Examples 5 on Powers

Click this to start.

Examples 5 on Powers

This set of examples is for advanced students. And it is of course, for practice on exponential algebra, too, and will help get more strength on calculations with powers.

0. Assuming that x, a, b, and c are real, show this:
$(1 + x^{a-b} + x^{a-c})^{-1} + (1 + x^{b-c} + x^{b-a})^{-1} + (1 + x^{c-a} + x^{c-b})^{-1} = 1$.

1. Assuming that $a > b > c > 0$, show that $a^{2a}b^{2b}c^{2c} > a^{b+c}b^{c+a}c^{a+b}$.

2. Assuming that $b^2 = ac$, simplify the expression below.
$a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3})$.

Suggestions or Solutions

To the Problem in the Example 0

Assuming that x, a, b, and c are real, show this:
$(1 + x^{a-b} + x^{a-c})^{-1} + (1 + x^{b-c} + x^{b-a})^{-1} + (1 + x^{c-a} + x^{c-b})^{-1} = 1.$

To begin with, setting $x^a = p$, $x^b = q$, and $x^c = r$, we get this:

$$(1 + x^{a-b} + x^{a-c})^{-1} + (1 + x^{b-c} + x^{b-a})^{-1} + (1 + x^{c-a} + x^{c-b})^{-1}$$

$$= (1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}+(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}+(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}.$$

Next, taking care of each term on the left hand side, we get these:

$$1+\tfrac{p}{q}+\tfrac{p}{r}=\tfrac{qr}{qr}+\tfrac{pr}{qr}+\tfrac{pq}{qr}=\tfrac{qr+pr+pq}{qr}\Rightarrow(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}=\tfrac{qr}{qr+pr+pq}.$$

$$1+\tfrac{q}{r}+\tfrac{q}{p}=\tfrac{pr}{pr}+\tfrac{pq}{pr}+\tfrac{qr}{pr}=\tfrac{pr+pq+qr}{pr}\Rightarrow(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}=\tfrac{pr}{pr+pq+qr}.$$

$$1+\tfrac{r}{p}+\tfrac{r}{q}=\tfrac{pq}{pq}+\tfrac{qr}{pq}+\tfrac{pr}{pq}=\tfrac{pq+qr+pr}{pq}\Rightarrow(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=\tfrac{pq}{pq+qr+pr}.$$

Thus, we get

$$(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}+(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}+(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=\tfrac{qr}{qr+pr+pq}+\tfrac{pr}{pr+pq+qr}+\tfrac{pq}{pq+qr+pr}=\tfrac{pq+qr+rp}{pq+qr+rp}=1.$$

If not quite sure of the idea behind the processes above, follow the steps below.

Examining first, the equality above, we can see that it is mainly made of three different powers, and the powers are x^a, x^b, and x^c. How?

For instance, we have $x^{a-b} = \dfrac{x^a}{x^b}$.

So other than 1 in the equality given in this problem, each of the terms can be taken as a fraction where each of the three powers above takes the position of the numerator or the denominator in alternate manner, and with the same frequency.

Thus, we can also, notice that every power is in the same situation.

So we can expect this:

The left hand side of the equality given will be eventually expressed in a fractional form. And since the right hand side is 1, what's in the numerator will be the same as what's in the denominator.

For instance, we may eventually be able to convert the left hand side of the equality into something like $\frac{x^{ab} + x^{bc} + x^{ca}}{x^{ab} + x^{bc} + x^{ca}}$, which equals 1.

What about the exponent -1 used in the term $(1 + x^{a-b} + x^{a-c})^{-1}$ and the other two terms?

We can just take the reciprocal of each of the three terms.

So for instance, we can set $(1 + x^{a-b} + x^{a-c})^{-1} = \frac{1}{1 + x^{a-b} + x^{a-c}}$.

And using the facts above, we should be able to get 1 doing some exponential algebra on the left hand side of the quality given. So let's now, begin the algebra.

We want to show $(1 + x^{a-b} + x^{a-c})^{-1} + (1 + x^{b-c} + x^{b-a})^{-1} + (1 + x^{c-a} + x^{c-b})^{-1} = 1$, which has quite a few powers.

Carrying around too many powers doing algebra, we are prone to errors.

So we may want to begin with simplification of those powers.

The powers used in the equality are x^a, x^b, and x^c, and are repeated often. So?

