^{2b}-1}{e^{2a}e^{2b}+1}=\frac{3\cdot 2-1}{3\cdot 2+1}=\frac{5}{7}$.

In short:

$f(a+b)=\frac{e^{a+b}-e^{-(a+b)}}{e^{(a+b)}+e^{-(a+b)}}$, $f(a)=\frac{e^a-e^{-a}}{e^a+e^{-a}}=\frac{1}{2}$, and $f(b)=\frac{e^b-e^{-b}}{e^b+e^{-b}}=\frac{1}{3}$.

$$\frac{e^a-e^{-a}}{e^a+e^{-a}}=\frac{e^{2a}-1}{e^{2a}+1}=\frac{1}{2}\Rightarrow 2(e^{2a}-1)=e^{2a}+1\Rightarrow e^{2a}=3.$$

$$\frac{e^b-e^{-b}}{e^b+e^{-b}}=\frac{e^{2b}-1}{e^{2b}+1}=\frac{1}{3}\Rightarrow 3(e^{2b}-1)=e^{2b}+1\Rightarrow e^{2b}=2.$$

$$f(a+b)=\frac{e^{a+b}-e^{-(a+b)}}{e^{(a+b)}+e^{-(a+b)}}=\frac{e^{2(a+b)}-1}{e^{2(a+b)}+1}=\frac{e^{2a}e^{2b}-1}{e^{2a}e^{2b}+1}=\frac{2\cdot 3-1}{2\cdot 3+1}=\frac{5}{7}.$$

Suggestions or Solutions

To the Problem in the Example 1

Suppose that x, y, and $z \neq 0$.

Suppose also, a and b are positive integers, $a^x = b^y = 51^z$, and $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

Then, find the value of $a + b$.

$$a^x = 51^z \Rightarrow (a^x)^{\frac{1}{xz}} = (51^z)^{\frac{1}{xz}} \Rightarrow a^{\frac{x}{xz}} = 51^{\frac{z}{xz}} \Rightarrow a^{\frac{1}{z}} = 51^{\frac{1}{x}}.$$

$$b^y = 51^z \Rightarrow (b^y)^{\frac{1}{yz}} = (51^z)^{\frac{1}{yz}} \Rightarrow b^{\frac{y}{yz}} = 51^{\frac{z}{yz}} \Rightarrow b^{\frac{1}{z}} = 51^{\frac{1}{y}}.$$

Thus, we get $a^{\frac{1}{z}}b^{\frac{1}{z}} = (ab)^{\frac{1}{z}} = 51^{\frac{1}{x}}51^{\frac{1}{y}} = 51^{\frac{1}{x}+\frac{1}{y}} = 51^{\frac{1}{z}}$. So, we get $ab = 51$.

By factorization of 51, we get $51 = 3 \cdot 17$. And we know that a and b are positive integers.

Thus, $a = 3$ and $b = 17$, or $a = 17$ and $b = 3$. Therefore, $a + b = 20$.

If not quite sure of the idea behind the processes above, follow the steps below.

Now, to begin with, we have a^x and b^y in the equation $a^x = b^y = 51^z$.

Next, the exponents x, y, and z are related to each other via this equality: $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

So we should be able to isolate (or extract) a and b each from the equation, and come up with an expression in terms of a and b using the equation and the equality. How?

Modifying $a^x = b^y = 51^z$, we can set up a relation between 51 and each of a and b. How?

Actually, we have two equations in the equation, $a^x = b^y = 51^z$.

One is $\boldsymbol{a^x = 51^z}$, and the other is $\boldsymbol{b^y = 51^z}$.

Now, we have an exponential identity where $\boldsymbol{(u^s)^t = u^{st}}$.

Therefore, we can get $\boldsymbol{A = u^s \Rightarrow A^t = (u^s)^t = u^{st}}$.

