Examples 3 on Definitions for Logarithms

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Find the value of $\boldsymbol{x}$ in each case below.

0. $\boldsymbol{x} = \mathbf{log_9}(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}})$ where $\boldsymbol{a} + \boldsymbol{b} = \mathbf{21}$ and $\boldsymbol{ab} = \mathbf{9}$.

1. $\boldsymbol{x} = \mathbf{log_9} \left| \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right|$ where $\boldsymbol{a} + \boldsymbol{b} = \mathbf{9}$ and $\boldsymbol{ab} = \mathbf{9}$.

2. $\boldsymbol{x} = \mathbf{log_{16}}(\frac{3b}{a^2+2} + \frac{3a}{b^2+2})$ where $\boldsymbol{a} + \boldsymbol{b} = \mathbf{6}$ and $\boldsymbol{ab} = \mathbf{2}$.

3. $\boldsymbol{x} = \mathbf{log}_{(3\sqrt{2}-2\sqrt{3})}(\boldsymbol{a}^2 - 4\boldsymbol{a} + 1)$ where $\boldsymbol{a} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

4. $\boldsymbol{x} = \mathbf{log_5}(\boldsymbol{a}^2 - \boldsymbol{ab} + \boldsymbol{b}^2)$ where $\boldsymbol{a} = \sqrt{3} - \sqrt{2}$, and $\boldsymbol{b} = \sqrt{3} + \sqrt{2}$.

5. Assuming $\dfrac{1}{\log_x y} + \log_x y = -\dfrac{10}{3}$, find the value of $(x^3y - 1)(y^3x - 1)$.

6. Show that $\mathbf{log_3 2}$ is not a rational number.

Suggestions or Solutions
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The definition for logs is $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$.

So using the definition, we can get the solutions the way below.

0. $\boldsymbol{x = \log_9(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}})}$ **where** $\boldsymbol{a + b = 21}$ **and** $\boldsymbol{ab = 9}$**.**

First, we can have $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}$, which is positive.

And next, squaring it, we get $(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}})^2 = \frac{a+b+2\sqrt{ab}}{ab} = \frac{21+2\sqrt{9}}{9} = \frac{27}{9} = \mathbf{3}$.

So we can get $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}} = \sqrt{\mathbf{3}}$. And we know $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}$.

Thus, we get $\boldsymbol{x = \log_9(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}) = \log_9 \sqrt{3}}$. So we get $\boldsymbol{x = \log_9 \sqrt{3}}$.

And by the definition, we get $\boldsymbol{x = \log_9 \sqrt{3} \Leftrightarrow \sqrt{3} = 9^x}$.

And we have $\boldsymbol{\sqrt{3} = 3^{0.5}}$, and $\boldsymbol{9^x = 3^{2x}}$. So we get $\boldsymbol{\sqrt{3} = 9^x \Rightarrow 3^{0.5} = 3^{2x}}$.

Thus, we get $\boldsymbol{2x = 0.5 \Rightarrow x = 1/4}$.

In short:

To begin with, we can get $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}$, and $(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}})^2 = \frac{a+b+2\sqrt{ab}}{ab} = \frac{21+2\sqrt{9}}{9} = \frac{27}{9} = \mathbf{3}$.

So we can get $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}} = \sqrt{\mathbf{3}} \Rightarrow \boldsymbol{x = \log_9(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}) = \log_9 \sqrt{3} \Rightarrow x = \log_9 \sqrt{3} \Leftrightarrow \sqrt{3} = 9^x}$.

And thus, we get $\boldsymbol{\sqrt{3} = 9^x \Rightarrow 3^{0.5} = 3^{2x} \Rightarrow 2x = 0.5 \Rightarrow x = 1/4}$.

1. $x = \log_9 |\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}|$ **where $a + b = 9$ and $ab = 9$.**

First, we can have $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}$, of which we don't know the sign.