So substituting each of those with a letter as $\boldsymbol{p}$, $\boldsymbol{q}$, or $\boldsymbol{r}$, we can make the equality simpler and clearer, that is, more succinct so that we can do the algebra more easily.
And thus, setting $x^a = p$, $x^b = q$, and $x^c = r$, we can simplify the left hand side, and get

$$(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}+(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}+(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=1.$$

Next, taking care of each term on the left hand side, we get these:

$$1+\tfrac{p}{q}+\tfrac{p}{r}=\tfrac{qr}{qr}+\tfrac{pr}{qr}+\tfrac{pq}{qr}=\tfrac{qr+pr+pq}{qr}\Rightarrow(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}=\tfrac{qr}{qr+pr+pq}.$$

$$1+\tfrac{q}{r}+\tfrac{q}{p}=\tfrac{pr}{pr}+\tfrac{pq}{pr}+\tfrac{qr}{pr}=\tfrac{pr+pq+qr}{pr}\Rightarrow(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}=\tfrac{pr}{pr+pq+qr}.$$

$$1+\tfrac{r}{p}+\tfrac{r}{q}=\tfrac{pq}{pq}+\tfrac{qr}{pq}+\tfrac{pr}{pq}=\tfrac{pq+qr+pr}{pq}\Rightarrow(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=\tfrac{pq}{pq+qr+pr}.$$

Thus, we get

$$(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}+(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}+(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=\tfrac{qr}{qr+pr+pq}+\tfrac{pr}{pr+pq+qr}+\tfrac{pq}{pq+qr+pr}=\tfrac{pq+qr+rp}{pq+qr+rp}=1.$$

In short:

To begin with, setting $x^a = p$, $x^b = q$, and $x^c = r$, we get this:

$$(1 + x^{a-b} + x^{a-c})^{-1} + (1 + x^{b-c} + x^{b-a})^{-1} + (1 + x^{c-a} + x^{c-b})^{-1}$$

$$= (1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}+(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}+(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}.$$

Next, taking care of each term on the left hand side, we get these:

$$1+\tfrac{p}{q}+\tfrac{p}{r}=\tfrac{qr}{qr}+\tfrac{pr}{qr}+\tfrac{pq}{qr}=\tfrac{qr+pr+pq}{qr}\Rightarrow(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}=\tfrac{qr}{qr+pr+pq}.$$

$$1+\tfrac{q}{r}+\tfrac{q}{p}=\tfrac{pr}{pr}+\tfrac{pq}{pr}+\tfrac{qr}{pr}=\tfrac{pr+pq+qr}{pr}\Rightarrow(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}=\tfrac{pr}{pr+pq+qr}.$$

$$1+\tfrac{r}{p}+\tfrac{r}{q}=\tfrac{pq}{pq}+\tfrac{qr}{pq}+\tfrac{pr}{pq}=\tfrac{pq+qr+pr}{pq}\Rightarrow(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=\tfrac{pq}{pq+qr+pr}.$$ Thus, we get

$$(1+\tfrac{p}{q}+\tfrac{p}{r})^{-1}+(1+\tfrac{q}{r}+\tfrac{q}{p})^{-1}+(1+\tfrac{r}{p}+\tfrac{r}{q})^{-1}=\tfrac{qr}{qr+pr+pq}+\tfrac{pr}{pr+pq+qr}+\tfrac{pq}{pq+qr+pr}=\tfrac{pq+qr+rp}{pq+qr+rp}=1.$$

Suggestions or Solutions

To the Problem in the Example 1

Assuming that $a > b > c > 0$, show that $a^{2a}b^{2b}c^{2c} > a^{b+c}b^{c+a}c^{a+b}$.

$a > b \Rightarrow \frac{a}{b} > 1$ and $a - b > 0$. Thus, $(\frac{a}{b})^{a-b} > 1$.

Then, $(\frac{a}{b})^{a-b} = (\frac{a}{b})^{a}(\frac{a}{b})^{-b} = a^a b^{-a} a^{-b} b^b > 1$. Thus, $a^a b^b > a^b b^a$.

$b > c \Rightarrow \frac{b}{c} > 1$ and $b - c > 0$. So $(\frac{b}{c})^{b-c} > 1$.

Then, $(\frac{b}{c})^{b-c} = (\frac{b}{c})^{b}(\frac{b}{c})^{-c} = b^b c^{-b} b^{-c} c^c > 1$. So $b^b c^c > b^c c^b$.

$a > c \Rightarrow \frac{a}{c} > 1$ and $a - c > 0$. So $(\frac{a}{c})^{a-c} > 1$.