So first, using $\boldsymbol{a^x = 51^z}$, we can get $\boldsymbol{a^{\frac{1}{z}} = 51^{\frac{1}{x}}}$. How?

$$\boldsymbol{a^x = 51^z \Rightarrow (a^x)^{\frac{1}{xz}} = (51^z)^{\frac{1}{xz}} \Rightarrow a^{\frac{x}{xz}} = 51^{\frac{z}{xz}} \Rightarrow a^{\frac{1}{z}} = 51^{\frac{1}{x}}}.$$

Next, using $\boldsymbol{b^y = 51^z}$, we can get $\boldsymbol{b^{\frac{1}{z}} = 51^{\frac{1}{y}}}$. How?

$$\boldsymbol{b^y = 51^z \Rightarrow (b^y)^{\frac{1}{yz}} = (51^z)^{\frac{1}{yz}} \Rightarrow b^{\frac{y}{yz}} = 51^{\frac{z}{yz}} \Rightarrow b^{\frac{1}{z}} = 51^{\frac{1}{y}}}.$$

Now, we can see these:

- In $\boldsymbol{a^{\frac{1}{z}}}$ and $\boldsymbol{b^{\frac{1}{z}}}$, the exponents are the same.
- In $\boldsymbol{51^{\frac{1}{x}}}$ and $\boldsymbol{51^{\frac{1}{y}}}$, the bases are the same.

Taking the product of two powers with the same base, we get another power, where the base is the same base, and the exponent is the sum of the two exponents.

That is, we have an exponential identity where $\boldsymbol{u^m u^n = u^{m+n}}$.

Taking a product of two powers with the same exponent, we get another power, where the base is the product of the two bases, and the exponent is the same exponent.

That is, we have another exponential identity where $\boldsymbol{u^m v^m = (uv)^m}$.

Now, we have these:

• In $\boldsymbol{a}^{\frac{1}{z}}$ and $\boldsymbol{b}^{\frac{1}{z}}$, the exponents are the same.

• In $\mathbf{51}^{\frac{1}{x}}$ and $\mathbf{51}^{\frac{1}{y}}$, the bases are the same.

Thus, we get $\boldsymbol{a}^{\frac{1}{z}}\boldsymbol{b}^{\frac{1}{z}} = \mathbf{51}^{\frac{1}{x}}\mathbf{51}^{\frac{1}{y}}$.

So we get $(\boldsymbol{ab})^{\frac{1}{z}} = \mathbf{51}^{\frac{1}{x}+\frac{1}{y}} = \mathbf{51}^{\frac{1}{z}}$.

Therefore, we can see that ***ab* = 51**.

What then about ***a* + *b***?

We know ***a*** and ***b*** both are positive integers. So?

So factorizing the number 51, we can get all the possible integers for ***a*** and ***b***.

We have 51 = 3·17.
And we know that ***a*** and ***b*** are positive integers.

Thus, ***a* = 3** and ***b* = 17**, or ***a* = 17** and ***b* = 3**.

Either way, we get ***a* + *b* = 20**.

In short:

$$\boldsymbol{a}^{x} = \mathbf{51}^{z} \Rightarrow (\boldsymbol{a}^{x})^{\frac{1}{xz}} = (\mathbf{51}^{z})^{\frac{1}{xz}} \Rightarrow \boldsymbol{a}^{\frac{x}{xz}} = \mathbf{51}^{\frac{z}{xz}} \Rightarrow \boldsymbol{a}^{\frac{1}{z}} = \mathbf{51}^{\frac{1}{x}}.$$

$$\boldsymbol{b}^{y} = \mathbf{51}^{z} \Rightarrow (\boldsymbol{b}^{y})^{\frac{1}{yz}} = (\mathbf{51}^{z})^{\frac{1}{yz}} \Rightarrow \boldsymbol{b}^{\frac{y}{yz}} = \mathbf{51}^{\frac{z}{yz}} \Rightarrow \boldsymbol{b}^{\frac{1}{z}} = \mathbf{51}^{\frac{1}{y}}.$$

Thus, we get $\boldsymbol{a^{\frac{1}{z}}b^{\frac{1}{z}} = (ab)^{\frac{1}{z}} = 51^{\frac{1}{x}}51^{\frac{1}{y}} = 51^{\frac{1}{x}+\frac{1}{y}} = 51^{\frac{1}{z}}}$.

So, we get $\boldsymbol{ab = 51}$.

By factorization of 51, we get $51 = 3 \cdot 17$.

And we know that $\boldsymbol{a}$ and $\boldsymbol{b}$ are positive integers.

Thus, $\boldsymbol{a = 3}$ and $\boldsymbol{b = 17}$, or $\boldsymbol{a = 17}$ and $\boldsymbol{b = 3}$.

Therefore, $\boldsymbol{a + b = 20}$.