And next, squaring it, we get $(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}})^2 = \frac{a+b-2\sqrt{ab}}{ab} = \frac{9-2\sqrt{9}}{9} = \frac{3}{9} = \frac{1}{3}$.

And we know $\left|\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}\right| > \mathbf{0}$ if $\boldsymbol{a \neq b}$, of course.

So we can get $\left|\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}\right| = \sqrt{\frac{1}{3}}$. And we know $\left|\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right| = \left|\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}\right|$.

Thus, we get $x = \log_9 \left|\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right| = \log_9 \sqrt{\frac{1}{3}}$. So we get $x = \log_9 \sqrt{\frac{1}{3}}$.

And by the definition, we get $x = \log_9 \sqrt{\frac{1}{3}} \Leftrightarrow \sqrt{\frac{1}{3}} = 9^x$.

And we have $\sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3^{0.5}} = 3^{-0.5}$, and $9^x = 3^{2x}$. So we get $\sqrt{\frac{1}{3}} = 9^x \Rightarrow 3^{-0.5} = 3^{2x}$.

Thus, we get $\mathbf{2x = -0.5 \Rightarrow x = -1/4}$.

In short:

To begin with, we can get $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}$, and $(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}})^2 = \frac{a+b-2\sqrt{ab}}{ab} = \frac{9-2\sqrt{9}}{9} = \frac{3}{9} = \frac{1}{3}$.

So we can get $\left|\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}\right| = \sqrt{\frac{1}{3}} \Rightarrow x = \log_9 \left|\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right| = \log_9 \sqrt{\frac{1}{3}} \Rightarrow x = \log_9 \sqrt{\frac{1}{3}} \Leftrightarrow \sqrt{\frac{1}{3}} = 9^x$.

And thus, we get $\sqrt{\frac{1}{3}} = 9^x \Rightarrow 3^{-0.5} = 3^{2x} \Rightarrow 2x = -0.5 \Rightarrow x = -1/4$.

2. $x = \log_{16}(\frac{3b}{a^2+2} + \frac{3a}{b^2+2})$ **where** $a + b = 6$ **and** $ab = 2$**.**

To begin with, we can have $\frac{3b}{a^2+2} + \frac{3a}{b^2+2} = \frac{3b(b^2+2)+3a(a^2+2)}{(a^2+2)(b^2+2)} = \frac{3b^3+3a^3+6(a+b)}{a^2b^2+2(a^2+b^2)+4} = \frac{3(b^3+a^3)+6(a+b)}{a^2b^2+2(a^2+b^2)+4}$.

Next, we can have $a^2 + b^2 = (a + b)^2 - 2ab$.

So we get $a^2 + b^2 = 6^2 - 2 \cdot 2 = 32$.

And next, we can have $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$.

So we get $a^3 + b^3 = 6^3 - 3 \cdot 2 \cdot 6 = 6^3 - 6^2 = 6^2(6 - 1) = 36 \cdot 5 = 180$.

Thus, we get $\frac{3(b^3+a^3)+6(a+b)}{a^2b^2+2(a^2+b^2)+4} = \frac{3 \cdot 180+6 \cdot 6}{2^2+2 \cdot 32+4} = \frac{540+36}{4+64+4} = \frac{576}{72} = 8$.

So we get $x = \log_{16}(\frac{3b}{a^2+2} + \frac{3a}{b^2+2}) = \log_{16} 8 \Rightarrow x = \log_{16} 8$.

And by the definition, we get $x = \log_{16} 8 \Leftrightarrow 8 = 16^x$.

And we have $8 = 2^3$, and $16^x = 2^{4x}$.

Thus, we get $4x = 3 \Rightarrow x = 3/4$.

3. $x = \log_{(3\sqrt{2}-2\sqrt{3})}(a^2 - 4a + 1)$ **where** $a = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

To begin with, we can have: $a = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{3+2+2\sqrt{6}}{3-2} = 5 + 2\sqrt{6}$.