$(\frac{a}{c})^{a-c} = (\frac{a}{c})^{a}(\frac{a}{c})^{-c} = a^a c^{-a} a^{-c} c^c > 1$. So $a^a c^c > a^c c^a$.

Now, we have $a^a b^b > a^b b^a$, $b^b c^c > b^c c^b$, and $a^a c^c > a^c c^a$.

The product of all the terms on the left hand side is $(a^a b^b)(b^b c^c)(a^a c^c) = a^{2a}b^{2b}c^{2c}$.

The product of all the terms on the right hand side is $(a^b b^a)(b^c c^b)(a^c c^a) = a^{b+c}b^{a+c}c^{b+a}$.

Therefore, $a^{2a}b^{2b}c^{2c} > a^{b+c}b^{a+c}c^{b+a}$.

If not quite sure of the idea behind the processes above, follow the steps below.

This example provides another practice on exponential algebra.
In particular, the practice is on relational expressions with powers.
That is, it is about exponential inequalities.

More specifically though, the expression $\boldsymbol{a^{2a}b^{2b}c^{2c} > a^{b+c}b^{c+a}c^{a+b}}$ we have to show can be called a conditional inequality. What then is the condition?

The statement that $\boldsymbol{a > b > c > 0}$ is the condition.

In this case though, the condition is not a premise as the ones we have to meet solving problems. That is, it is rather a means we can take advantage of than a burden to us.

In fact, it is the base on which the conditional inequality can hold. So we should begin with it, and also, be able to make use of it.

We want to know first though, what we need to show, and how we are going to do it.

So to begin with, examining the inequality we want to show, we can see these:

• The expression is made of some powers, and those powers have three bases, which are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, which are all positive.

• The expression is basically composed of $\boldsymbol{a^a}$, $\boldsymbol{a^b}$, $\boldsymbol{a^c}$, $\boldsymbol{b^a}$, $\boldsymbol{b^b}$, $\boldsymbol{b^c}$, $\boldsymbol{c^a}$, $\boldsymbol{c^b}$, and $\boldsymbol{c^c}$, so each of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is used as a base and an exponent in a power. And also, the bases are used in the condition we want to take advantage of. So we may want to consider using each of the three as both, a base and an exponent.

• In addition, the bases are used in alternate manner, and with the same frequency. So all the bases repeat themselves in the same manner.

Next, examining the condition given, we can see it is not just an inequality, but can give us three different relational expressions as follows: $\boldsymbol{a > b > 0}$, $\boldsymbol{b > c > 0}$, and $\boldsymbol{a > c > 0}$.

Let’s now, come up with the relational expression, $\boldsymbol{a^{2a}b^{2b}c^{2c} > a^{b+c}b^{c+a}c^{a+b}}$.

We can’t just do so, of course. So what do we begin with?

By means of the fact that $\boldsymbol{a > b > c > 0}$, we are going to make $\boldsymbol{a^a}$, $\boldsymbol{a^b}$, $\boldsymbol{a^c}$, $\boldsymbol{b^a}$, $\boldsymbol{b^b}$, etc.

So first, we may want to begin with the inequality where $\boldsymbol{a > b > 0}$.

Then, since $\boldsymbol{a > b}$, we can get $\boldsymbol{\frac{a}{b} > 1}$, and $\boldsymbol{a - b > 0}$. What then can we get taking $\boldsymbol{\frac{a}{b}}$ as a base, and taking $(\boldsymbol{a - b})$ as an exponent?

We can get $\boldsymbol{(\frac{a}{b})^{a-b} > 1}$.

Is the inequality above still true though, even if we have this: $\boldsymbol{0 < a - b < 1}$?

For instance, we know $\boldsymbol{3 > 2}$, and $\boldsymbol{\sqrt{3} > \sqrt{2}}$, so $\boldsymbol{(\frac{3}{2})^{\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{2}} > 1}$. And we have $\boldsymbol{1^{\frac{1}{4}} = \sqrt[4]{1} = 1}$.

So since $\boldsymbol{\frac{a}{b} > 1}$, we get $\boldsymbol{(\frac{a}{b})^{\frac{1}{4}} > 1^{\frac{1}{4}} = 1}$.

Next, we can make use of the inequality where $\boldsymbol{b > c > 0}$.

Then, we can get $\boldsymbol{\frac{b}{c} > 1}$, and $\boldsymbol{b - c > 0}$.

What then can we get taking $\boldsymbol{\frac{b}{c}}$ as a base, and taking $(\boldsymbol{b - c})$ as an exponent?

We can get $\boldsymbol{(\frac{b}{c})^{b-c} > 1}$.

And next, we have $\boldsymbol{a > c > 0}$. So we get $\boldsymbol{\frac{a}{c} > 1}$, and $\boldsymbol{a - c > 0}$.

What then can we get taking $\boldsymbol{\frac{a}{c}}$ as a base, and taking $(\boldsymbol{a - c})$ as an exponent?

We can get $\boldsymbol{(\frac{a}{c})^{a-c} > 1}$.

So we now have $(\frac{a}{b})^{a-b} > 1$, $(\frac{b}{c})^{b-c} > 1$, and $(\frac{a}{c})^{a-c} > 1$.

Next, we are going to refine each of the expressions above.

To begin with, we have

$(\frac{a}{b})^{a-b} = (\frac{a}{b})^{a}(\frac{a}{b})^{-b} = (a^1b^{-1})^a(a^1b^{-1})^{-b} = a^ab^{-a}a^{-b}b^b = a^{a-b}b^{-a+b} > 1$. Thus,

multiplying both sides by a^bb^a respectively, we get $a^{a-b+b}b^{-a+b+a} > a^bb^a \Rightarrow a^ab^b > a^bb^a$.

Next, we have

$(\frac{b}{c})^{b-c} = (\frac{b}{c})^{b}(\frac{b}{c})^{-c} = (b^1c^{-1})^b(b^1c^{-1})^{-c} = b^bc^{-b}b^{-c}c^c = b^{b-c}c^{-b+c} > 1$. Thus,

multiplying both sides by b^cc^b respectively, we get $b^{b-c+c}c^{-b+c+b} > b^cc^b \Rightarrow b^bc^c > b^cc^b$.

Next, we have

$(\frac{a}{c})^{a-c} = (\frac{a}{c})^{a}(\frac{a}{c})^{-c} = (a^1c^{-1})^a(a^1c^{-1})^{-c} = a^ac^{-a}a^{-c}c^c = a^{a-c}c^{-a+c} > 1$. Thus,

multiplying both sides by a^cc^a respectively, we get $a^{a-c+c}c^{-a+c+a} > a^cc^a \Rightarrow a^ac^c > a^cc^a$.

Now, we have $a^ab^b > a^bb^a$, $b^bc^c > b^cc^b$, and $a^ac^c > a^cc^a$.

So taking the product of all the terms on the left hand side, we get $a^ab^bb^bc^ca^ac^c$, which equals $a^{2a}b^{2b}c^{2c}$.

Next, taking the product of all the terms on the right hand side, we get $a^bb^ab^cc^ba^cc^a$, which equals $a^{b+c}b^{a+c}c^{b+a}$.

Therefore, we get $a^{2a}b^{2b}c^{2c} > a^{b+c}b^{a+c}c^{b+a}$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming that $b^2 = ac$, simplify the expression below.
$a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3})$.

We have $a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3}) = \dfrac{a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})}{a^3+b^3+c^3}$.

Meanwhile, $a^2b^2c^2(a^{-3} + b^{-3} + c^{-3}) = a^{-1}b^2c^2 + a^2b^{-1}c^2 + a^2b^2c^{-1}$.

So we get these:

$b^2 = ac \Rightarrow a^{-1}b^2c^2 = a^{-1}(ac)c^2 = c^3$.

$b^2 = ac \Rightarrow b^4 = a^2c^2 \Rightarrow a^2b^{-1}c^2 = a^2c^2b^{-1} = b^4b^{-1} = b^3$.

$b^2 = ac \Rightarrow a = b^2c^{-1} \Rightarrow a^2b^2c^{-1} = a^2a = a^3$.

Thus, $a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3}) = a^2b^2c^2(a^{-3} + b^{-3} + c^{-3})(a^3 + b^3 + c^3)^{-1}$
$= (c^3 + b^3 + a^3)(a^3 + b^3 + c^3)^{-1} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

The expression to be simplified is the same as $a^2b^2c^2(a^{-3} + b^{-3} + c^{-3})/(a^3 + b^3 + c^3)$, which equals $\dfrac{a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})}{a^3+b^3+c^3}$, which however, does not look like we can simplify any further. It just seems to be no more than a fractional expression if we just look at it.

Examining the expression closely though, we can see that the constants ***a***, ***b***, and ***c*** are used with the same frequency, and exactly in the same manner. So we can expect some similarity between the constants.