So next, we can have $a^2 - 4a + 1 = (5 + 2\sqrt{6})^2 - 4(5 + 2\sqrt{6}) + 1$

$= 25 + 24 + 20\sqrt{6} - 20 - 8\sqrt{6} + 1 = 30 - 12\sqrt{6} = 2(15 - 6\sqrt{6}) = 2(15 - 2\sqrt{9 \cdot 6})$

$= 2(9 + 6 - 2\sqrt{9 \cdot 6}) = 2(\sqrt{9} - \sqrt{6})^2 = 2(3 - \sqrt{6})^2$

$= (\sqrt{2})^2(3 - \sqrt{6})^2 = (3\sqrt{2} - \sqrt{12})^2 = (3\sqrt{2} - 2\sqrt{3})^2$.

So we get $x = \log_{(3\sqrt{2}-2\sqrt{3})}(a^2 - 4a + 1) = \log_{(3\sqrt{2}-2\sqrt{3})}(3\sqrt{2} - 2\sqrt{3})^2$.

And next, setting $b = 3\sqrt{2} - 2\sqrt{3}$, we get $x = \log_b b^2$.

And by the definition, we get $x = \log_b b^2 \Leftrightarrow b^2 = b^x$. Thus, we get $x = 2$.

4. $x = \log_5 (a^2 - ab + b^2)$ **where** $a = \sqrt{3} - \sqrt{2}$, and $b = \sqrt{3} + \sqrt{2}$.

To begin with, we can have $a^2 - ab + b^2 = (a - b)^2 + ab$.

So next, we can have $a^2 - ab + b^2 = \{(\sqrt{3} - \sqrt{2}) - (\sqrt{3} + \sqrt{2})\}^2 + (\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

$= (-\sqrt{2})^2 + (3 - 2) = 4 + 1 = 5$.

So we get $x = \log_5 (a^2 - ab + b^2) = \log_5 5 \Rightarrow x = \log_5 5$.

And by the definition, we get $x = \log_5 5 \Leftrightarrow 5 = 5^x$. Thus, we get $x = 1$.

5. Assuming $\frac{1}{\log_x y} + \log_x y = -\frac{10}{3}$, find the value of $(x^3y - 1)(y^3x - 1)$.

Assuming first, $a = \log_x y$, we can set $1/a + a = -10/3$.

Next, we can solve for a the equation above the way below.

$1/a + a = -10/3 \Rightarrow 1 + a^2 = -10a/3 \Rightarrow 3 + 3a^2 = -10a \Rightarrow 3a^2 + 10a + 3 = 0$.

$3a^2 + 10a + 3 = 3a^2 + 9a + a + 3 = 3a(a + 3) + (a + 3) = (a + 3)(3a + 1) = 0$.

So we get $a = -3$ or $-1/3$. And we know $a = \log_x y$.

So first, we can get $\log_x y = -3 \Rightarrow y = x^{-3}$ by the definition for logs.
And next, by the definition for logs again, we can get $\log_x y = -1/3 \Rightarrow y = x^{-1/3} \Rightarrow x = y^{-3}$.

So putting threads together, we have $y = x^{-3}$ or $x = y^{-3}$.
And we can put them this way, too: $x^3y = 1$ or $y^3x = 1$.

So first, $x^3y = 1 \Rightarrow (x^3y - 1)(y^3x - 1) = 0$.
And next, $y^3x = 1 \Rightarrow (x^3y - 1)(y^3x - 1) = 0$.

6. Show that $\log_3 2$ is not a rational number.

Assuming $\log_3 2$ is a rational number, we can set $\log_3 2 = m/n$ where m and n are integers prime to each other.

Then, by the definition for logs, we get $2 = 3^{m/n}$. So we get $2^n = 3^m$, which is not possible, because 2^n is even, but 3^m is odd.

So it is not the case where $2^n = 3^m$, which means it cannot be the case where we can set $\log_3 2 = m/n$ where m and n are integers prime to each other, which means $\log_3 2$ is not a rational number.