Besides, we have a condition on the expression, which provides us with a connective expression among the constants. So we should take advantage of it. Taking the benefit of it though, we may want to begin with expanding the numerator in the expression below.

$$\frac{a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})}{a^3+b^3+c^3}$$

Then, we get $a^2b^2c^2(a^{-3} + b^{-3} + c^{-3}) = a^{-1}b^2c^2 + a^2b^{-1}c^2 + a^2b^2c^{-1}$. What then is the next?

We can now use this condition: $b^2 = ac$.

We may not just want to take though, the condition as is.
But we may want to take it in many different ways, too.
In other words, we should be able to convert the condition as many ways as necessary.

For instance, we can convert it into $c = b^2a^{-1}$, $a = b^2c^{-1}$, $b^4 = a^2c^2$, etc.

Let's now, apply the condition to the numerator of $\frac{a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})}{a^3+b^3+c^3}$.

To begin with, we have $a^{-1}b^2c^2$ term in the numerator.

So we can get $b^2 = ac \Rightarrow a^{-1}b^2c^2 = a^{-1}(ac)c^2 = c^3$.

Next, we have $a^2b^{-1}c^2$ term.
And also, we can have $b^2 = ac \Rightarrow b^4 = a^2c^2$.

Thus, we can get $b^4 = a^2c^2 \Rightarrow a^2b^{-1}c^2 = a^2c^2b^{-1} = b^4b^{-1} = b^3$.

Next, we have $a^2b^2c^{-1}$ term, and also, can have $b^2 = ac \Rightarrow a = b^2c^{-1}$.

Thus, we can get $a = b^2c^{-1} \Rightarrow a^2b^2c^{-1} = a^2a = a^3$.

So we can see that $a^2b^2c^2(a^{-3} + b^{-3} + c^{-3}) = a^{-1}b^2c^2 + a^2b^{-1}c^2 + a^2b^2c^{-1} = a^3 + b^3 + c^3$.

Thus, we can get $\dfrac{a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})}{a^3+b^3+c^3} = \dfrac{a^3+b^3+c^3}{a^3+b^3+c^3} = 1$.

Therefore, we get $a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3}) = 1$.

Quite often, we can put an expression in some different ways.

So the same expressions can look different.

And thus, we can put $a^{-3} + b^{-3} + c^{-3}$ the way below, too.

$$a^{-3} + b^{-3} + c^{-3} = \frac{b^3c^3 + a^3c^3 + a^3b^3}{a^3b^3c^3}.$$

Next, we can get $a^2b^2c^2 = a^2acc^2 = a^3c^3$, since $b^2 = ac$.
So we get this:

$$a^2b^2c^2(a^{-3} + b^{-3} + c^{-3}) = \frac{a^3c^3(b^3c^3 + a^3c^3 + a^3b^3)}{a^3b^3c^3} = \frac{b^3c^3 + a^3c^3 + a^3b^3}{b^3}$$

$$= \frac{b^3c^3 + b^6 + a^3b^3}{b^3} \quad \text{(since } b^2 = ac \Rightarrow b^6 = a^3c^3.\text{)}$$

$$= c^3 + b^3 + a^3.$$

Therefore, we get $a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3}) = 1$.

In short:

We have $a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3}) = \dfrac{a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})}{a^3+b^3+c^3}$.

Meanwhile, $\boldsymbol{a^2b^2c^2(a^{-3} + b^{-3} + c^{-3}) = a^{-1}b^2c^2 + a^2b^{-1}c^2 + a^2b^2c^{-1}}$.

So we get these:

$\boldsymbol{b^2 = ac \Rightarrow a^{-1}b^2c^2 = a^{-1}(ac)c^2 = c^3}$.

$\boldsymbol{b^2 = ac \Rightarrow b^4 = a^2c^2 \Rightarrow a^2b^{-1}c^2 = a^2c^2b^{-1} = b^4b^{-1} = b^3}$.

$\boldsymbol{b^2 = ac \Rightarrow a = b^2c^{-1} \Rightarrow a^2b^2c^{-1} = a^2a = a^3}$.

Thus, $\boldsymbol{a^2b^2c^2(a^3 + b^3 + c^3)^{-1}(a^{-3} + b^{-3} + c^{-3}) = a^2b^2c^2(a^{-3} + b^{-3} + c^{-3})(a^3 + b^3 + c^3)^{-1}}$

$\boldsymbol{= (c^3 + b^3 + a^3)(a^3 + b^3 + c^3)^{-1} = 1}$.